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WEAK SEPARATION PROPERTIES FOR SELF-SIMILAR SETS

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ABSTRACT. We develop a theory for self-similar sets in $\mathbb{R}^s$ that fulfil the weak separation property of Lau and Ngai, which is weaker than the open set condition of Hutchinson.

1. INTRODUCTION AND NOTATION

A simple method of constructing subsets K of $\mathbb{R}^s$ with fractal features is the following: Take contracting similitudes $f_1, \dots, f_N : \mathbb{R}^s \longrightarrow \mathbb{R}^s$ with contraction ratios $r_i \in]0,1[$, i.e. $|f_i(x) - f_i(y)| = r_i|x-y|$ for all $x, y \in \mathbb{R}^s$. Then the unique non-empty compact set K with

$$K = \bigcup_{i=1}^{N} f_i(K)$$

is in general a fractal [4]. K is called a self-similar set since it is made up of smaller similar copies of itself. Interesting properties of K like its topology, Hausdorff measure or dimension seem to depend on the extent to which these small copies overlap each other. The most celebrated condition that ensures that there is not too much overlapping is the open set condition (OSC) of Hutchinson. It is fulfilled iff there is a non-empty open set V such that the sets $f_i(V)$ are disjoint and contained in V.

However appealing this condition may be, it is not easy to check since except for some simple examples the open set V, if it exists, may be almost as exotic as K itself. For this reason a variety of equivalent conditions has been developed. Schief [8] proved that the OSC is equivalent to $\mathcal{H}^\alpha(K) > 0$, where $\mathcal{H}^\alpha$ is Hausdorff measure of dimension α, and α is the so called similarity dimension of $\{f_i\}_{i=1}^N$, i.e. the unique solution of the equation $\sum_{i=1}^N r_i^\alpha = 1$. This result shows that an algebraic condition developed previously by Bandt and Graf [2] is equivalent to the OSC, too. To formulate this condition we need some more notation. Let

$$I := \{\mathbf{i} = (i_1, \dots, i_n) : n \geq 0,\ i_1, \dots, i_n \in \{1, \dots, N\}\}$$

be the set of all finite words over the alphabet $\{1, \dots, N\}$. For $\mathbf{i} \in I$ define $f_{\mathbf{i}} :=$

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$f_{i_1} \dots f_{i_n}$. Bandt and Graf [2] considered essentially the subset

$$\mathcal{E} := \left\{ f_{\mathbf{i}}^{-1} f_{\mathbf{j}} : \ \mathbf{i}, \mathbf{j} \in I, \mathbf{i} \neq \mathbf{j} \right\}$$

of the group of all similitudes on $\mathbb{R}^s$ endowed with the topology of pointwise convergence. Actually they investigated a certain subset of $\mathcal{E}$ that contains all $f \in \mathcal{E}$ with $r_f \in]r_{\max}, r_{\max}^{-1}[$ as an open subset, where r_f denotes the contraction ratio of f and $r_{\max} := \max\{r_1, \dots, r_N\}$. They proved that $\mathcal{H}^\alpha(K) > 0$ iff the identity id is not contained in the closure $\mathrm{cl}(\mathcal{E})$ of $\mathcal{E}$. Thus by Schief's work [8] this means that the OSC is equivalent to $\mathrm{id} \notin \mathrm{cl}(\mathcal{E})$. This condition has made the OSC more handy.

However, things are still not as simple as one might hope. For example consider similitudes $f_1, \dots, f_N$ of the form $f_i(x) = x/n + y_i$, where $n, y_i \in \mathbb{Z}$ with $|n| \geq 2$. Clearly the OSC does not hold for $N > n$. Since even for $N \leq n$ the OSC may fail, the theory of OSC does not help us to calculate the Hausdorff measure and dimension of K in many cases. Moreover, to the best of our knowledge, there does not exist a general classification of those parameters $(n; y_1, \dots, y_N)$ for which the OSC is satisfied.

That was the reason why we looked for a separation condition that is weaker than the OSC but still strong enough to obtain good results. A natural way to create a condition that is less restrictive than the OSC is the following: We say that the weak separation property (WSP) is fulfilled iff $\mathrm{id} \notin \mathrm{cl}(\mathcal{E} \backslash \{\mathrm{id}\})$. I.e. the WSP holds iff the identity is not an accumulation point of $\mathcal{E}$. We adopt the term WSP from Lau and Ngai [6] who gave a definition that is different but turns out to be equivalent to ours. Note that the OSC is valid iff the WSP holds and $f_{\mathbf{i}} \neq f_{\mathbf{j}}$ for all $\mathbf{i} \neq \mathbf{j}$.

At first glance this might seem to be merely a minor generalization of OSC but in fact it increases the class of tractable self-similar sets considerably. Indeed, in the example mentioned above the WSP is fulfilled for any choices of $n, y_i \in \mathbb{Z}$ with $|n| \geq 2$.

For the investigation of the WSP we may simplify the notation a little bit, i.e. we may forget about the indices of the similitudes. Thus we define $F := \{f_{\mathbf{i}} : \mathbf{i} \in I\}$ to be the semigroup with identity generated by $\{f_i\}_{i=1}^N$. The set $F^{-1}F := \{f^{-1}g : f, g \in F\}$ will replace $\mathcal{E}$. To formulate some conditions that are equivalent to the WSP we set

$$F_b := \{f \in F : r_f \in \,]br_{\min}, b] \ \} \text{ for } b \geq 0 \quad \text{and} \quad \mathcal{F} := \bigcup_{b>0} \{f^{-1}g : f, g \in F_b\},$$

where $r_{\min} := \min\{r_1, \dots, r_N\}$. Note that $K = \bigcup_{f \in F_b} f(K)$ for all $b \in]0,1[$. Finally, we adapt some of the definitions given in [2], [8] to our requirements and set

$$F_{a,U,M} := \{g \in F_{a \ \mathrm{diam}\, U} : g(M) \cap U \neq \emptyset\} \quad \text{and} \quad \gamma_{a,M} := \sup_{U \subseteq \mathbb{R}^s} \sharp F_{a,U,M}$$

for $a > 0$ and $U, M \subseteq \mathbb{R}^s$.

The dimension β of self-similar sets that fulfil the WSP but not the OSC is strictly less than α. Indeed, β is the exponential growth rate of $\sharp F_b$ for $b \searrow 0$, i.e. $\beta = -\lim_{b \searrow 0} \log \sharp F_b / \log b$. Alternatively β may be characterized as the limit of the similarity dimensions α_b of F_b for $b \searrow 0$. We shall show how to compute β for a generalization of the one-dimensional example given above to higher dimensions. As in the case of OSC, K has positive Hausdorff measure at its dimensional value β when the WSP holds. Schief [8] proved that if $\alpha = s$ and K has positive Lebesgue measure $\lambda(K)$ then K has a non-empty interior. We will show that if the WSP is satisfied and K has dimension s then K contains interior points too. This might be a further step towards answering the question (e.g. [8]) whether every self-similar set with positive Lebesgue measure has interior points.

2. Results

We start with a list of conditions that are all equivalent to the weak separation property, if K is in general position, i.e. not contained in a hyperplane.

Theorem 1. *If K is not contained in a hyperplane then the following conditions are equivalent.*

(1a) *There are $x \in K$ and $\varepsilon > 0$ such that for all $h \in \mathcal{F}$ either $h(x) = x$ or $|h(x) - x| \geq \varepsilon$.*
(1b) *There are $x \in \mathbb{R}^s$ and $\varepsilon > 0$ such that for all $h \in \mathcal{F}$ either $h(x) = x$ or $|h(x) - x| \geq \varepsilon$.*
(2a) *There are $x_0, \ldots, x_s$ in general position and $\varepsilon > 0$ such that for any $h \in \mathcal{F}$ and any $j \in \{0, \ldots, s\}$ either $h(x_j) = x_j$ or $|h(x_j) - x_j| \geq \varepsilon$.*
(2b) *There are $x_0, \ldots, x_s$ in general position and $\varepsilon > 0$ such that for any $h \in \mathcal{F} \backslash \{\mathrm{id}\}$ there is $j \in \{0, \ldots, s\}$ with $|h(x_j) - x_j| \geq \varepsilon$.*
(3a) *The identity is an isolated point of $F^{-1}F$.*
(3b) *The identity is an isolated point of $\mathcal{F}$.*
(4a) *$\gamma_{a,M} < \infty$ for all $a > 0$ and all bounded $M \subseteq \mathbb{R}^s$.*
(4b) *$\gamma_{a,M} < \infty$ for some $a > 0$ and some non-empty $M \subseteq \mathbb{R}^s$.*
(5a) *For all $x \in \mathbb{R}^s$ there is some $\ell \in \mathbb{N}$ such that for any $f \in F$ and any $b > 0$, every ball with radius b contains at most l elements of $F_b(f(x))$.*
(5b) *There are $x \in \mathbb{R}^s$ and $\ell \in \mathbb{N}$ such that for any $f \in F$ and any $b > 0$, every ball with radius b contains at most l elements of $F_b(f(x))$.*

Here $F_b(y) := \{g(y) : \ g \in F_b\}$.

(1a) – (3b) are analogues of the conditions developed by Bandt and Graf [2]. (4a) and (4b) follow Schief's approach [8] implicitly contained in [2]. (5b) is the original version of the weak separation property of Lau and Ngai [6] with only minor modification. In fact Lau and Ngai considered $F_b^* := \{f_{\mathbf{i}} : \mathbf{i} \in I_b\}$ instead of F_b, where

$$I_b := \left\{\mathbf{i} = (i_1, \ldots, i_n) : r_{i_1} \cdot \ldots \cdot r_{i_n} \leq b < r_{i_1} \cdot \ldots \cdot r_{i_{n-1}}\right\}.$$

But since

$$F_b^* \subseteq F_b \subseteq F_b^* \cup F_{br_{\max}}^* \cup \ldots \cup F_{br_{\max}^m}^* \tag{6}$$

for m with $r_{\max}^{m+1} \leq r_{\min}$ the two definitions are equivalent. We prefer F_b since F_b does not depend on the way the elements of F are generated by $f_1, \ldots, f_N$.

Proof of Theorem 1. The implications $(i\mathrm{a}) \Rightarrow (i\mathrm{b})$ for $i = 1, \ldots, 5$ are trivial.

<u>(1b) ⇒ (2a)</u> : With respect to the Hausdorff metric, $F_b(x)$ converges towards K as $b \searrow 0$. Thus since K is not contained in a hyperplane, there are $g_0, \ldots, g_s \in F$ such that the points $x_j := g_j(x), j = 0, \ldots, s$, are in general position. So for all $j \in \{0, \ldots, s\}$ any $h \in \mathcal{F}$ either keeps x_j fixed or fulfills

$$|h(x_j) - x_j| = r_{g_j} |g_j^{-1} h g_j(x) - x| \geq \min_j r_{g_j} \, \varepsilon$$

since $g_j^{-1} h g_j \in \mathcal{F}$.

(2b) $\Leftrightarrow$ (3b) : This follows from the observation in [2] that for any $x_0, \ldots, x_s$ that are in general position the sets $W_\varepsilon, \varepsilon > 0$, of all similtudes h on $\mathbb{R}^s$ with $|h(x_j) - x_j| < \varepsilon$ for all $j = 0, \ldots, s$, form a neighborhood base of the identity.

(3b) $\Rightarrow$ (3a) : Note that $\mathcal{F}$ is the intersection of $F^{-1}F$ and the open set of all similitudes h on $\mathbb{R}^s$ with $r_h \in]r_{\min}, r_{\min}^{-1}[$.

(2b) $\Rightarrow$ (4a) : Without loss of generality we may assume $x_0, \ldots, x_s \in M$. Let $U \subseteq \mathbb{R}^s$ with $\operatorname{diam} U > 0$. We again follow Bandt and Graf [2] and observe that for any $f, g \in F_{a,U,M}$ that are not identical we have $|f(x_j) - g(x_j)| \geq a \operatorname{diam} U\ r_{\min}\varepsilon$ for some $j = j(f, g) \in \{0, \ldots, s\}$. Now suppose, F' is a subfamily of $F_{a,U,M}$ such that $j(f, g)$ is the same for all $f, g \in F'$. Then for $\rho := a\ r_{\min}\varepsilon/2$ the open balls $B\left(f(x_j), \rho \operatorname{diam} U\right), f \in F'$, with centers $f(x_j)$ and radius $\rho \operatorname{diam} U$ are pairwise disjoint and their centers are contained in a ball of radius $\operatorname{diam} U(1 + 2a \operatorname{diam} M)$. Using Lebesgue measure we see that $\sharp F' \leq (1 + 2a \operatorname{diam} M + 2\rho)^s/\rho^s =: N_{a,M}$. Since $N_{a,M}$ is independent of U, Ramsey's theorem yields that there is a finite constant $C_{a,M}$ with $\sharp F_{a,U,M} \leq C_{a,M}$ for all $U \subseteq \mathbb{R}^s$.

(4a) $\Rightarrow$ (5a) : Denote by $\overline{B}(y, b)$ the closed ball with center y and radius b. Then for any $x \in \mathbb{R}^s$

$$\sup_{b>0} \sup_{y\in\mathbb{R}^s} \sup_{f\in F} \sharp F_b\left(f(x)\right) \cap \overline{B}(y, b) \ \leq\ \sup_{b,y,f} \sharp \left\{g \in F_b : g\left(f(x)\right) \in \overline{B}(y, b)\right\}$$

$$\leq \sup_{b,y} \sharp \left\{g \in F_b : g\left(F(x)\right) \cap \overline{B}(y, b) \neq \emptyset\right\} \ \leq \gamma_{1/2, F(x)}$$

which is finite since $F(x)$ is bounded.

(5b) $\Rightarrow$ (4b) : Denote by c the maximal number of unit balls needed to cover a set of diameter two. Then

$$\begin{aligned} \gamma_{1/2,\{x\}} &\leq \sup_{U\subseteq\mathbb{R}^s} c \sup_{z\in\mathbb{R}^s} \sharp \left\{f \in F_{\operatorname{diam} U/2} : f(x) \in \overline{B}(z, \operatorname{diam} U/2)\right\} \\ &= c \sup_{b>0} \sup_{z\in\mathbb{R}^s} \sum_{y\in F_b(x)\cap\overline{B}(z,b)} \sharp \left\{f \in F_b : f(x) = y\right\} \\ &\leq c\ C \sup_{b,z} \sharp F_b(x) \cap \overline{B}(z, b) \ \leq\ c\ C\ \ell, \end{aligned}$$

where $C < \infty$ exists by the following observation.

Lemma 1. *If K is not contained in a hyperplane, then (5b) implies*

$$\sup_{y\in\mathbb{R}^s} \sup_{b>0} \sharp \left\{f \in F_b : f(x) = y\right\} < \infty, \tag{7}$$

where $x \in \mathbb{R}^s$ is as in (5b).

Proof. Let $g_0, \ldots, g_s \in F$ be such that the points $x_i := g_i(x), i = 0, \ldots, s$, are in general position and set $\rho := \max_i |x_i - x|$. Since any similarity h is determined by its values $h(x_i), i = 0, \ldots, s$, we get

$$\sup_{y,b} \sharp \left\{f \in F_b : f(x) = y\right\} \leq \sup_{y,b} \left[\sharp \left\{f(x_i) : f \in F_b, i \in \{0, \ldots, s\}, f(x) = y\right\}\right]_s,$$

where $[m]_s := m \cdot (m - 1) \cdot \ldots \cdot (m - s)$. This is less than or equal to

$$\left[\sum_{i=0}^{s} \sup_{y,b} \sharp \left\{f(x_i) : f \in F_b, f(x) = y\right\}\right]_s \leq \left[\sum_{i=0}^{s} \sup_{y,b} \sharp \left[F_b(x_i) \cap \overline{B}\left(y, b\rho\right)\right]\right]_s$$

because $|f(x_i) - y| = r_f |x_i - x| \leq b\rho$. Thus if we denote by c_ρ the number of unit balls, needed to cover a ball of radius ρ, we get $\sup_{y,b} \sharp \{f \in F_b : f(x) = y\} \leq [(s+1)c_\rho \ell]_s < \infty$. $\square$

(4b) $\Rightarrow$ (1b) : Without loss of generality, assume $M = \{y\}$ for some $y \in \mathbb{R}^s$ and set $U := B(y, 1/2a)$. Modifying an idea of Schief [8] we define

$$I_U(f) := \left\{g \in F_{r_f} : g(M) \cap f(U) \neq \emptyset\right\} \quad \text{for } f \in F. \tag{8}$$

Observe that $I_U(f)$ is just $F_{a,f(U),M}$ and thus there exists some $f \in F$ with

$$\sharp I_U(f) = \max \{\sharp I_U(g) : g \in F\}. \tag{9}$$

We show that for arbitrary $v \in F$

$$I_U(vf) = v\ I_U(f). \tag{10}$$

By (9) we only have to show $\supseteq$. But this is clear, since $vg \in F_{r_{vf}}$ for $g \in F_{r_f}$ and $g(M) \cap f(U) \neq \emptyset$ implies $vg(M) \cap vf(U) \neq \emptyset$.

Now set $x := f(y)$. We construct $\varepsilon > 0$ such that (1b) is fulfilled. Let $h = v^{-1}w \in \mathcal{F}$, where $v, w \in F_b$ for some $b > 0$. We may assume $r_w \leq r_v$ because if not we have $|h(x) - x| \geq \left|x - h^{-1}(x)\right|$ and the following works for h^{-1} instead of h. First, we consider the case $wf \in I_U(vf)$. Then by (10) $wf = vg'$ for some $g' \in I_U(f)$. Hence $|h(x) - x| = \left|v^{-1}wf(y) - f(y)\right| = |g'(y) - f(y)|$. If $g'(y) \neq f(y)$ this is not smaller than $\varepsilon_1 := \min \{|g(y) - f(y)| : g \in I_U(f), g(y) \neq f(y)\}$.

Now we consider the case $wf \notin I_U(vf)$. This means $v^{-1}wf(y) \notin f(U)$ since by $w \in F_{r_v}$ we have $wf \in F_{r_{vf}}$. Due to $f(U) = B(x, r_f/2a)$ this implies $|h(x) - x| \geq r_f/2a =: \varepsilon_2$. Finally set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$.

(4a) $\Rightarrow$ (1a) : In the same manner as in the proof of (4b) $\Rightarrow$ (1b) we construct for some arbitrary $y \in K$ some f and ε such that $x = f(y)$ and ε satisfy (1b). But now we have $x \in K$ since $f(K) \subseteq K$. $\square$

If K is not in general position, the conditions of Theorem 1 need not be equivalent any more. For instance if g_1 and g_2 are rotations in the plane that keep the origin fixed then the self-similar set K which is generated by $f_i(x) = rg_i(x)$ consists of the origin only. Thus (1a) if fulfilled trivially. But if the angles of rotation of g_1 and g_2 are incommensurable one easily sees that (3b) is no longer valid.

So if K is contained in a hyperplane, one has to consider the set of similitudes restricted to the subspace spanned by K.

Definition. *A family $\{f_i\}_{i=1}^N$ of contracting similitudes is said to have the* ***weak separation property (WSP)*** *if one of the conditions of Theorem 1 holds.*

As already mentioned in the introduction, OSC and WSP are related in the following manner.

Proposition 1. *Assume that K is not contained in a hyperplane. Then the OSC holds if and only if the WSP holds and $f_{\mathbf{i}} \neq f_{\mathbf{j}}$ for all $\mathbf{i} \neq \mathbf{j}$.*

What about the dimension of K? From [3] we know that for self-similar sets all usual definitions of dimensions such as Hausdorff, box-counting and packing dimensions give the same value. Thus we write simply $\dim K$ for the dimension of K. In this context it is convenient to use box-counting dimensions. Recall [4] that the lower and the upper box-counting dimensions of a subset S of $\mathbb{R}^s$ may be defined by

$$\underline{\dim}_B S := \liminf_{b\searrow 0} \frac{\log N_b(S)}{-\log b} \quad \text{and} \quad \overline{\dim}_B S := \limsup_{b\searrow 0} \frac{\log N_b(S)}{-\log b},$$

where $N_b(S)$ is the smallest number of open balls with radius b that cover S.

The following theorem shows that if the WSP holds, $\dim K$ may also be characterized by the exponential growth rate of $\sharp F_b$ for $b \searrow 0$.

Theorem 2. *For any arbitrary set $\{f_i\}_{i=1}^N$ of contracting similitudes*

$$\dim K \leq \beta := \lim_{b\searrow 0} \frac{\log \sharp F_b}{-\log b} = \lim_{b\searrow 0} \alpha_b,$$

where $\alpha_b, b \in]0,1[$, is the similarity dimension of F_b, i.e. the solution of $\sum_{f\in F_b} r_f^{\alpha_b} = 1$. If $\{f_i\}_{i=1}^N$ has the WSP and K is not contained in a hyperplane

$$\dim K = \beta = \lim_{b\searrow 0} \frac{\log \sharp F_b(x)}{-\log b} \quad \textit{for all } x \in \mathbb{R}^s.$$

The proof makes use of the following lemmas.

Lemma 2. *If the WSP holds and K is not contained in a hyperplane, then for any $x \in \mathbb{R}^s$ there exists some finite C such that $\sharp F_b(x) \leq \sharp F_b \leq C \sharp F_b(x)$ for all $b > 0$.*

Proof. The first inequality is obvious. Since (5a) and (5b) are equivalent, (7) holds for all $x \in \mathbb{R}^s$ by Lemma 1. This implies the second inequality. □

Lemma 3. *If $\{B_1, \ldots, B_n\}$ is a family of open balls in $\mathbb{R}^s$, there is a disjoint subfamily $\{B_{i_1}, \ldots, B_{i_r}\}$ such that $\lambda\left(\bigcup_{j=1}^n B_j\right) \leq 3^s \sum_{p=1}^r \lambda(B_{i_p})$.*

This may be found in [1, Lemma 2.3.6], formulated for cubes instead of balls.

Proof of Theorem 2. First, we prove the existence of β. Note that $\sharp F_b$ increases monotonically when b decreases and furthermore $F_{b_1 b_2} \subseteq F_{b_1}(F_{b_2} \cup F_{b_2 r_{\min}^{-1}})$ for any $b_1, b_2 \in]0,1[$. Hence $\sharp F_{b_1 b_2} \leq 2 \sharp F_{b_1} \sharp F_{b_2}$. Now we follow an idea of Milnor [7, page 2]. For each fixed $a \in]0,1[$, setting $k = [\log b / \log a] + 1$, the inequalities $\sharp F_b \leq \sharp F_{a^k} \leq (2\sharp F_a)^k \leq (2\sharp F_a)^{\log b/\log a + 1}$ imply that $\limsup_{b\searrow 0} (\sharp F_b)^{-1/\log b} \leq (2\sharp F_a)^{-1/\log a}$. Therefore $\limsup_{b\searrow 0} (\sharp F_b)^{-1/\log b} \leq \inf_a (2\sharp F_a)^{-1/\log a} \leq \liminf_{a\searrow 0} (\sharp F_a)^{-1/\log a}$ and the existence of β follows.

Since $\sharp F_b (b r_{\min})^{\alpha_b} < \sum_{f\in F_b} r_f^{\alpha_b} = 1 \leq \sharp F_b b^{\alpha_b}$, we have

$$\frac{\log \sharp F_b}{-\log b r_{\min}} < \alpha_b \leq \frac{\log \sharp F_b}{-\log b} \qquad \text{for } b \in]0,1[$$

and conclude $\beta = \lim_{b\searrow 0} \alpha_b$.

For arbitrary $x \in \mathbb{R}^s$ the family of sets $F_b(x), b \in]0,1[$, approximates K, in the sense that for some finite c_x

$$\mathrm{d}_H(K, F_b(x)) < c_x b \qquad \text{for all } b \in]0,1[, \tag{11}$$

where d_H is the Hausdorff metric [4]. This implies that K may be covered by the balls $B(y, c_x b), y \in F_b(x)$. Thus $N_{c_x b}(K) \leq \sharp F_b(x) \leq \sharp F_b$, giving $\dim K \leq \beta$.

Now suppose that the WSP holds and K is in general position. In view of Lemma 2 it suffices to prove

$$\liminf_{b \searrow 0} \frac{\log \sharp F_b(x)}{-\log b} \le \dim K \tag{12}$$

for some x. Choose x and ε according to (1b). Fix $b \in]0,1[$ and consider the family $B(y, 2c_x b), y \in F_b(x)$, of open balls. By Lemma 3 there exists a subset Y of $F_b(x)$, such that the balls of the corresponding subfamily are pairwise disjoint and fulfil

$$3^s \sharp Y \lambda\left(B(0, 2c_x b)\right) \ge \lambda\left(\bigcup_{y \in F_b(x)} B(y, 2c_x b)\right) \ge \lambda\left(\bigcup_{y \in F_b(x)} B\left(y, b r_x\right)\right),$$

where $r_x := \min\{2c_x, r_{\min}\varepsilon/2\}$. Due to the choice of x and ε the union above is disjoint. Hence we get

$$\sharp Y \ge \frac{\lambda\left(B\left(0, r_x\right)\right)}{3^s \lambda\left(B(0, 2c_x)\right)} \sharp F_b(x) = c'_x \sharp F_b(x), \tag{13}$$

where $c'_x > 0$ does not depend on b. (11) implies that for each $y \in F_b(x)$ there is some $z_y \in K$ such that $|z_y - y| < c_x b$. This and the disjointness of the balls $B(y, 2c_x b), y \in Y$, yield that z_{y_1} and z_{y_2} have distance at least $2c_x b$ for different $y_1, y_2 \in Y$. Hence $N_{c_x b}(K) \ge N_{c_x b}\left(\{z_y \in K : y \in Y\}\right) = \sharp Y \ge c'_x \sharp F_b(x)$ by (13), and (12) follows. □

The problem of determining growth rates of finitely generated groups has been treated first by Milnor [7]. Since then the growth functions of a wide variety of groups have been studied (e.g. [9], [10]). However, to the best of our knowledge, there does not exist a general algorithm yet that applies to the problem of determining the exponential growth rate β. For special cases we will describe in section 3 how to find a linear recurrence relation for $\sharp F_b$ that enables us to compute β.

We call β the *growth dimension* of K or more correctly of $\{f_i\}_{i=1}^N$, in contrast to the *similarity dimension* α. We do not know whether there exists some self-similar set K in general position with $\dim K < \min\{\beta, s\}$.

The following proposition clarifies the relation between α and β.

Proposition 2. *In general $\beta \le \alpha$. Equality holds if and only if $f_{\mathbf{i}} \ne f_{\mathbf{j}}$ for all $\mathbf{i} \ne \mathbf{j}$.*

Proof. Due to (6) and $\sharp F^*_{b r_{\max}} \le N \sharp F^*_b$ there is some finite c such that $\sharp F^*_b \le \sharp F_b \le c \sharp F^*_b$ for all $b \in]0,1[$. Hence the exponential growth rates of $\sharp F_b$ and $\sharp F^*_b$ agree. Thus by the same argument as in the proof of Theorem 2 we have $\beta = \lim_{b \searrow 0} \alpha^*_b$, where α^*_b is the similarity dimension of F^*_b.

Note that $\sum_{\mathbf{i} \in I_b} r^{\alpha}_{f_{\mathbf{i}}} = 1$ and hence $\alpha^*_b \le \alpha$ for all $b \in]0,1[$. Thus $\beta \le \alpha$. Moreover it follows that $\beta = \alpha$ if $f_{\mathbf{i}} \ne f_{\mathbf{j}}$ for all $\mathbf{i} \ne \mathbf{j}$ since in this case $\alpha^*_b = \alpha$ for all $b \in]0,1[$. For the converse, suppose there are different $\mathbf{i}, \mathbf{j} \in I_b$ for some b with $f_{\mathbf{i}} = f_{\mathbf{j}}$. If we replace $\{f_1, \ldots, f_N\}$ by F^*_b, the growth dimension β does not change but the similarity dimension decreases to α^*_b. Consequently $\beta \le \alpha^*_b < \alpha$. □

We now turn to the Hausdorff measure of dimension β.

Corollary. *If K is not contained in a hyperplane then*

$$0 \le \sup_{a > 0} \frac{1}{a^\beta \gamma_{a,K}} \le \mathcal{H}^\beta(K) < \infty.$$

In particular the WSP implies $\mathcal{H}^\beta(K) > 0$ when K is in general position.

Proof. Since $\dim K \le \beta$ by Theorem 2, $\mathcal{H}^\beta(K)$ turns out to be finite by a result of Falconer [3, Theorem 4]. If the WSP is not satisfied, the lower bound degenerates into 0 according to (4b), so nothing more has to be shown. Otherwise the claim follows from Falconer [3, Theorem 2]. □

We conjecture that also the converse is true and $\mathcal{H}^\beta(K) > 0$ implies the WSP, but cannot prove it. This would generalize Schief's theorem [8]. However, Schief's observation that K contains interior points if $\dim K = s$ and the OSC is fulfilled is still true if we replace the OSC by the WSP as the following theorem shows.

Theorem 3. *If the WSP holds and K has dimension s then K contains interior points.*

Proof. 1. By the corollary we know $\lambda(K) > 0$. Choose $x \in K$ and $\varepsilon > 0$ according to (1a), set $U = B(x, \varepsilon/2)$ and define $I_U(f)$ as done in (8) with K instead of M. By (4a) we can choose $f \in F$ such that (9) is valid. $f(U)$ intersects $g(K)$ for all $g \in I_U(f)$. Therefore $\rho := (r_f\varepsilon/2 - \max\{\mathrm{d}(f(x), g(K)) : g \in I_U(f)\})/3$ is positive. Note that by self-similarity and $f(x) \in K$ the intersection of K and the ball $B = B(f(x), \rho)$ has positive Lebesgue measure. For abbreviation we set $\hat{r} := r_f\varepsilon/2 - \rho$. We claim, that there is some $y \in B$ with $B(y, \hat{r}) \subseteq K$. This would prove the theorem.

2. First observe that $U_y := f^{-1}(B(y, \hat{r}))$ is contained in U for all $y \in B$ and therefore $I_{U_y}(g) \subseteq I_U(g)$ for all $g \in F$. On the other hand, by the definition of ρ, we have $I_U(f) \subseteq I_{U_y}(f)$ and consequently $\sharp I_{U_y}(f) \le \max_{g\in F} \sharp I_{U_y}(g) \le \max_{g\in F} \sharp I_U(g) = \sharp I_U(f) \le \sharp I_{U_y}(f)$. It follows that f is maximal not only with respect to U, but also with respect to each U_y and hence we have as in (10)

$$I_{U_y}(gf) = g\, I_{U_y}(f) \qquad \text{for all } y \in B, g \in F. \tag{14}$$

3. Now assume, that our claim made in part 1 is false and hence $b(y, \hat{r}) < 1$ for all $y \in B$ where

$$b(z, r) := \frac{\lambda(B(z, r) \cap K)}{\lambda(B(z, r))} \qquad \text{for } z \in \mathbb{R}^s, r > 0.$$

Then there is some positive δ such that $\lambda(B_\delta) > 0$ for $B_\delta := \{y \in B \cap K : b(y, \hat{r}) \le 1 - \delta\}$. (14) yields for all $y \in B_\delta$ and $g \in F$

$$\begin{aligned} b(g(y), r_g\hat{r}) &= \frac{\lambda(B(g(y), r_g\hat{r}) \cap K)}{\lambda(B(0, r_g\hat{r}))} = \frac{\lambda\left(gf[U_y] \cap \bigcup\{v(K) : v \in I_{U_y}(gf)\}\right)}{r_g^s \lambda(B(0, \hat{r}))} \\ &= \frac{\lambda\left(f[U_y] \cap \bigcup\{v(K) : v \in I_{U_y}(f)\}\right)}{\lambda(B(0, \hat{r}))} = \frac{\lambda(B(y, \hat{r}) \cap K)}{\lambda(B(0, \hat{r}))} \\ &= b(y, \hat{r}) \le 1 - \delta. \end{aligned} \tag{15}$$

In part 4 we will construct some positive c such that

$$\lambda\left(\bigcup_{g \in F_b} g(B_\delta)\right) \ge c \qquad \text{for all } 0 < b < 1. \tag{16}$$

This yields the desired contradiction. Indeed, by Lebesgue's density theorem [4] and Egoroff's theorem there is a measurable subset A of K such that $\lambda(K \setminus A) < c$ and $b(a, r)$ converges uniformly to 1 on A as r tends to 0. In particular there is some $0 < r_0 < 1$ such that $b(a, r) > 1 - \delta$ for all $a \in A$ and $0 < r \le r_0$. But due to (16) and (15) there are $y \in B_\delta$ and $g \in F_{r_0}$ with $g(y) \in A$ and $b(g(y), r_g\hat{r}) \le 1 - \delta$.

4. It remains to show (16). (1a) implies that for any $b \in]0,1[$ and any $g_1, g_2 \in F_b$ the balls $g_1(B)$ and $g_2(B)$ are either concentric or disjoint, depending on whether $g_1(f(x))$ and $g_2(f(x))$ coincide or not. Let F'_b be a subset of F_b with different $g(f(x))$ for different $g \in F'_b$ and the same cardinality as $F_b(f(x))$. Then we have

$$\begin{aligned}\lambda\Big(\bigcup_{g\in F_b} g(B_\delta)\Big) &\geq \sum_{g\in F'_b} \lambda\left(g(B_\delta)\right) = \lambda(B_\delta)\sum_{g\in F'_b} r_g^s \\ &\geq \lambda(B_\delta)(br_{\min})^s \sharp F_b/C \geq \lambda(B_\delta) r_{\min}^s/C =: c > 0.\end{aligned}$$

The third to last inequality follows from Lemma 2 and the second to last inequality is a consequence of

$$0 < \lambda(K) = \lambda\Big(\bigcup_{g\in F_b} g(K)\Big) \leq \sum_{g\in F_b} \lambda\left(g(K)\right) \leq \lambda(K) b^s \sharp F_b.$$

□

3. Example

We give a generalization of the class of examples mentioned in the introduction. Another type of examples, involving P.V. numbers, is treated in [6].

Proposition 3. *Let h be a contracting similitude on $\mathbb{R}^s$ and Γ a discrete group of isometries with $\Gamma h \subseteq h\Gamma$. Furthermore let $e_1, \ldots, e_N$ be positive integers and $s_1, \ldots, s_N \in \Gamma$. Then $\{h^{e_i} s_i\}_{i=1}^N$ has the WSP.*

Proof. Since all $h^{e_i} s_i$ are in $G = \left\{h^k : k \in \mathbb{N}_0\right\}\Gamma$ and G is a semigroup we have $F \subseteq G$. Therefore $F_b \subseteq \left\{h^k : r_h^k \leq b < r_h^{k-e_1}\right\}\Gamma$, if we assume $e_1 \geq e_i$ for all i. Consequently

$$\begin{aligned}\mathcal{F} &\subseteq \bigcup_{b>0} \Gamma\left\{h^{-k} : r_h^k \leq b < r_h^{k-e_1}\right\}\left\{h^\ell : r_h^\ell \leq b < r_h^{\ell-e_1}\right\}\Gamma \\ &\subseteq \Gamma\left\{h^k : -e_1 \leq k \leq e_1\right\}\Gamma \;\subseteq\; \left\{h^k : 0 \leq k \leq e_1\right\}\Gamma \cup \Gamma\left\{h^k : -e_1 \leq k < 0\right\}.\end{aligned}$$

Since Γ is discrete it follows that $\mathcal{F}$ is discrete also and (3b) holds. □

In the following we shall indicate how one may determine the dimension of self-similar sets of this kind. For the sake of simplicity we assume $e_1, \ldots, e_N = 1$. Set $G_n := F_{r_h^n}$ for $n \geq 0$. Then we have

$$\beta = \lim_{n\to\infty} \frac{\log \sharp G_n}{-n \log r_h} \quad \text{and} \quad G_n = \bigcup_{i=1}^N h s_i G_{n-1} \quad \text{for } n \geq 1. \tag{17}$$

We shall derive a linear recurrence relation for $\sharp G_n$ which is given by the characteristic polynomial of an integer matrix A. The entries of A are indexed by elements of the index set $\mathcal{Q} := \{Q \subseteq S : \mathrm{id} \in Q\}$, where $S = \{s \in \Gamma : sG_n \cap G_n \neq \emptyset$ for some $n \geq 0\}$. $\mathcal{Q}$ is finite since S is finite. Indeed, if $f, g \in G_n$ with $sf = g$ for some $s \in S$ then $|s(0)| = |gf^{-1}(0)| \leq |gf^{-1}(0) - gf^{-1}(y)| + |gf^{-1}(y)| = |y| + |gf^{-1}(y)|$ for all $y \in \mathbb{R}^s$. Thus if we choose $y \in f(K)$ then $gf^{-1}(y) \in g(K) \subseteq K$. Hence $s(0) \in \overline{B}(0, 2\max\{|y| : y \in K\})$ for all $s \in S$. Since Γ is discrete there are only finitely many $s \in \Gamma$ that fulfil this relation [5, Folgerung 3.11] and consequently S is finite.

For $Q \subseteq \Gamma$ we define

$$z(Q,n) := \sharp \bigcap_{s\in Q} sG_n.$$

We are interested in determining $z(\{\mathrm{id}\},n) = \sharp G_n$. To this end we use a generalization of the ordinary sieve formula:

Lemma 4. *Let $X, Y, A_{x,y}$ $(x \in X, y \in Y)$ be finite sets with $X, Y \neq \emptyset$. Then*

$$\sharp \bigcap_{x\in X} \bigcup_{y\in Y} A_{x,y} = \sum_{P\subseteq X\times Y,\ \pi_1(P)=X} (-1)^{\sharp P+\sharp X}\ \sharp \bigcap_{(x,y)\in P} A_{x,y},$$

where π_1 is the projection to the first coordinate.

This lemma may be proved by induction over $\sharp X$. Thus we obtain by (17)

$$z(Q,n) = \sum_{P\subseteq Q\times\{1,\dots,N\},\ \pi_1(P)=Q} (-1)^{\sharp P+\sharp Q}\ z(Q'_P, n-1), \tag{18}$$

where $Q'_P := \{h^{-1}shs_i : (s,i) \in P\} \subseteq \Gamma$. The identity need not be in Q'_P yet, so we replace Q'_P by $Q_P = s_P^{-1}Q'_P$, where $s_P \in Q'_P$ is arbitrary. We may omit those summands with $Q_P \notin \mathcal{Q}$ since they vanish by definition of S. Note that there may be different P with the same Q_P. Thus by summing up we get $z(Q,n) = \sum_{R\in\mathcal{Q}'} a_{Q,R}\ z(R,n-1)$ for particular integers $a_{Q,R}$ and some $\mathcal{Q}' \subseteq \mathcal{Q}$. Starting with $Q = \{\mathrm{id}\}$ we may construct a set $\mathcal{Q}' \subseteq \mathcal{Q}$ with $\{\mathrm{id}\} \in \mathcal{Q}'$ and an integer matrix $A = (a_{Q,R})_{Q,R\in\mathcal{Q}'}$ such that

$$(z(Q,n))_{Q\in\mathcal{Q}'} = A\ \ (z(Q,n-1))_{Q\in\mathcal{Q}'} \qquad \text{for all } n \geq 1.$$

Induction over n yields $\sharp G_n = (A^n)_{\{\mathrm{id}\},\{\mathrm{id}\}}$. Hence $\sharp G_n$ fulfills the difference equation that belongs to the characteristic polynomial of A, and β may be computed by determining the eigenvalues of A.

In order to illustrate this algorithm, we consider the simplest non-trivial example. Let $f_i(x) = x/3 + y_i$ where $y_1 = 0, y_2 = 1$ and $y_3 = 3$. Since $f_1f_3 = f_2f_1$, the OSC is not fulfilled. However the WSP holds. Indeed, let Γ be the group of all translations t_z by $z \in \mathbb{Z}$, choose $s_1 = \mathrm{id}, s_2 = t_3, s_3 = t_9$ and $h(x) = x/3$. Then $f_i = hs_i$ and the assumptions of Proposition 3 are fulfilled. Since K is contained in the intervall $E = [0, 9/2]$ we have $s(E) \cap E \neq \emptyset$ for all $s \in S$. Hence $S \subseteq \{t_{-4}, \dots, t_4\}$. Thus starting with $Q = \{\mathrm{id}\}$ we get by (18)

$$\begin{aligned} z(\{\mathrm{id}\},n) &= z(\{\mathrm{id}\},n-1) + z(\{t_3\},n-1) + z(\{t_9\},n-1) - z(\{\mathrm{id},t_3\},n-1) \\ &\quad - z(\{\mathrm{id},t_9\},n-1) - z(\{t_3,t_9\},n-1) + z(\{\mathrm{id},t_3,t_9\},n-1) \\ &= 3\ z(\{\mathrm{id}\},n-1) - z(\{\mathrm{id},t_3\},n-1). \end{aligned}$$

With some more effort we get $z(\{\mathrm{id},t_3\},n) = z(\{\mathrm{id}\},n-1)$. Consequently we obtain for $n \geq 2$ the recurrence relation $\sharp G_n - 3\ \sharp G_{n-1} + \sharp G_{n-2} = 0$ which belongs to the characteristic polynomial of the matrix $A = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$. Since $\sharp G_0 = 1$ and $\sharp G_1 = 3$ we get

$$\sharp G_n = \frac{5+3\sqrt{5}}{10}\left(\frac{3+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{3-\sqrt{5}}{2}\right)^n.$$

Consequently by (17) $\dim K = \beta = \bigl(\log(3+\sqrt{5}) - \log 2\bigr) / \log 3 \approx 0.876036$.

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