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THE RELATIONSHIP BETWEEN FRAGMENTABLE SPACES AND CLASS $\mathcal{L}$ SPACES

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ABSTRACT. In this note, we show that each fragmentable space introduced by Jayne and Rogers in 1985 is of class $\mathcal{L}$ which was introduced by Kenderov in 1984. Our example shows that a space which is of class $\mathcal{L}$ may not be a fragmentable space.

1. INTRODUCTION AND PRELIMINARIES

In order to investigate the existence of 'nice' selections for upper semicontinuous set-valued mappings with compact values, Jayne and Rogers in [1] (see also Ribarska in [4, p.243]) introduced the concept called '*fragmentable spaces*'. On the other hand, for the purpose of study on the uniqueness of optimization problems, Kenderov in [2] introduced another kind of spaces called '*class* $\mathcal{L}$'. Since then, these two concepts have been playing a very important role in the study of the geometry of Banach spaces and the generic stability and uniqueness of nonlinear analysis related to optimization problems and mathematical programming (e.g., Jane and Rogers [1], Hansell et al. [7]–[8]; Kenderov [2], Beer [6] and references therein). However, to the best of the authors knowledge, so far, there has been no discussion on the relationship between these two classes of spaces. It is our purpose in this note, to discuss the relationship between fragmentable spaces and spaces of class $\mathcal{L}$. Precisely, we prove that each fragmentable space is of class $\mathcal{L}$. But the converse is not true in general, i.e., there exists an example which shows that a space which is of class $\mathcal{L}$ is not a fragmentable space.

Now we recall some notation and definitions. Let Y be a space with metric ρ, and U a nonempty subset of Y. We shall denote by ρ-$diam(U)$ the diameter of U determined by the metric ρ. The following definition was first introduced by Jayne and Rogers in [1]:

Definition 1. Let Y be a Hausdorff topological space and ρ a metric (it may not have any relationship with the topology of Y). For each given $\epsilon > 0$, if for any non-empty subset G of Y, there exists a non-empty relatively open subset U of G such that ρ-$diam(U) \leq \epsilon$, then Y is said to be *a fragmentable space*.

Let X and Y be two Hausdorff topological spaces and $F : X \to 2^Y \setminus \{\emptyset\}$ a set-valued mapping. Then **(1):** F is said to be *upper* (resp., *lower*) *semicontinuous*

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if the set $\{x \in X : F(x) \subset G\}$ is open (resp., closed) for each open (resp., closed) subset G of Y; **(2):** F is said to be *almost lower semicontinuous at* $x_0 \in X$ if there exists $y_0 \in F(x_0)$ such that for any non-empty open neighborhood $N(y_0)$, there exists a non-empty open neighborhood $N(x_0)$ of x_0 such that $N(y_0) \cap F(x)$ for each $x \in N(x_0)$; **(3):** F is said to be *almost lower semicontinuous* if F is almost lower semicontonuous at each point $x \in X$; and **(4):** F is said to be *USCO* if F is upper semicontinuous with non-empty and compact values. We recall that a topological space X is said to be Čech-complete (for instance, see Engelking [3]) if X is homeomorphic to a G_δ subset of a Hausdorff compact space. The following definition was first introduced by Kenderov in [2]:

Definition 2. A topological space Y is said to be of '*class* $\mathcal{L}$' if for each Čech-complete metric space X and each USCO mapping $F : X \to 2^Y$, there exists a dense G_δ subset Q of X such that F is almost lower semicontinuous on Q.

2. Main results

Now we have the following relationship between fragmentable spaces and spaces of class $\mathcal{L}$.

Theorem 1. *Each fragmentable topological space Y is of class $\mathcal{L}$. The converse is not true, i.e., there exists a space which is of class $\mathcal{L}$, but not fragmentable.*

Proof. We divide the proof into two parts. Let us first prove the first statement of Theorem 1. For each given Čech-complete metric space X, by Theorem 3.9.3 of Engelking [3], X must be a Baire space. Now for each USCO set-valued mapping $F : X \to 2^Y$, by Ribarska [4, p.249], it follows that F has a minimum mapping $F_1 : X \to 2^Y$ such that $F_1(x) \subset F(x)$ for each $x \in X$. By Proposition 2.5 in [4], there exists a dense G_δ subset Q of X such that $F_1(x)$ is a singleton set for each $x \in G_\delta$. Let $y_0 := F_1(x_0) \subset F(x_0)$ for each $x_0 \in Q$. Then for each non-empty open neighborhood $N(y_0)$ of y_0, we have $F_1(x_0) \subset N(y_0)$. By the upper semicontinuity of F_1 at x_0, there exists a non-empty open neighborhood $N(x_0)$ of x_0 such that $F_1(x) \subset N(y_0)$ for each $x \in N(x_0)$. As $F_1(x) \subset F(x)$ for all $x \in X$, it follows that $N(y_0) \cap F(x) \neq \emptyset$ for all $x \in N(x_0)$. Thus F is almost lower semicontinuous on Q, and we have proved that each fragmentable space is of class $\mathcal{L}$. The second part of Theorem 1 will be verified by the following two lemmas.

In order to verify the second statement of Theorem 1, we need some notation. Let Γ be an uncountable set and $\mathcal{L}^\infty(\Gamma)$ a space consisting of all real-valued functions defined on Γ with supremum norm. Then it is clear that $\mathcal{L}^\infty(\Gamma)$ is a Banach space. Let $\mathcal{L}^\infty_c(\Gamma)$ be a subspace of $\mathcal{L}^\infty(\Gamma)$ consisting of all real-valued bounded functions with countable supports. We have the following:

Lemma 1. *$\mathcal{L}^\infty_c(\Gamma)$ is a Banach space.*

Proof. Clearly $\mathcal{L}^\infty_c(\Gamma)$ is a linear space. To claim our conclusion, it suffices to show that $\mathcal{L}^\infty_c(\Gamma)$ is closed in $\mathcal{L}^\infty(\Gamma)$. Suppose not; then there exist a net $\{f_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{L}^\infty_c(\Gamma)$ and $f \in \mathcal{L}^\infty(\Gamma)$ such that $\|f_\alpha - f\| := \sup_{x\in\Gamma} |f_\alpha(x) - f(x)| \to 0$, where $f \notin \mathcal{L}^\infty_c(\Gamma)$. Then the set $\{x \in \Gamma : |f(x)| > 0\}$ is uncountable. Note that $\{x \in \Gamma : f(x) > 0\} = \cup_{n=1}^\infty \{x \in \Gamma : f(x) > \frac{1}{n}\}$, and there exists $n_0 \in \mathbb{N}$ such that the set $A := \{x \in \Gamma : f(x) > \frac{1}{n_0}\}$ is uncountable. As $\|f_\alpha - f\| \to 0$, without loss of generality, we may assume that there exists $\alpha_0 \in \Lambda$ such that for each $\alpha \in \Lambda$ with $\alpha \geq \alpha_0$, $\sup_{x\in\Gamma} |f_\alpha(x) - f(x)| < \frac{1}{2n_0}$. If $x \in A$, then $f_{\alpha_0}(x) > \frac{1}{n_0} - \frac{1}{2n_0} = \frac{1}{2n_0}$

which contradicts our assumption that f_{α_0} has a countable support. Thus the set $\{x \in \Gamma : |f(x)| > 0\}$ must be countable, i.e., $f \in \mathcal{L}_c^\infty(\Gamma)$ and we complete the proof. $\square$

Let B denote the closed unit ball in the space $\mathcal{L}_c^\infty(\Gamma)$. Then the following lemma shows that the second part of Theorem 1 is true:

Lemma 2. *The unit ball B with weaker topology in $\mathcal{L}_c^\infty(\Gamma)$ is of class $\mathcal{L}$, but it is not a fragmentable space.*

Proof. By Theorem 1 of Christensen [5], the Banach space $\mathcal{L}_c^\infty(\Gamma)$ with the weaker topology is of class $\mathcal{L}$ (see also Kenderov [2, p.207] and Beer [6, p.653]). By Proposition 2(a) of Kenderov [2], it follows that B with weaker topology is of class $\mathcal{L}$. On the other hand, the discussion in §5 of Hansell et al. [7]–[8] shows that B with weaker topology in $\mathcal{L}_c^\infty(\Gamma)$ is not a fragmentable space. Therefore we complete the proof of Theorem 1. $\square$

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