Let $\mathcal{A}$ be a Banach algebra and let $A, B \in \mathcal{A}$. A multiplication (operator) $A \cdot B : \mathcal{A} \to \mathcal{A}$ is defined by the equation $(A \cdot B)(X) = AXB, X \in \mathcal{A}$. An elementary operator $R$ is an operator of the form $R = \sum_{i=1}^{n} A_i \cdot B_i$, where $n$ is finite. Clearly, $R$ can have different representations $\sum_{t=1}^{m} C_i \cdot D_t$, the length of $R$ is defined to be the smallest number of multiplication terms required for any representation of $R$.

Elementary operators have been studied in great detail in recent years. Both their spectral theory and their structural theory have received much attention (cf. [AF, Cu, F]). The range inclusion problem for elementary operators [F] can be stated as follows: If $I$ is an ideal in $\mathcal{A}$, and if $\text{Ran}(R) \subseteq I$, must $R$ be expressible as $\sum_{i=1}^{n} A_i \cdot B_i$ with $\text{Ran}(A_i \cdot B_i) \subseteq I, i = 1, 2, \ldots, n$? Positive results have been obtained with certain hypotheses [AF].

Let $\mathcal{C}$ be a subalgebra of $\mathcal{A}$. David Larson has suggested the study of $\mathcal{E}_C(\mathcal{A})$, the algebra of elementary operators on $\mathcal{A}$ which leave $\mathcal{C}$ invariant. With $\mathcal{A} = B(\ell_2)$ and $\mathcal{C} = \mathcal{T}_\infty$, the nest algebra of upper triangular operators with respect to the standard basis in $\ell_2$, we provide in Theorem 3 an example of an $R \in \mathcal{E}_C(\mathcal{A})$ which has no representation $\sum_{t=1}^{m} C_t \cdot D_t$ such that $C_t \cdot D_t \in \mathcal{E}_C(\mathcal{A}), t = 1, 2, \ldots, m$. (This example was discussed, without proof, in [Co].) We also demonstrate in Proposition 4 that $R$ lies in the completely bounded norm closed linear span of the length one elements of $\mathcal{E}_C(\mathcal{A})$.

$\mathcal{T}_\infty$ is a commutative subspace lattice (CSL) algebra. Recall that a CSL algebra is a reflexive algebra whose lattice of invariant subspaces is commutative. For finite dimensional CSL algebras there is no operator like the operator $R$ of the previous paragraph.

**Proposition 1.** Let $\mathcal{C}$ be a CSL subalgebra of $\mathcal{M}_n$, and let $R \in \mathcal{E}_C(\mathcal{M}_n) (R \neq 0)$. Then there exists a representation $R = \sum_{t=1}^{m} C_t \cdot D_t$ such that $C_t \cdot D_t \in \mathcal{E}_C(\mathcal{M}_n), 1 \leq t \leq m$.

**Proof.** Let $R = \sum_{s=1}^{p} A_s \cdot B_s$. Since $\mathcal{C}$ is a CSL algebra, it admits a star diagram, that is, there exists a subset $I \times J \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that $\mathcal{C} = \{(a_{ij}) : a_{ij} \in C, (i,j) \in I \times J\}$. 

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Let \( A_1 = (a_{ij}) = \sum a_{ij}E_{ij}, \ B_1 = (b_{ij}) = \sum b_{ij}E_{ij} \). With this, we see that we can expand \( A_1 \cdot B_1 \) as a linear combination of multiplications involving only pairs of matrix units: \( A_1 \cdot B_1 = \sum_{i,j,k,l} \alpha_{ijkl}(E_{ij} \cdot E_{kl}) \). Expanding \( A_2 \cdot B_2, \ldots, A_p \cdot B_p \) in like manner and combining like terms, we arrive at

\[
R = \sum \beta_{ijkl}(E_{ij} \cdot E_{kl})
\]

(1)

Let \( E_{ab} \in \mathcal{C} \) and assume that \( R(E_{ab}) \neq 0 \). Then

\[
R(E_{ab}) = \sum_{i,j} \beta_{iabj}(E_{ia} \cdot E_{bj})E_{ab} = \sum_{i,j} \beta_{iabj}E_{ij}
\]

and there is a \((c, d)\) such that \( \beta_{cabd} \neq 0 \). Then \( E_{cd} \in \mathcal{C} \), for if not, we would have that \( \sum \beta_{cabd}E_{ij} \notin \mathcal{C} \), contradicting our hypothesis. So every nonzero multiplication \( \beta_{iabj}(E_{ia} \cdot E_{bj}) \) maps \( E_{ab} \) into \( \mathcal{C} \) and every multiplication in (1) either annihilates \( E_{ab} \) or maps it onto a nonzero element of \( \mathcal{C} \). Thus \( \sum \beta_{ijkl}(E_{ij} \cdot E_{kl}) \) is an expansion of \( R \) such that each multiplication lies in \( \mathcal{E}_\mathcal{C}(\mathcal{M}_n) \).

\[\Box\]

Note: The conclusion of Proposition 1 can be false if \( \mathcal{C} \) is not a CSL subalgebra of \( \mathcal{M}_n \) (see Example 5).

Clearly, the number of terms in (1) can be arbitrarily large for arbitrarily large \( n \). The observation of this fact in a specific instance will facilitate the proof of Theorem 3.

**Lemma 2.** Let \( n > 2 \), and let \( T_n \subseteq \mathcal{M}_n \) be the algebra of upper triangular \( n \times n \) matrices. There exists an \( R \in \mathcal{E}_{T_n}(\mathcal{M}_n) \) of length two such that for any representation \( R = \sum_{t=1}^m C_t \cdot D_t \) with each \( C_t \cdot D_t \in \mathcal{E}_{T_n}(\mathcal{M}_n) \), we must have \( m \geq \log_2 n \).

**Proof.** Let \( \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \) be a sequence of distinct complex numbers and define the following four matrices in \( \mathcal{M}_n \):

\[
\begin{align*}
A_1 &= \sum_{i=2}^n E_{ii}, & B_1 &= \sum_{j=2}^n \alpha_{j-1}E_{j(j-1)}, \\
A_2 &= \sum_{k=2}^n \alpha_{k-1}E_{kk}, & B_2 &= \sum_{l=2}^n -E_{l(l-1)}.
\end{align*}
\]

Consider the elementary operator \( R = A_1 \cdot B_1 + A_2 \cdot B_2 \). By Proposition 1, \( R \) expands as

\[
\sum_{i=2}^n \sum_{j=2}^n (\alpha_{j-1} - \alpha_{i-1})(E_{ii} \cdot E_{j(j-1)});
\]

\( R \in \mathcal{E}_{T_n}(\mathcal{M}_n) \) since it takes strictly upper triangular matrix units to upper triangular matrix units, and the diagonal matrix units are annihilated.

Since the \( \alpha_k \)'s are all distinct, any expansion of \( R \) must include nonzero multiples of every one of the terms

\[
E_{ii} \cdot E_{j(j-1)}
\]

for \( i \neq j, 2 \leq i, j \leq n \). On the other hand, if \( C \cdot D \in \mathcal{E}_{T_n}(\mathcal{M}_n) \) is to contain a nonzero multiple of \( E_{ii} \cdot E_{j(j-1)} \) then \( C \) must contain a nonzero multiple of \( E_{ii} \) and \( D \) must contain a nonzero multiple of \( E_{j(j-1)} \). Thus, if \( C \cdot D \) contains nonzero multiples of \( E_{ii} \cdot E_{j(j-1)} \) and \( E_{kk} \cdot E_{l(l-1)} \) then it must also contain a nonzero multiple of \( E_{kk} \cdot E_{j(j-1)} \). If \( C \cdot D \) is to leave \( T_n \) invariant, we must have \( k \neq j \).
Now suppose that $R = \sum_{i=1}^{m} C_i \cdot D_i$. For each $i$ let $W_i$ be the set of $t$'s for which $C_i \cdot D_i$ contains a nonzero multiple of $E_{ii} \cdot E_{kk}$ of multiplications with $C_i \cdot D_i \in \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$.

We claim the sets $W_i$ are all distinct. If $i \neq j$, the last paragraph implies that there is a $t_0 \in W_i$ such that $C_{t_0} \cdot D_{t_0}$ contains a nonzero multiple of $E_{ii} \cdot E_{jj}$. But, also by the last paragraph, this means that $C_{t_0} \cdot D_{t_0}$ cannot contain nonzero multiples of any operator of the form $E_{jj} \cdot E_{ii}$. This tells us that $t_0 \notin W_j$, so $W_j \neq W_i$.

Thus, the total number of sets $W_i$ (that is, $n - 1$) is no greater than the total number of distinct nonempty subsets of $\{1, 2, \ldots, m\}$, which is $2^m - 1$. The result follows.

**Theorem 3.** There exists an elementary operator $R \in \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$ of length two which cannot be written as a finite sum $\sum_{i=1}^{m} C_i \cdot D_i$ of multiplications with $C_i \cdot D_i \in \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$ for $1 \leq t \leq m$.

**Proof.** Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be a sequence in $l_\infty$ with the $\alpha_i$'s all distinct. Form the following four operators in $B(l_2)$:

$$
A_1 = \sum_{i=2}^{\infty} E_{ii}, \quad B_1 = \sum_{j=2}^{\infty} \alpha_{j-1} E_{j(j-1)},
$$

$$
A_2 = \sum_{k=2}^{\infty} \alpha_{k-1} E_{kk}, \quad B_2 = \sum_{l=2}^{\infty} -E_{l(l-1)}.
$$

Let $R = A_1 \cdot B_1 + A_2 \cdot B_2$. Again, routine computations show that $R \in \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$, but $A_1 \cdot B_1 \notin \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$, $A_2 \cdot B_2 \notin \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$.

Suppose we can write $R$ as a finite sum $\sum_{t=1}^{m} C_t \cdot D_t$ with $C_t \cdot D_t \in \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$, $1 \leq t \leq m$. With $n$ fixed and $P_n$ the projection onto the first $n$ basis vectors, let $X = P_n X P_n \in \mathcal{M}_n$ for $X \in B(l_2)$. Let $R' = A'_1 \cdot B'_1 + A'_2 \cdot B'_2$. Then $R'$ is an elementary operator of the type constructed in Lemma 2. Further, $R' = \sum_{t=1}^{n} C'_t \cdot D'_t$ with $C'_t \cdot D'_t \in \mathcal{E}_{\mathcal{T}_{\infty}}(\mathcal{M}_n)$ for $1 \leq t \leq m$. Lemma 2 therefore implies that $m \geq \log_2 n$. But this must be true for all $n > 2$ with $m$ fixed, an impossibility.

We conclude that $R$ cannot be written as a finite sum $\sum_{t=1}^{m} C_t \cdot D_t$ with $C_t \cdot D_t \in \mathcal{E}_{\mathcal{T}_{\infty}}(B(l_2))$.

If $A$ is a C*-algebra and $\phi : A \to A$ is a bounded linear operator, define $\phi_n : M_n(A) \to M_n(A)$ by the equation $\phi_n((a_{ij})) = (\phi(a_{ij}))_{ij}$. If $\sup_n \|\phi_n\| < \infty$, say that $\phi$ is completely bounded and define the completely bounded norm $\|\phi\|_{cb}$ of $\phi$ to be this supremum. $CB(A)$, the algebra of completely bounded operators, contains the elementary operators as a proper (nonclosed) subalgebra.

If $A \odot B$ is the algebraic tensor product of two operator algebras $A, B$, define $\|u\|_h$, the Haagerup norm of $u \in A \odot B$, by

$$
\|u\|_h = \inf \left\{ \|(a_1, a_2, \ldots, a_n)\| \|(b_1, b_2, \ldots, b_n)'\| : u = \sum_{i=1}^{n} a_i \otimes b_i \right\}.
$$

The closure of $A \odot B$ in the topology defined by this norm is known as the Haagerup tensor product of $A$ and $B$, denoted by $A \otimes_h B$. For an excellent overview of the theory of completely bounded operators and the Haagerup tensor product, see [ChrSi].
There are rich connections between the structures of $CB(A)$ and of $A \odot A$. In [CSi] it is shown that the map $\theta : A \odot A \to CB(A)$, given by $\theta(a \odot b)(x) = axb$, $x \in A$, is an isometry.

**Proposition 4.** Let $F$ be the linear span of the length one elements of $\mathcal{E}_{T_\infty}(B(l_2))$, and let $\overline{F}$ be the closure of $F$ in the completely bounded norm topology. Then $R \in \overline{F}$, where $R$ is the operator constructed in Theorem 3.

**Proof.** Without loss of generality, assume $\|\alpha\|_\infty \leq 1$. Let $\epsilon$ be given, and let $\beta = (\beta_1, \beta_2, \ldots) \in l_\infty$ be a simple (i.e., finitely valued) $\epsilon/3$ approximation of $\alpha$ such that $\|\beta\|_\infty \leq 1$. Let $\{\beta_1, \beta_2, \ldots, \beta_m\}$ be the set of distinct values which $\beta$ assumes. Form the following four operators in $B(l_2)$:

$$C_1 = \sum_{i=2}^\infty E_{ii}, \quad D_1 = \sum_{j=2}^\infty \beta_{j-1}^* E_{j(j-1)},$$

$$C_2 = \sum_{k=2}^\infty \beta_{k-1}^* E_{kk}, \quad D_2 = \sum_{l=2}^\infty -E_{l(l-1)},$$

and let $S = C_1 \cdot D_1 + C_2 \cdot D_2$. We first show that $S \in F$. Let $Q_1, Q_2, \ldots, Q_m$ be diagonal projections such that $\sum_{i=1}^m \beta_i Q_i = C_2$. Let $T_1, T_2, \ldots, T_m$ be operators with only ones and zeroes on the first subdiagonal and zeroes elsewhere such that $\sum_{i=1}^m \beta_i T_i = D_1$.

Now, for $E_{ij} \in T_\infty$, $E_{ii}(E_{ij})E_{j(j-1)} = E_{i(j-1)} \in T_\infty$ as long as $i \neq j$. Thus, $Q_r \cdot T_s \in \mathcal{E}_{T_\infty}(B(l_2))$ for $r \neq s$. We have

$$C_1 \cdot D_1 + C_2 \cdot D_2 = \left( \sum_{r=1}^m Q_r \cdot \left( \sum_{s=1}^m \beta_s T_s \right) \right) - \left( \sum_{r=1}^m \beta_r Q_r \cdot \left( \sum_{s=1}^m T_s \right) \right),$$

Each term in the above (finite) sum lies in $\mathcal{E}_{T_\infty}(B(l_2))$, so that $S = C_1 \cdot D_1 + C_2 \cdot D_2 \in F$.

We now show that $\|S - R\|_{cb} < \epsilon$. We have that $\|A_1 - C_1\| = \|B_2 - D_2\| = 0$, and $\|A_2 - C_2\| < \epsilon/3$, $\|B_1 - D_1\| < \epsilon/3$. Consider the following operator vectors:

\[
\left( 0, \sqrt{\frac{\epsilon}{3}} C_1, \sqrt{\frac{\epsilon}{3}} (A_2 - C_2), \sqrt{\frac{\epsilon}{3}} C_2, \left( \sqrt{\frac{\epsilon}{3}} B_1, \sqrt{\frac{\epsilon}{3}} (B_1 - D_1), \sqrt{\frac{\epsilon}{3}} B_2, 0 \right)^t \right).
\]

By construction, each has norm less than $\sqrt{\epsilon}$. Thus, $\epsilon$ is larger than

\[
\left\| 0 \odot \sqrt{\frac{\epsilon}{3}} B_1 + \sqrt{\frac{\epsilon}{3}} C_1 \odot \sqrt{\frac{\epsilon}{3}} (B_1 - D_1) + \sqrt{\frac{\epsilon}{3}} (A_2 - C_2) \odot \sqrt{\frac{\epsilon}{3}} B_2 + \sqrt{\frac{\epsilon}{3}} C_2 \odot 0 \right\|_h
\]

\[
= \left\| (A_1 - C_1) \odot B_1 + C_1 \odot (B_1 - D_1) + (A_2 - C_2) \odot B_2 + C_2 \odot (B_2 - D_2) \right\|_h
\]

\[
= \left\| [(A_1 \odot B_1) + (A_2 \odot B_2)] - [(C_1 \odot D_1) + (C_2 \odot D_2)] \right\|_h.
\]

Theorem 4.3 in [CSi] now implies that $\|S - R\|_{cb} < \epsilon$.

Since $\epsilon$ was arbitrary, $R \in \overline{F}$. \hfill \Box
We close with a finite dimensional example, which demonstrates that if $C$ is not a CSL subalgebra of $M_n$, then the conclusion of proposition 1 can be false.

**Example 5.** Let $\mathcal{A} = M_2$, the algebra of $2 \times 2$ matrices over $C$. Let $C$ be the triangular Toeplitz subalgebra

$$C = \left\{ \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix} : \lambda, \mu \in C \right\}.$$ 

Let

$$A_1 = B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

and let $R = A_1 \cdot B_1 + A_2 \cdot B_2$. It is not difficult to verify that

$$R \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d - a \end{pmatrix}$$

(so $R \neq 0$), and

$$R \begin{pmatrix} \lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda - \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Thus, $R \in \mathcal{E}_C(\mathcal{A})$ (in fact, $R$ annihilates $C$).

Routine computations now show that $R$ does not lie in (the norm closure of) the linear span of length one elements of $\mathcal{E}_C(\mathcal{A})$.

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