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ON THE DEFINITION OF REAL W^*-ALGEBRAS

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ABSTRACT. We prove that, if A is a real C^*-algebra having a predual A_*, then A_* is the unique predual of A and the product of A is $\sigma(A, A_*)$-continuous.

0. INTRODUCTION

Real W^*-algebras are usually defined as those real C^*-algebras A having a complete predual A_* such that the product of A is separately $\sigma(A, A_*)$-continuous (see for instance [2]). We prove in this note that the requirement of the separate $\sigma(A, A_*)$-continuity of the product is superfluous.

Notation and preliminaries. If X is a Banach space over the field $\mathbb{F}$ (that is, either $\mathbb{R}$ or $\mathbb{C}$) and $S \subset X$, then $X_\star$ and S° denote the dual space and the polar of S in $X^\star$ respectively. If X is a dual Banach space and $S \subset X$, then $X_\star$ and $S_\circ$ denote respectively a predual of X and the prepolar of S in $X_\star$. By $J\colon X \to X^{\star\star}$ we denote the cononical embedding; usually we identify every set $S \subset X$ with $J(S) \subset X^{\star\star}$ and no reference to J is made. For $S \subset X$, the symbol $\widetilde{S}$ means the weak* closure of $J(S)$ in $X^{\star\star}$. For Banach spaces X, Y over the same field $\mathbb{F}, X \cong Y$ means that there exists a surjective linear isometric isomorphism of X onto Y. If the fact that $\mathbb{F} = \mathbb{C}$ is to be emphasized, then the symbols $X^\star$, $X_\star$, S°, $S_\circ$, $\widetilde{S}$ are correspondingly replaced by X^*, X_*, $S^{\circ_\mathbb{C}}$, $S_{\circ_\mathbb{C}}$, $\widetilde{S}^\mathbb{C}$, and to emphasize that $\mathbb{F} = \mathbb{R}$ they are replaced by X', $X_\prime$, $S^{\circ_\mathbb{R}}$, $S_{\circ_\mathbb{R}}$, $\widetilde{S}^\mathbb{R}$. If X is a complex Banach space, then X_r denotes its underlying real Banach space, and we recall that then $(X^*)_r \cong (X_r)'$ by the mapping $f \mapsto \mathfrak{Re}\, f$. We also have $(X^{**})_r \cong (X_r)''$, and this isomorphism induces a homeomorphism between $(X^{**}, w^*_\mathbb{C})$ and $((X_r)'', w'_\mathbb{R})$, where $w^*_\mathbb{C} := \sigma(X^{**}, X^*)$ and $w'_\mathbb{R} := \sigma(X''_r, X'_r)$ are the corresponding weak* topologies on X^{**} and X''_r. Therefore, for $S \subset X$ we have $\widetilde{S}^\mathbb{C} = \widetilde{S}^\mathbb{R}$ as topological spaces. If X is complex and $\tau\colon X \to X$ is an involutive conjugate-linear isometry, then $X^\tau := \{x \in X \mid \tau(x) = x\}$ is a real Banach space that is called a *real form* on X. In that case τ^* is the conjugate-linear involutive isometry $\tau^*\colon X^* \to X^*$ given by $\tau^*(f)(x) := \overline{f(\tau(x))}$ for all x in X and f in X^*. Finally, $\mathcal{L}(X)$ is the Banach algebra of all bounded linear operators on X.

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1. The main result

1.1. Definition. For a Banach space X over $\mathbb{F}$ we define $\mathcal{B}(X)$ to be the set of all functionals $\varphi \in X^{\star\star\star}$ such that for every nonempty closed convex subset $C \subset X$, the mapping $\varphi\colon (\widetilde{C}, w^*) \to \mathbb{F}$ has at least one point of continuity in $\widetilde{C}$.

It is known that $\mathcal{B}(X)$ is a closed subspace of $X^{\star\star\star}$. If X is complex, then X and X_r have the same closed convex subsets; if C is one of them, then $(\widetilde{C}^{\mathbb{C}}, w^*_{\mathbb{C}})$ is homeomorphic to $\widetilde{C}^{\mathbb{R}}, w'_{\mathbb{R}})$. Hence we have

$$\mathcal{B}(X_r) = \mathfrak{Re}(\mathcal{B}(X)). \tag{1}$$

1.2. Definition. For a Banach space X over $\mathbb{F}$ we define its *universal frame* $\gamma(X)$ by

$$\gamma(X) := (\mathcal{B}(X) \cap J(X)^{\circ})_{\circ}. \tag{2}$$

Therefore by omitting the reference to J, we have $X \subset \gamma(X) \subset X^{\star\star}$, and $\gamma(X)$ is a closed subspace of $X^{\star\star}$. Assume that X is complex; the definition of polarity together with $iX \subset X$ yield $(X_r)^{\circ} = \mathfrak{Re}(X^{\circ})$ and as $\mathfrak{Re}\colon X^{**} \to (X_r)''$ is injective, by (1)

$$\mathcal{B}(X_r) \cap (X_r)^{\circ} = \mathfrak{Re}\,\mathcal{B}(X) \cap \mathfrak{Re}(X^{\circ}) = \mathfrak{Re}[\mathcal{B}(X) \cap X^{\circ}].$$

Let $S \subset X^*$; then $\mathfrak{Re}\,S \subset (X_r)'$ and if $iS \subset S$, the prepolar $S_{\circ_{\mathbb{R}}}$ in X_r satisfies $(\mathfrak{Re}\,S)_{\circ} = \mathfrak{Re}(S_{\circ})$. Applying this to the set $S := \mathcal{B}(X) \cap (X)^{\circ} \subset X^{***}$ we get

$$\begin{aligned}\mathfrak{Re}\,\gamma(X) &= \mathfrak{Re}[(\mathcal{B}(X) \cap (X)^{\circ})_{\circ}] = [\mathfrak{Re}(\mathcal{B}(X) \cap (X)^{\circ})]_{\circ}\\ &= [\mathcal{B}(X_r) \cap (X_r)^{\circ}]_{\circ} = \gamma(X_r).\end{aligned}$$

1.3. Definition. A Banach space X over $\mathbb{F}$ is said to be *well framed* if $\gamma(X) = X$.

1.4. Lemma. *If X is a complex Banach space and X is well framed, then X_r is also well framed.*

Proof. It follows from the previous discussion and [3], Theorem 15. ☐

The class of the well framed Banach spaces is hereditary and invariant under surjective isometric isomorphisms ([3], Theorem 16).

1.5. Definition. A *real C^*-algebra* is a real Banach *-algebra A such that for all a in A, $\|a^*a\| = \|a\|^2$ and $\mathbf{1} + a^*a$ is invertible in A if A has a unit. If A is not unital, then we require that $\mathbf{1} + a^*a$ be invertible for all a in the unital extension A_1 of A.

A real Banach *-algebra is a real C^*-algebra if and only if $\|a\|^2 \leq \|a^*a + b^*b\|$ holds for all a, b in A, which occurs if and only if A is isometrically *-isomorphic to a norm-closed selfadjoint algebra of bounded operators on a real Hilbert space ([2], Lemma 1.1, and [4], Theorem 15.3 and Lemma 15.4). If A is a real C^*-algebra, then there exists a (complex) C^*-algebra B with an involutive conjugate-linear *-automorphism τ such that $A = B^{\tau}$ (see [7], 4.1.13, and [4], 15.4). Of course, algebraically regarded, B is nothing but the complexification of A.

1.6. Proposition. *If A is a real C^*-algebra with a (complete) predual A_*, then A_* is well framed.*

Proof. Let B be a (complex) C^*-algebra with an involutive conjugate-linear *-isomorphism τ such that $A = B^\tau$. Then B^{**} is a W^*-algebra and, as Banach spaces, we have $A^* \cong (B^*)^{\tau^*}$. Since B^* is well framed ([3], Theorem 18) we may apply Lemma 1.4 to obtain that $(B^*)_r$ is also well framed. Since the property of being well framed is hereditary and, by the isometry $A^* \cong (B^*)^{\tau^*}$, A^* can be seen as a closed subspace of $(B^*)_r$, it follows that A^* is well framed too. Considering A_* as a closed subspace of A^* by means of the canonical embedding into the bidual, we get the result. □

1.7. *Remark.* Let A be a real C^*-algebra having a predual A_*. By 1.6 and [3], Theorem 15, every linear isometry of A onto A is $\sigma(A, A_*)$-continuous and, if E is a Banach space with $E^* \cong A$, then $E \cong A_*$. It is well known and easy to see that this implies (in fact, it is equivalent to the fact) that A_* is the unique predual of A in the familiar sense of C^*-theory [8], namely, if E is any predual of A and if E and A_* are canonically regarded as subspaces of A^*, then $E = A_*$. Indeed if E is a predual of A, then by the above there exists a surjective linear isometry $\phi\colon A_* \to E$, and the linear isometries ϕ^* and $\phi^{*^{-1}}$ from A onto A are $\sigma(A, A_*)$-continuous. By standard theory of duality this implies $\phi^{**}(A_*) = A_*$. But clearly ϕ^* is $\sigma(A, E) - \sigma(A, A_*)$-continuous and $\phi^{*^{-1}}$, i.e. $(\phi^{-1})^*$, is $\sigma(A, A_*) - \sigma(A, E)$-continuous, and therefore $\phi^{**}(A_*) = E$. Thus $E = A_*$. □

1.8. Definition. We recall that a *real W^*-algebra* is a real C^*-algebra A having a complete predual A_* such that the product of A is separately $\sigma(A, A_*) - \sigma(A, A_*)$-continuous.

By [2], Theorem 1.6, the bidual A^{**} of a real C^*-algebra A, endowed with a product and an involution that extend those of A, is a real W^*-algebra. We also recall that the selfadjoint part of a real C^*-algebra is a JB-algebra in a natural way (for JB-algebras and JBW-algebras see [5], §3 and §4).

1.9. Proposition. *If A is a real C^*-algebra with a predual A_*, then A has a unit and A is the norm-closed linear hull of the set of its unitary elements.*

Proof. By the Alaoglu and Krein-Milman theorems, there exists an extreme point (say x) in the closed unit ball of A. But real C^*-algebras are examples of real JB*-triples in the sense of [6] and by Lemma 3.2 therein, x is a complete tripotent in the JB*-triple associated to A, hence $a - (xx^*a + ax^*x) + xx^*ax^*x = 0$ holds for all a in A. Regarding A as a real C^*-subalgebra of the real W^*-algebra A^{**} and taking into account that A^{**} has a unit $\mathbf{1}$ (see [2], Corollary 2.8) we can rewrite the above as $(\mathbf{1} - xx^*)A(\mathbf{1} - x^*x) = 0$. By the w^*-density of A in A^{**} and the separate w^*-continuity of the product of A^{**}, we actually have $(\mathbf{1} - xx^*)A^{**}(\mathbf{1} - x^*x) = 0$ and, as a consequence, $(\mathbf{1} - xx^*)(\mathbf{1} - x^*x) = 0$. It follows that $\mathbf{1} = xx^* + x^*x - xx^*x^*x$ lies in A and is indeed a unit for A. If a is in A and satisfies $a^* = -a$, then $\exp(ta)$ is a unitary element of A for every t in $\mathbb{R}$, and the equality $a = \lim_{t\to 0} \frac{1}{t}(\exp(ta) - \mathbf{1})$, where the limit is taken in the norm topology of A, shows that a lies in the norm-closed linear hull of the unitary elements of A. To see the same in the case $a = a^*$, take into account that by Remark 1.7, the involution * is $\sigma(A, A_*)$-continuous and therefore the selfadjoint part of A is $\sigma(A, A_*)$-closed in A, hence it is a JBW-algebra. But JBW-algebras coincide with the norm-closed linear hull of their idempotents [5], Proposition 4.2.3, and if e is a selfadjoint idempotent of A, then $2e - \mathbf{1}$ is a selfadjoint unitary element of A. Finally, we note that every element a in A can be written in the form $a = b + c$ with $b = b^*$ and $c = -c^*$. □

1.10. *Remark.* The above proof shows that the closed unit ball of a real C^*-algebra A has extreme points if and only if A has a unit $\mathbf{1}$ and, if this is the case, then the extreme points of the closed unit ball of A are precisely those elements x in A satisfying $(\mathbf{1} - xx^*)A(\mathbf{1} - x^*x) = 0$. This is a real variant of a well known result for complex C^*-algebras ([8], Theorem 1.6.4).

1.11. Theorem. *If A is a real C^*-algebra having a predual A_*, then A is a real W^*-algebra.*

Proof. If a is a unitary element in A, then the operator L_a of left multiplication by a in A is a surjective linear isometry, hence by Remark 1.7 it is $\sigma(A, A_*)-\sigma(A, A_*)$-continuous. Now, the mapping $a \mapsto L_a$ from A into the algebra $\mathcal{L}(A)$ is linear and norm-continuous, and the set of all $\sigma(A, A_*)$-continuous operators on A is norm-closed in $\mathcal{L}(A)$, whence by Proposition 1.9 it follows that L_a is $\sigma(A, A_*)$-continuous for every a in A. In other words, the product of A is $\sigma(A, A_*)$-continuous in the second variable. The $\sigma(A, A_*)$-continuity of the product of A in the first variable is obtained by simple argument of symmetry. □

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