PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 11, November 1996

POSITION OF SINGULARITIES AND FUNDAMENTAL GROUP OF THE COMPLEMENT OF A UNION OF LINES

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(Communicated by Peter Li)

ABSTRACT. In this paper we give two examples of complex line arrangements in CP^2 with 7 lines, that both have 3 triple points and 12 double points, and their complements have nonisomorphic global fundamental groups. These two line arrangements are, in some sense, a much simpler example of a pair of plane algebraic curves that have the same local topology but have complements with different global topology—compare with the example given by Zariski, or the recent example given by Artal-Bartolo.

1. Introduction

We are interested in the question of the dependence between the topology of the complement of a complex plane curve and the position of the singularities. In [7], Zariski showed that the fundamental group of the complement of an irreducible sextic curve in the projective plane with six cusps lying on a conic is isomorphic to the free product of Z_2 and Z_3 (see also [4]). In [8], he then showed that the complement of an irreducible sextic curve with six cusps that do not lie on a conic has abelian fundamental group. This is the first pair of curves known to have identical local topology but to have complements with different global topology. Recently, in 1990, Artal-Bartolo constructed a pair of complex line arrangements in the projective plane that have the same local topology, both have 9 lines, 9 triple points, and their complements have different Alexander polynomials [1] (see also [2], 212-213).

The two examples of line arrangements given by Artal-Bartolo are as follow: the first arrangement V_1 is given by

$$xyz(x-y)(y+bz)(x-y-z)(ax+y+z)(ax+y+bz)$$

$$\times(abx+(a-ab+1)y+bz)=0,$$

where $ab(a+1)(b-1)(a+b)(a-ab+1)(ab+b-1)\neq 0$.

The second line arrangement V_2 is given by

$$xyz(x+y)(x+az)(y+bz)(cx+(c+1)y+bz)$$

$$\times(c(a+b)x+a(c+1)y+ab(c+1)z)(cx+(c+1)y+acz)=0,$$

Received by the editors May 1, 1995.
1991 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H20.

where $abc(c+1)(ac-b)(a+b) \neq 0$ and $c(a-bc)-b(c+1)=0$. It was shown in [1] (see also [2]) that the Alexander polynomials $\Delta_{V_i}(t)$ of the curves V_i, $i=1,2$, are:

$$\Delta_{V_1}(t) = (t-1)^8(t^2+t+1),$$

$$\Delta_{V_2}(t) = (t-1)^8.$$

These two examples are used to exhibit a change in the first Betti number of the corresponding Milnor fiber. It is believed that the example of Artal-Bartolo is among the simplest examples showing the dependence of the topology of the complement of a curve on the position of the singularities.

By studying the fundamental groups of some line arrangements that have less than 8 lines, the author found a pair of line arrangements with 7 lines, that both have 3 triple points and 12 double points and their complements have nonisomorphic global fundamental groups. This provides a simpler pair of line arrangements that have the same local topology but have complements with different global topology.

In section 2, we shall describe a product theorem of a reducible curve of M. Oka and Sakamoto [5] and use it to study the fundamental groups of complements of some line arrangements. In section 3, we shall give two line arrangements that both have 7 lines, 3 triple points and 12 double points, and then study the fundamental groups of their complements in CP^2. We shall show that the fundamental group of the complement of the first example of line arrangement is isomorphic to the direct product of 3 copies of free groups on two generators, hence has a trivial center. On the other hand, the fundamental group of the complement of the second example of line arrangement has a nontrivial center. Hence these two groups cannot be isomorphic.

2. π_1 of complements of some line arrangements

We are going to use the following result of Oka and Sakamoto [5]:

Theorem 2.1. *Let C_1 and C_2 be two distinct algebraic curves in the complex affine plane C^2 of degree n_1 and n_2 respectively. Suppose that C_1 and C_2 intersect at $n_1 n_2$ distinct points. Then $\pi_1(C^2 \setminus (C_1 \cup C_2)) \cong \pi_1(C^2 \setminus C_1) \oplus \pi_1(C^2 \setminus C_2)$.*

Denote a free group on t generators by F_t. Let $\Sigma_0 = \bigcup_{i=1}^{n} L_i$ be a union of n distinct projective lines in the projective plane CP^2 such that all L_i pass through a point O, and $C^2 = CP^2 \setminus L_n$. If we transform the line L_n to become the line at infinity to C^2, then $C^2 \cap (\bigcup_{i=1}^{n-1} L_i)$ is a set of $n-1$ parallel lines, and it is easy to see that $\pi_1(CP^2 \setminus \Sigma_0) \cong \pi_1(C^2 \setminus (C^2 \cap (\bigcup_{i=1}^{n-1} L_i))) \cong F_{n-1}$. Denote the multiplicity of a point P of a curve by $m(P)$. By using Theorem 2.1, we have:

Theorem 2.2. *Let $\Sigma = \bigcup_{i=1}^{n} L_i$ be a line arrangement in CP^2 and assume that there is a line L of Σ such that for any singular point S of Σ with $m(S) \geq 3$, we have $S \in L$. Then $\pi_1(CP^2 \setminus \Sigma)$ is isomorphic to a direct product of free groups.*

Proof. Let $\{S_1, S_2, \cdots, S_t\}$ be the set of all singular points of Σ with $m(S_i) \geq 3$; then, by our assumption, $\{S_1, \cdots, S_t\} \subset L$. Let the number of nodes of Σ that lie on L be r, and note that $n = r + 1 - t + \sum_{i=1}^{t} m(S_i)$.

Suppose $t = 0$; then Σ is $r+1$ lines in general position and $\pi_1(CP^2 \setminus \Sigma) \cong A_r$, where A_r is a free abelian group on r generators [7].

Suppose that $t > 0$. Let $C^2 = CP^2 \setminus L$; then $CP^2 \setminus \Sigma = C^2 \setminus (\Sigma \setminus L)$. Note that $(\Sigma \setminus L) \cap C^2 = \sigma_0 \cup \sigma_1 \cup \cdots \cup \sigma_t$, where

a) σ_0 is r lines in general position,
b) $\sigma_i, i > 0$, is $m(S_i) - 1$ distinct parallel lines in C^2.

Lines in σ_i and σ_j intersect if $i \neq j$, $0 \leq i, j \leq t$, and $(\Sigma \setminus L) \cap C^2$ has only nodes. Note that the fundamental group of the complement of k parallel lines in C^2 is isomorphic to a free group on k generators. By Theorem 2.1, we have:

$$\pi_1(C^2 \setminus (\sigma_0 \cup \sigma_1 \cup \cdots \cup \sigma_t)$$

$$\cong \pi_1(C^2 \setminus \sigma_0) \oplus \pi_1(C^2 \setminus \sigma_1) \oplus \cdots \oplus \pi_1(C^2 \setminus \sigma_t)$$

$$\cong A_r \bigoplus_{i=1}^{t} F_{m(S_i)-1}.$$

Note that in the above proof, we actually showed that:

Corollary 2.1. *Let E be the set $\{S_1, S_2, \cdots, S_t\}$ of all singular points of Σ with multiplicity ≥ 3, and the number of nodes of Σ on L be r. Suppose that the line arrangement Σ has a component L such that $E \subset L$. Then*

$$\pi_1(CP^2 \setminus \Sigma) \cong A_r \bigoplus_{i=1}^{t} F_{m(S_i)-1},$$

where A_r is a free abelian group on r generators and $m(S_i)$ is the multiplicity of S_i.

A line arrangement that satisfies the conditions of Corollary 2.1 can be constructed easily. For example, take L to be the X-axis, and choose 3 points S_1, S_2, S_3 on L, and at each S_i, choose two distinct lines $L_1^i, L_2^i \neq L$ that go through S_i , and require that these lines intersect in nodes at points that do not lie on L. The union of these seven lines gives a line arrangement in CP^2, and we call it Σ'. By Corollary 4.1, we have $\pi_1(CP^2 \setminus \Sigma') \cong F_2 \oplus F_2 \oplus F_2$.

We are going to use the following lemma (cf. [9]) in the next section:

Lemma 2.1. *Let $\Sigma = L_1 \cup L_2 \cup \cdots \cup L_n$, $n > 1$, be a line arrangement in CP^2 with n components. Suppose that L_1 intersects $L_2 \cup \cdots \cup L_n$ at $n-1$ distinct points. Then $\pi_1(CP^2 \setminus \Sigma)$ splits as a direct product of two groups Z and H, where Z is an infinite cyclic group on one generator.*

Proof. Let $C^2 = CP^2 \setminus L_n$; then $L_1 \cap C^2$ intersects $(L_2 \cup \cdots \cup L_{n-1}) \cap C^2$ at $n-2$ distinct points. By Theorem 2.1, we have

$$\pi_1(CP^2 \setminus \Sigma) \cong \pi_1(C^2 \setminus (L_1 \cup L_2 \cup \cdots \cup L_{n-1}) \cap C^2)$$

$$\cong \pi_1(C^2 \setminus L_1 \cap C^2) \oplus \pi_1(C^2 \setminus (L_2 \cup \cdots \cup L_{n-1}) \cap C^2) \cong Z \oplus H,$$

where $Z = \pi_1(C^2 \setminus L_1 \cap C^2)$ is an infinite cyclic group on one generator, and $H = \pi_1(C^2 \setminus (L_2 \cup \cdots \cup L_{n-1}) \cap C^2)$.

3. Two line arrangements with 7 lines

Let $\Sigma = \bigcup_{i=1}^{n} L_i$ be a line arrangement in CP^2, and $\{A_1, A_2, \cdots, A_b\}$ be the set of all singular points of Σ. Then by counting intersection numbers, we have:

$$\frac{n(n-1)}{2} = \sum_{i=1}^{b} \frac{m(A_i)(m(A_i)-1)}{2}.$$

Below we are going to give two line arrangements with 7 lines that have the same local topology but have complements with different global topology.

I. Let W_1 be a line arrangement in CP^2 given by the following homogeneous equation:

$$y(x-y)(x+y)(2x-y-2z)(2x+y-2z)(3x-y-6z)(3x+y-6z) = 0.$$

Put $z = 1$; we see that W_1 is given by the X-axis, together with 3 pairs of lines such that each pair intersects the X-axis at a triple point, and W_1 has 3 triple points and 12 double points. Coordinates of triple points are: $[x, y, z] = [0, 0, 1], [1, 0, 1], [2, 0, 1]$. Coordinates of double points are: $[2, 2, 1], [\frac{2}{3}, \frac{2}{3}, 1], [3, 3, 1], [\frac{3}{2}, \frac{3}{2}, 1], [4, 6, 1], [\frac{8}{5}, \frac{6}{5}, 1], [2, -2, 1], [\frac{2}{3}, \frac{-2}{3}, 1], [3, -3, 1], [\frac{3}{2}, \frac{-3}{2}, 1], [4, -6, 1], [\frac{8}{5}, \frac{-6}{5}, 1]$. By Corollary 2.1, we have

$$\pi_1(CP^2 \setminus W_1) \cong F_2 \oplus F_2 \oplus F_2. \tag{3.1}$$

II. Let W_2 be a line arrangement in CP^2 given by the following homogeneous equation:

$$xy(x-y)(x+y-2z)(x-2y-2z)(2x+y-2z)(3x-y-9z) = 0.$$

Put $z = 1$. We find that coordinates of triple points are: $[x, y, z] = [0, 0, 1], [2, 0, 1], [0, 2, 1]$. Coordinates of double points are: $[1, 0, 1], [0, -1, 1], [1, 1, 1], [-2, -2, 1], [\frac{2}{3}, \frac{2}{3}, 1], [\frac{6}{5}, \frac{-2}{5}, 1]$, $[3, 0, 1], [0, -9, 1], [\frac{9}{2}, \frac{9}{2}, 1], [\frac{11}{4}, \frac{-3}{4}, 1], [\frac{11}{5}, \frac{-12}{5}, 1], [\frac{16}{5}, \frac{3}{5}, 1]$. Note that the component of W_2 given by $3x - y - 9z = 0$ intersects all other lines at normal crossings, hence by Lemma 2.1 we have

$$\pi_1(CP^2 \setminus W_2) \cong Z \oplus N, \tag{3.2}$$

where Z is an infinite cyclic group on one generator. In particular, this shows that $\pi_1(CP^2 \setminus W_2)$ has a nontrivial center.

Theorem 3.1. *There exist two line arrangements with 7 lines in CP^2 that both have 3 triple points and 12 double points, yet the fundamental groups of their complements are not isomorphic.*

Proof. By (3.1), we have $G_1 = \pi_1(CP^2 \setminus W_1) \cong F_2 \oplus F_2 \oplus F_2$. We see that the center of G_1 is trivial since the free group F_2 has a trivial center. On the other hand, (3.2) implies that $G_2 = \pi_1(CP^2 \setminus W_2)$ has a nontrivial center, hence G_1 and G_2 cannot be isomorphic. □

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