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STRONGLY π-REGULAR RINGS HAVE STABLE RANGE ONE

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Abstract. A ring R is said to be *strongly π-regular* if for every $a \in R$ there exist a positive integer n and $b \in R$ such that $a^n = a^{n+1}b$. For example, all algebraic algebras over a field are strongly π-regular. We prove that every strongly π-regular ring has stable range one. The stable range one condition is especially interesting because of Evans' Theorem, which states that a module M cancels from direct sums whenever $\mathrm{End}_R(M)$ has stable range one. As a consequence of our main result and Evans' Theorem, modules satisfying Fitting's Lemma cancel from direct sums.

Introduction

Let R be a ring, associative with unity. Recall that R has *stable range one* provided that, for any $a, b \in R$ with $aR + bR = R$, there exists $y \in R$ such that $a+by$ is invertible in R. See [17] and [18]. In this note we will prove that strongly π-regular rings have stable range one. As a consequence we shall obtain that modules satisfying Fitting's Lemma (over any ring) cancel from direct sums.

A ring R is said to be *strongly π-regular* if for each $a \in R$ there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$. By results of Azumaya [3] and Dischinger [8], the element x can be chosen to commute with a. In particular, this definition is left-right symmetric. Strongly π-regular rings were introduced by Kaplansky [12] as a common generalization of algebraic algebras and artinian rings.

In [13], Menal proved that a strongly π-regular ring whose primitive factor rings are artinian has stable range one. In [11], various results concerning algebraic algebras and strongly π-regular rings were obtained. In particular, Goodearl and Menal showed that algebraic algebras over an infinite field have stable range one [11, Theorem 3.1] (in fact they showed the somewhat stronger condition called *unit* 1-*stable range*), and, in [11, p.271], they conjectured that any algebraic algebra has stable range one. Our Corollary 5 proves this conjecture. Further, they ask whether all strongly π-regular rings have stable range one [11, p.279], proving that the answer is affirmative in several cases. For instance, the strongly π-regular ring

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R has stable range one when either R is a von Neumann regular ring [11, Theorem 5.8] or every element of R is a sum of a unit plus a central unit [11, Corollary 6.2].

More recently, Yu [20],[21] and Camillo and Yu [5] proved that strongly π-regular rings have stable range one under some additional hypothesis. For example they show that a strongly π-regular ring such that every power of a regular element is regular, has stable range one [5, Theorem 5], generalizing [11, Theorem 5.8].

Goodearl and Menal also proved that a strongly π-regular ring has stable range one if and only if every nilpotent regular element of any corner of the ring is unit-regular in that corner [11, Theorem 6.1]. We will prove that in fact every nilpotent regular element of every *exchange ring* is unit-regular.

Let R be any ring. An element $a \in R$ is said to be *regular* if there exists $b \in R$ such that $a = aba$. It is easy to see that a is regular if and only if the right annihilator of a ("the kernel of a") and the right ideal generated by a ("the image of a") are both direct summands of R_R. The element $a \in R$ is said to be *unit-regular* if there exists an invertible element $u \in R$ such that $a = aua$. It is easy to see that a is unit-regular if and only if a is regular and $\mathrm{rann}(a) \cong E$ as right R-modules, where E denotes a complement of aR in R_R ("the kernel of a" is isomorphic to "the cokernel of a").

An element $a \in R$ is said to be *strongly π-regular* if there exists a positive integer n and $b \in R$ such that $a^n = a^{n+1}b$ and $ab = ba$. The ring R is said to be *strongly π-regular* if every element of R is strongly π-regular. By combining results of Dischinger [8] and Azumaya [3], one obtains the characterization of strongly π-regular rings as either the left π-regular rings or the right π-regular rings; see [11, p. 300] or [5, Lemma 6].

A right R-module M has the *exchange property* (see [7]) if for every module A_R and any decompositions

$$A = M' \oplus N = \bigoplus_{i \in I} A_i$$

with $M' \cong M$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus (\bigoplus_{i \in I} A'_i).$$

M has the *finite exchange property* if the above condition is satisfied whenever the index set I is finite. Clearly a finitely generated module satisfies the exchange property if and only if it satisfies the finite exchange property.

Following [19], we say that a ring R is an *exchange ring* if R_R satisfies the (finite) exchange property. By [19, Corollary 2], this definition is left-right symmetric.

Every strongly π-regular ring is an exchange ring [16, Example 2.3]. A great deal is known about strongly π-regular rings and exchange rings; see for example [4], [15], [16], [20] and [1].

The results

The following technical lemma is the key to obtain our main results.

Lemma 1. *Let R be an exchange ring and let a be a regular element of R. Let K denote the right annihilator of a, and E be a complement of aR in R_R. Then there exist right ideals A_i, A'_i, B_i, B'_i, C_i, C'_i, for $i \geq 1$, such that the following conditions are satisfied:*

(1) $R = K \oplus (\bigoplus_{j=1}^{i} (A_j \oplus B_j)) \oplus C_i$ *for all $i \geq 1$. Hence,* $C_i \cong A_{i+1} \oplus B_{i+1} \oplus C_{i+1}$.

(2) *$E \cong (A_i \oplus B_i) \oplus (A'_i \oplus B'_i)$ for all $i \geq 1$.*

(3) *$K \cong A'_i \oplus B'_i \oplus C'_i$ for all $i \geq 1$.*

(4) *$A'_i \oplus B'_i = A_{i+1} \oplus A'_{i+1}$ for all $i \geq 1$.*

(5) *$aR = C_1 \oplus C'_1$ and, for $i \geq 1$, $aA_i \oplus aB_i = B_{i+1} \oplus B'_{i+1}$ and $aC_i = C_{i+1} \oplus C'_{i+1}$. Hence, $C_{i+1} \subseteq a^{i+1}R$.*

Proof. Write $R = K \oplus L = E \oplus aR$. By using the exchange property we obtain A_1, C_1, A'_1, C'_1 such that $R = K \oplus A_1 \oplus C_1$, and $E = A_1 \oplus A'_1$, and $aR = C_1 \oplus C'_1$. Note that $K \oplus A_1 \oplus C_1 = A'_1 \oplus C'_1 \oplus A_1 \oplus C_1$, and so $K \cong A'_1 \oplus C'_1$. Set $B_1 = B'_1 = 0$.

Now assume that, for some $n \geq 1$, we have constructed right ideals A_i, A'_i, B_i, B'_i, C_i, C'_i, with $i \leq n$, satisfying the desired conditions. We will construct A_{n+1}, A'_{n+1}, B_{n+1}, B'_{n+1}, C_{n+1}, C'_{n+1}. Using (4) repeatedly and the fact that $B'_1 = 0$, we obtain

(6) $A'_1 \oplus (\bigoplus_{i=1}^{n-1} B'_{i+1}) = (\bigoplus_{i=2}^{n} A_i) \oplus A'_n \oplus B'_n$.

From (1) we have $aR = (\bigoplus_{i=1}^{n}(aA_i \oplus aB_i)) \oplus aC_n$. By using this and relations (5) and (6) we obtain

$$\begin{aligned}
R = E \oplus aR &= A_1 \oplus A'_1 \oplus (\bigoplus_{i=1}^{n}(aA_i \oplus aB_i)) \oplus aC_n \\
&= (A_1 \oplus B_1) \oplus A'_1 \oplus (\bigoplus_{i=1}^{n-1}(B_{i+1} \oplus B'_{i+1})) \oplus aA_n \oplus aB_n \oplus aC_n \\
&= A_1 \oplus (\bigoplus_{i=1}^{n} B_i) \oplus (A'_1 \oplus (\bigoplus_{i=1}^{n-1} B'_{i+1})) \oplus aA_n \oplus aB_n \oplus aC_n \\
&= A_1 \oplus (\bigoplus_{i=1}^{n} B_i) \oplus (\bigoplus_{i=2}^{n} A_i) \oplus (A'_n \oplus B'_n) \oplus aA_n \oplus aB_n \oplus aC_n \\
&= (\bigoplus_{i=1}^{n}(A_i \oplus B_i)) \oplus (A'_n \oplus B'_n) \oplus aA_n \oplus aB_n \oplus aC_n.
\end{aligned}$$

Now applying the exchange property to the decompositions

$$\begin{aligned}
R &= K \oplus (\bigoplus_{i=1}^{n}(A_i \oplus B_i)) \oplus C_n \\
&= (\bigoplus_{i=1}^{n}(A_i \oplus B_i)) \oplus (A'_n \oplus B'_n) \oplus (aA_n \oplus aB_n) \oplus aC_n,
\end{aligned}$$

we obtain a decomposition

$$R = K \oplus (\bigoplus_{i=1}^{n}(A_i \oplus B_i)) \oplus A_{n+1} \oplus B_{n+1} \oplus C_{n+1}$$

such that

$$A_{n+1} \oplus A'_{n+1} = A'_n \oplus B'_n$$

for some right ideal A'_{n+1}, while

$$B_{n+1} \oplus B'_{n+1} = aA_n \oplus aB_n$$

for some right ideal B'_{n+1}, and

$$C_{n+1} \oplus C'_{n+1} = aC_n$$

for some right ideal C'_{n+1}. So we obtain (1), (4) and (5).

Since $(\bigoplus_{i=1}^{n+1}(A_i \oplus B_i)) \oplus C_{n+1}$ is a common complement of both K and $A'_{n+1} \oplus B'_{n+1} \oplus C'_{n+1}$, we obtain (3).

Now we will prove (2). We have

$$\begin{aligned} E &\cong (A_n \oplus B_n) \oplus (A'_n \oplus B'_n) \\ &\cong aA_n \oplus aB_n \oplus A_{n+1} \oplus A'_{n+1} \\ &= B_{n+1} \oplus B'_{n+1} \oplus A_{n+1} \oplus A'_{n+1} \\ &= (A_{n+1} \oplus B_{n+1}) \oplus (A'_{n+1} \oplus B'_{n+1}). \end{aligned}$$

This completes the inductive step. □

Theorem 2. *Let R be an exchange ring and let a be a nilpotent regular element of R. Then a is unit-regular.*

Proof. Assume that $a^{n+2} = 0$ for some $n \geq 0$. Let A_i, A'_i, B_i, B'_i, C_i, C'_i be right ideals as in Lemma 1. Then $C_{n+1} \subseteq K$ by (5) of Lemma 1, and so $C_{n+1} = 0$ by (1). By (5), we have $C'_{n+1} = aC_n \cong C_n$ so that, using (1), we obtain $C'_{n+1} \cong C_n \cong A_{n+1} \oplus B_{n+1} \oplus C_{n+1} = A_{n+1} \oplus B_{n+1}$. Now using this fact and (2), (3), we have

$$\begin{aligned} K &\cong A'_{n+1} \oplus B'_{n+1} \oplus C'_{n+1} \\ &\cong (A'_{n+1} \oplus B'_{n+1}) \oplus (A_{n+1} \oplus B_{n+1}) \cong E. \end{aligned}$$

We conclude that a is unit-regular. □

Theorem 3. *Let R be an exchange ring and let a be a regular element of R. If a is strongly π-regular, then a is unit-regular.*

Proof. Let a be a regular, strongly π-regular element of R. Let $b \in R$ be such that $a^n = a^{n+1}b$ for some $n \geq 1$, and $ab = ba$. Set $e = a^n b^n$ and note that e is idempotent. Moreover, $ea = ae$ is invertible in eRe, with inverse $a^n b^{n+1}$, and $a(1-e) = (1-e)a \in (1-e)R(1-e)$ is a regular nilpotent element with $(a(1-e))^n = 0$. Since $(1-e)R(1-e)$ is an exchange ring [19, Theorem 2], it follows from Theorem 2 that $a(1-e)$ is unit-regular in $(1-e)R(1-e)$. Consequently, a is unit-regular in R. □

Theorem 4. *Strongly π-regular rings have stable range one.*

Proof. By [16, Example 2.3], any strongly π-regular ring is an exchange ring. So the result follows from Theorem 3 and [5, Theorem 3]. Alternatively, one can use Theorem 2 and [11, Theorem 6.1]. □

Our next result proves the conjecture made by Goodearl and Menal in [11, p.271].

Corollary 5. *Any algebraic algebra over a field has stable range one.*

Proof. Clearly, an algebraic algebra over a field is strongly π-regular. So, the result follows from Theorem 4. □

A module M is said to *satisfy Fitting's Lemma* if for each $f \in \mathrm{End}_R(M)$ there exists an integer $n \geq 1$ such that $M = \mathrm{Ker}(f^n) \oplus f^n(M)$. By [2, Proposition 2.3], M satisfies Fitting's Lemma if and only if $\mathrm{End}_R(M)$ is strongly π-regular. It was proved in [6] that modules satisfying Fitting's Lemma have power cancellation. Theorem 4 enables us to improve this result, as follows.

Corollary 6. *Let M be a module satisfying Fitting's Lemma. Then M cancels from direct sums.*

Proof. By [2, Proposition 2.3], $E := \mathrm{End}_R(M)$ is a strongly π-regular ring. Now, Theorem 4 gives that the stable range of E is one and, by Evans' Theorem [9, Theorem 2], M cancels from direct sums. □

Now we will obtain some cancellation results for finitely generated modules over certain strongly π-regular rings. We need the following concept, introduced by Goodearl [10].

Definition ([10])**.** An element u of a ring R is said to be *right repetitive* provided that for each finitely generated right ideal I of R, the right ideal $\sum_{n=0}^{\infty} u^n I$ is finitely generated. The ring R is *right repetitive* if each element of R is right repetitive.

Note that every algebraic algebra is (right and left) repetitive.

Corollary 7. *Let R be a strongly π-regular, right repetitive ring. Then any cyclic right R-module cancels from direct sums.*

Proof. Apply [14, Theorem 19], Theorem 4 and Evans' Theorem [9, Theorem 2]. □

Corollary 8. *If R is a ring such that all matrix rings $M_n(R)$ are strongly π-regular and right repetitive, then any finitely generated right R-module cancels from direct sums.*

Proof. Apply [10, Theorem 8], Theorem 4 and Evans' Theorem [9, Theorem 2]. □

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