STRONGLY $\pi$-REGULAR RINGS
HAVE STABLE RANGE ONE

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Abstract. A ring $R$ is said to be strongly $\pi$-regular if for every $a \in R$ there exist a positive integer $n$ and $b \in R$ such that $a^n = a^{n+1}b$. For example, all algebraic algebras over a field are strongly $\pi$-regular. We prove that every strongly $\pi$-regular ring has stable range one. The stable range one condition is especially interesting because of Evans' Theorem, which states that a module $M$ cancels from direct sums whenever $\text{End}_R(M)$ has stable range one. As a consequence of our main result and Evans' Theorem, modules satisfying Fitting’s Lemma cancel from direct sums.

Introduction

Let $R$ be a ring, associative with unity. Recall that $R$ has stable range one provided that, for any $a, b \in R$ with $aR + bR = R$, there exists $y \in R$ such that $a + by$ is invertible in $R$. See [17] and [18]. In this note we will prove that strongly $\pi$-regular rings have stable range one. As a consequence we shall obtain that modules satisfying Fitting’s Lemma (over any ring) cancel from direct sums.

A ring $R$ is said to be strongly $\pi$-regular if for each $a \in R$ there exist a positive integer $n$ and $x \in R$ such that $a^n = a^{n+1}x$. By results of Azumaya [3] and Dischinger [8], the element $x$ can be chosen to commute with $a$. In particular, this definition is left-right symmetric. Strongly $\pi$-regular rings were introduced by Kaplansky [12] as a common generalization of algebraic algebras and artinian rings.

In [13], Menal proved that a strongly $\pi$-regular ring whose primitive factor rings are artinian has stable range one. In [11], various results concerning algebraic algebras and strongly $\pi$-regular rings were obtained. In particular, Goodearl and Menal showed that algebraic algebras over an infinite field have stable range one [11, Theorem 3.1] (in fact they showed the somewhat stronger condition called unit 1-stable range), and, in [11, p.271], they conjectured that any algebraic algebra has stable range one. Our Corollary 5 proves this conjecture. Further, they ask whether all strongly $\pi$-regular rings have stable range one [11, p.279], proving that the answer is affirmative in several cases. For instance, the strongly $\pi$-regular ring

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A ring $R$ has stable range one when either $R$ is a von Neumann regular ring [11, Theorem 5.8] or every element of $R$ is a sum of a unit plus a central unit [11, Corollary 6.2].

More recently, Yu [20] and Camillo and Yu [5] proved that strongly $\pi$-regular rings have stable range one under some additional hypothesis. For example, they show that a strongly $\pi$-regular ring such that every power of a regular element is regular, has stable range one [5, Theorem 5], generalizing [11, Theorem 5.8].

Goodearl and Menal also proved that a strongly $\pi$-regular ring has stable range one if and only if every nilpotent regular element of any corner of the ring is unit-regular in that corner [11, Theorem 6.1]. We will prove that in fact every nilpotent regular element of every exchange ring is unit-regular.

Let $R$ be any ring. An element $a \in R$ is said to be regular if there exists $b \in R$ such that $a = aba$. It is easy to see that $a$ is regular if and only if the right annihilator of $a$ (“the kernel of $a$”) and the right ideal generated by $a$ (“the image of $a$”) are both direct summands of $R_R$. The element $a \in R$ is said to be unit-regular if there exists an invertible element $u \in R$ such that $a = auu$. It is easy to see that $a$ is unit-regular if and only if $a$ is regular and $\text{rann}(a) \cong E$ as right $R$-modules, where $E$ denotes a complement of $aR$ in $R_R$ (“the kernel of $a$” is isomorphic to “the cokernel of $a$”).

An element $a \in R$ is said to be strongly $\pi$-regular if there exists a positive integer $n$ and $b \in R$ such that $a^n = a^{n+1}b$ and $ab = ba$. The ring $R$ is said to be strongly $\pi$-regular if every element of $R$ is strongly $\pi$-regular. By combining results of Dischinger [8] and Azumaya [3], one obtains the characterization of strongly $\pi$-regular rings as either the left $\pi$-regular rings or the right $\pi$-regular rings; see [11, p. 300] or [5, Lemma 6].

A right $R$-module $M$ has the exchange property (see [7]) if for every module $A_R$ and any decompositions

$$A = M' \oplus N = \bigoplus_{i \in I} A_i$$

with $M' \cong M$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus (\bigoplus_{i \in I} A'_i).$$

$M$ has the finite exchange property if the above condition is satisfied whenever the index set $I$ is finite. Clearly a finitely generated module satisfies the exchange property if and only if it satisfies the finite exchange property.

Following [19], we say that a ring $R$ is an exchange ring if $R_R$ satisfies the (finite) exchange property. By [19, Corollary 2], this definition is left-right symmetric.

Every strongly $\pi$-regular ring is an exchange ring [16, Example 2.3]. A great deal is known about strongly $\pi$-regular rings and exchange rings; see for example [4], [15], [16], [20] and [1].

The results

The following technical lemma is the key to obtain our main results.

**Lemma 1.** Let $R$ be an exchange ring and let $a$ be a regular element of $R$. Let $K$ denote the right annihilator of $a$, and $E$ be a complement of $aR$ in $R_R$. Then there exist right ideals $A_i$, $A'_i$, $B_i$, $B'_i$, $C_i$, $C'_i$, for $i \geq 1$, such that the following conditions are satisfied:

1. $R = K \oplus (\bigoplus_{j=1}^i (A_j \oplus B_j)) \oplus C_i$ for all $i \geq 1$. Hence, $C_i \cong A_{i+1} \oplus B_{i+1} \oplus C_{i+1}$. 


(2) \( E \cong (A_i \oplus B_i) \oplus (A'_i \oplus B'_i) \) for all \( i \geq 1 \).

(3) \( K \cong A'_i \oplus B'_i \oplus C'_i \) for all \( i \geq 1 \).

(4) \( A'_i \oplus B'_i = A'_{i+1} \oplus A'_{i+1} \) for all \( i \geq 1 \).

(5) \( aR = C_i \oplus C'_i \) and, for \( i \geq 1 \), \( aA_i \oplus aB_i = B_{i+1} \oplus B'_{i+1} \) and \( aC_i = C_{i+1} \oplus C'_{i+1} \).

Hence, \( C_{i+1} \subseteq a^{i+1}R \).

Proof. Write \( R = K \oplus L = E \oplus aR \). By using the exchange property we obtain \( A_1, C_1, A'_1, C'_1 \) such that \( R = K \oplus A_1 \oplus C_1 \), and \( E = A_1 \oplus A'_1 \), and \( aR = C_1 \oplus C'_1 \).

Note that \( K \oplus A_1 \oplus C_1 = A'_1 \oplus C'_1 \oplus A_1 \oplus C_1 \), and so \( K \cong A'_1 \oplus C'_1 \). Set \( B_1 = B'_1 = 0 \).

Now assume that, for some \( n \geq 1 \), we have constructed right ideals \( A_i, A'_i, B_i, B'_i, C_i, C'_i \) with \( i \leq n \), satisfying the desired conditions. We will construct \( A_{n+1}, A'_{n+1}, B_{n+1}, B'_{n+1}, C_{n+1}, C'_{n+1} \). Using (4) repeatedly and the fact that \( B'_1 = 0 \), we obtain

(6) \( A'_1 \oplus (\bigoplus_{i=1}^{n-1} B'_{i+1}) = \bigoplus_{i=1}^{n} (aA_i \oplus aB_i) \oplus aC_n \).

From (1) we have \( aR = (\bigoplus_{i=1}^{n} (aA_i \oplus aB_i)) \oplus aC_n \). By using this and relations (5) and (6) we obtain

\[
R = E \oplus aR = A_1 \oplus A'_1 \oplus \left( \bigoplus_{i=1}^{n} (aA_i \oplus aB_i) \right) \oplus aC_n
\]

\[
= (A_1 \oplus B_1) \oplus A'_1 \oplus \left( \bigoplus_{i=1}^{n-1} (B_{i+1} \oplus B'_{i+1}) \right) \oplus aA_n \oplus aB_n \oplus aC_n
\]

\[
= A_1 \oplus \left( \bigoplus_{i=1}^{n} B_i \right) \oplus A'_1 \oplus \left( \bigoplus_{i=1}^{n-1} B'_{i+1} \right) \oplus aA_n \oplus aB_n \oplus aC_n
\]

\[
= A_1 \oplus \left( \bigoplus_{i=1}^{n} B_i \right) \oplus \left( \bigoplus_{i=2}^{n} A_i \right) \oplus \left( \bigoplus_{i=1}^{n} A'_n \oplus B'_n \right) \oplus aA_n \oplus aB_n \oplus aC_n
\]

\[
= \left( \bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus \left( \bigoplus_{i=1}^{n} A'_n \oplus B'_n \right) \oplus aA_n \oplus aB_n \oplus aC_n.
\]

Now applying the exchange property to the decompositions

\[
R = K \oplus \left( \bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus C_n
\]

\[
= \left( \bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus \left( A'_n \oplus B'_n \right) \oplus (aA_n \oplus aB_n) \oplus aC_n,
\]

we obtain a decomposition

\[
R = K \oplus \left( \bigoplus_{i=1}^{n} (A_i \oplus B_i) \right) \oplus A_{n+1} \oplus B_{n+1} \oplus C_{n+1}
\]

such that

\[
A_{n+1} \oplus A'_{n+1} = A'_n \oplus B'_n
\]

for some right ideal \( A'_{n+1} \), while

\[
B_{n+1} \oplus B'_{n+1} = aA_n \oplus aB_n
\]
for some right ideal $B'_{n+1}$, and
$$C_{n+1} \oplus C'_{n+1} = aC_n$$
for some right ideal $C'_{n+1}$. So we obtain (1), (4) and (5).

Since $(\bigoplus_{i=1}^{n+1} (A_i \oplus B_i)) \oplus C_{n+1}$ is a common complement of both $K$ and $A'_{n+1} \oplus B'_{n+1} \oplus C'_{n+1}$, we obtain (3).

Now we will prove (2). We have
$$E \cong (A_n \oplus B_n) \oplus (A'_n \oplus B'_n)$$
$$\cong aA_n \oplus aB_n \oplus A_{n+1} \oplus A'_{n+1}$$
$$= B_{n+1} \oplus B'_{n+1} \oplus A_{n+1} \oplus A'_{n+1}$$
$$= (A_{n+1} \oplus B_{n+1}) \oplus (A'_{n+1} \oplus B'_{n+1}).$$

This completes the inductive step. \hfill \Box

**Theorem 2.** Let $R$ be an exchange ring and let $a$ be a nilpotent regular element of $R$. Then $a$ is unit-regular.

**Proof.** Assume that $a^{n+2} = 0$ for some $n \geq 0$. Let $A_1, A'_1, B_1, B'_1, C_i, C'_i$ be right ideals as in Lemma 1. Then $C_{n+1} \subseteq K$ by (5) of Lemma 1, and so $C_{n+1} = 0$ by (1). By (5), we have $C'_{n+1} = aC_n \cong C_n$ so that, using (1), we obtain $C'_{n+1} \cong C_n \cong A_{n+1} \oplus B_{n+1} \oplus C_{n+1} = A_{n+1} \oplus B_{n+1}$. Now using this fact and (2), (3), we have
$$K \cong A'_{n+1} \oplus B'_{n+1} \oplus C'_{n+1}$$
$$\cong (A'_{n+1} \oplus B'_{n+1}) \oplus (A_{n+1} \oplus B_{n+1}) \cong E.$$

We conclude that $a$ is unit-regular. \hfill \Box

**Theorem 3.** Let $R$ be an exchange ring and let $a$ be a regular element of $R$. If $a$ is strongly $\pi$-regular, then $a$ is unit-regular.

**Proof.** Let $a$ be a regular, strongly $\pi$-regular element of $R$. Let $b \in R$ be such that $a^n = a^{n+1}b$ for some $n \geq 1$, and $ab = ba$. Set $e = a^n b^n$ and note that $e$ is idempotent. Moreover, $ea = ae$ is invertible in $eRe$, with inverse $a^{n-1} b^{n+1}$, and $a(1-e) = (1-e) a \in (1-e)R(1-e)$ is a regular nilpotent element with $(a(1-e))^n = 0$. Since $(1-e)R(1-e)$ is an exchange ring [19, Theorem 2], it follows from Theorem 2 that $a(1-e)$ is unit-regular in $(1-e)R(1-e)$. Consequently, $a$ is unit-regular in $R$. \hfill \Box

**Theorem 4.** Strongly $\pi$-regular rings have stable range one.

**Proof.** By [16, Example 2.3], any strongly $\pi$-regular ring is an exchange ring. So the result follows from Theorem 3 and [5, Theorem 3]. Alternatively, one can use Theorem 2 and [11, Theorem 6.1]. \hfill \Box

Our next result proves the conjecture made by Goodearl and Menal in [11, p.271].

**Corollary 5.** Any algebraic algebra over a field has stable range one.

**Proof.** Clearly, an algebraic algebra over a field is strongly $\pi$-regular. So, the result follows from Theorem 4. \hfill \Box
A module $M$ is said to satisfy Fitting’s Lemma if for each $f \in \text{End}_R(M)$ there exists an integer $n \geq 1$ such that $M = \text{Ker}(f^n) \oplus f^n(M)$. By [2, Proposition 2.3], $M$ satisfies Fitting’s Lemma if and only if $\text{End}_R(M)$ is strongly \(\pi\)-regular. It was proved in [6] that modules satisfying Fitting’s Lemma have power cancellation. Theorem 4 enables us to improve this result, as follows.

**Corollary 6.** Let $M$ be a module satisfying Fitting’s Lemma. Then $M$ cancels from direct sums.

*Proof.* By [2, Proposition 2.3], $E := \text{End}_R(M)$ is a strongly \(\pi\)-regular ring. Now, Theorem 4 gives that the stable range of $E$ is one and, by Evans’ Theorem [9, Theorem 2], $M$ cancels from direct sums. \(\square\)

Now we will obtain some cancellation results for finitely generated modules over certain strongly \(\pi\)-regular rings. We need the following concept, introduced by Goodearl [10].

**Definition** ([10]). An element $u$ of a ring $R$ is said to be right repetitive provided that for each finitely generated right ideal $I$ of $R$, the right ideal $\sum_{n=0}^{\infty} u^n I$ is finitely generated. The ring $R$ is right repetitive if each element of $R$ is right repetitive.

Note that every algebraic algebra is (right and left) repetitive.

**Corollary 7.** Let $R$ be a strongly \(\pi\)-regular, right repetitive ring. Then any cyclic right $R$-module cancels from direct sums.

*Proof.* Apply [14, Theorem 19], Theorem 4 and Evans’ Theorem [9, Theorem 2]. \(\square\)

**Corollary 8.** If $R$ is a ring such that all matrix rings $M_n(R)$ are strongly \(\pi\)-regular and right repetitive, then any finitely generated right $R$-module cancels from direct sums.

*Proof.* Apply [10, Theorem 8], Theorem 4 and Evans’ Theorem [9, Theorem 2]. \(\square\)

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**References**


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