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ON VOICULESCU'S DOUBLE COMMUTANT THEOREM

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ABSTRACT. For a separable infinite-dimensional Hilbert space H, we consider the full algebra of bounded linear transformations $B(H)$ and the unique non-trivial norm-closed two-sided ideal of compact operators $\mathcal{K}$. We also consider the quotient C^*-algebra $\mathcal{C} = B(H)/\mathcal{K}$ with quotient map

$$\pi\colon B(H) \to \mathcal{C}.$$

For $\mathcal{A}$ any C^*-subalgebra of $\mathcal{C}$, the relative commutant is given by $\mathcal{A}' = \{C \in \mathcal{C}\colon CA = AC \text{ for all } A \text{ in } \mathcal{A}\}$. It was shown by D. Voiculescu that, for $\mathcal{A}$ any *separable* unital C^*-subalgebra of $\mathcal{C}$,

(VDCT) $$\mathcal{A}'' = \mathcal{A}.$$

In this note, we exhibit a *non-separable* unital C^*-subalgebra $\mathcal{A}_0$ of $\mathcal{C}$ for which (VDCT) fails.

1. Introduction

For a separable infinite-dimensional Hilbert space H, we consider the full algebra of bounded linear transformations $B(H)$ and the unique non-trivial norm-closed two-sided ideal of compact operators $\mathcal{K}$. We also consider the quotient C^*-algebra $\mathcal{C} = B(H)/\mathcal{K}$ with quotient map

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For $\mathcal{A}$ any C^*-subalgebra of $\mathcal{C}$, the relative commutant is given by $\mathcal{A}' = \{C \in \mathcal{C}\colon CA = AC \text{ for all } A \text{ in } \mathcal{A}\}$. It was shown by D. Voiculescu in [7] that, for $\mathcal{A}$ any *separable* unital C^*-subalgebra of $\mathcal{C}$,

(VDCT) $$\mathcal{A}'' = \mathcal{A}.$$

In this note, we exhibit a *non-separable* unital C^*-subalgebra $\mathcal{A}_0$ of $\mathcal{C}$ for which (VDCT) fails.

The construction of $\mathcal{A}_0$ involves Berezin-Toeplitz operators on the Segal-Bargmann Hilbert space of Gaussian square-integrable entire functions on the complex plane **C**. The necessary analysis was done in [1, 2] and we simply put the pieces together here in order to construct the desired example.

The analytic preliminaries are discussed in §2. In §3, we exhibit the algebra $\mathcal{A}_0$ and show that (VDCT) fails. In §4, we discuss examples of non-separable C^*-subalgebras of $\mathcal{C}$ for which (VDCT) holds. In §5, there are some additional remarks.

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2. Preliminary results

As customary, for $\mathcal{B}$ a subalgebra of $B(H)$, we write $\mathcal{B}' = \{T \in B(H)\colon TB = BT$ for all B in $\mathcal{B}\}$. We will consider the particular Hilbert space

$$H^2 = H^2(\mathbf{C}^n, d\mu)$$

where $d\mu(z) = (2\pi)^{-n} e^{-|z|^2/2}\, dv(z)$ is Gaussian measure ($dv(z)$ is ordinary Lebesgue measure on $\mathbf{C}^n$) and H^2 consists of all the $d\mu$ square-integrable entire functions. This space is a Bergman space, with reproducing kernel functions $e^{z\cdot a/2}$. Here,

$$z \cdot a = z_1\overline{a}_1 + \cdots + z_n\overline{a}_n$$

for $z = (z_1, \ldots, z_n)$, $a = (a_1, \ldots, a_n)$ in $\mathbf{C}^n$ and

$$|z|^2 = z \cdot z.$$

The kernel functions have the property that

$$h(a) = \langle h, e^{z\cdot a/2}\rangle \equiv \int h(z) e^{a\cdot z/2} d\mu(z)$$

for all h in H^2 and a in $\mathbf{C}^n$. H^2 is a closed subspace of $L^2 = L^2(\mathbf{C}^n, d\mu)$ and the orthogonal projection from L^2 onto H^2 is given by

$$(Pg)(a) = \langle g, e^{z\cdot a/2}\rangle$$

for all g in L^2.

For f in $L^\infty(\mathbf{C}^n)$, the full algebra of bounded measurable functions, we can define a bounded Berezin-Toeplitz operator on H^2 by

$$(T_f h)(a) = P(fh)(a) = \langle f(z)h(z), e^{z\cdot a/2}\rangle.$$

In [1, 2], a detailed study of these operators was carried out. In this connection, two C^*-subalgebras of L^∞ are especially noteworthy: the algebras AP and ESV. AP consists of uniform limits of finite linear combinations of characters

$$\chi_a(z) = e^{i\mathrm{Im}(z\cdot a)}$$

($\mathrm{Im}(z \cdot a) = (z \cdot a - a \cdot z)/2i$) while ESV consists of all f in L^∞ for which (ignoring sets of measure zero)

$$(*) \qquad \operatorname*{Lim}_{R\to\infty} \sup_{\{z\colon |z|\geq R\}} \sup_{\{w\colon |z-w|\leq 1\}} \{|f(z) - f(w)|\} = 0.$$

The condition $(*)$ is uniformly closed and says that the function f is "slowly varying at infinity". The algebras AP and ESV have only the constant functions in common. ESV contains, for example, all functions

$$\hat{f}(z) = f(z/|z|), \qquad z \neq 0,$$

where f is continuous on the unit sphere S^{2n-1}.

We denote by $\tau(AP)$, $\tau(ESV)$, $\tau(L^\infty)$ the C^*-algebras on H^2 generated, respectively, by all Berezin-Toeplitz operators T_f with f in AP, ESV, L^∞. The algebra $\tau\{AP(\mathbf{C}^n)\}$ was shown in [1] to be exactly the "canonical commutation relation" algebra $CCR(\mathbf{C}^n)$ described in [3, pp. 19–22]. It follows that $\tau(AP)$ is a *simple* C^*-algebra.

In the discussion which follows, we will need to recall that an element A of $B(H)$ is *Fredholm* if and only if $\pi(A)$ is invertible in $\mathcal{C}$. Since $\tau(AP)$ is *simple*, π restricted to $\tau(AP)$ must be a $*$-isomorphism and it follows easily that the only Fredholm elements in $\tau(AP)$ must be *invertible*. For $\tau\{ESV(\mathbf{C})\}$, on the other hand, it was checked in [1] that, for

$$\theta(z) = \begin{cases} z, & |z| \leq 1, \\ z/|z|, & |z| \geq 1, \end{cases}$$

T_θ is Fredholm with

$$\text{index}(T_\theta) = -1 = \dim \ker(T_\theta) - \dim\ \text{coker}(T_\theta).$$

It follows from standard operator theory that T_θ is neither invertible nor, even, a compact perturbation of an invertible. In fact, $T_\theta + K$ is Fredholm with

$$\text{index}(T_\theta + K) = -1$$

for all K in $\mathcal{K}$.

3. Main result

We can now demonstrate the failure of (VDCT) in the non-separable case.

Theorem. (VDCT) *fails for* $\mathcal{A}_0 = \pi\tau\{AP(\mathbf{C})\}$.

Proof. By [2, Theorem D],

$$\{\pi\tau(AP)\}' = \pi\tau(ESV).$$

Moreover, by [2, Proposition A and Theorem B],

$$\pi\tau(L^\infty) \subset \{\pi\tau(ESV)\}'.$$

It follows that

$$\pi\tau(AP) \subset \pi\tau(L^\infty) \subset \{\pi\tau(AP)\}''.$$

To show that (VDCT) fails, we need only check that

$$\pi\tau(AP) \neq \pi\tau(L^\infty).$$

For $n = 1$ ($\mathbf{C}^n = \mathbf{C}$), this is easy. Suppose that

$$\pi\tau\{AP(\mathbf{C})\} = \pi\tau\{L^\infty(\mathbf{C})\}.$$

Then, since $\mathcal{K} \subset \tau\{L^\infty(\mathbf{C})\}$ [2, Theorem 16], we must have

$$\tau\{L^\infty(\mathbf{C})\} = \tau\{AP(\mathbf{C})\} + \mathcal{K}.$$

But, by the discussion at the end of §2, T_θ is in $\tau\{L^\infty(\mathbf{C})\}$ and is Fredholm with $\text{index}(T_\theta) = -1$ while

$$T_\theta = A_\theta + K_\theta$$

for some A_θ in $\tau\{AP(\mathbf{C})\}$ and K_θ in $\mathcal{K}$. It follows that $A_\theta = T_\theta - K_\theta$ must be Fredholm with

$$\text{index}(A_\theta) = \text{index}(T_\theta - K_\theta) = -1.$$

This is contradicted by the observation that A_θ is invertible.

Remark. In fact, (VDCT) fails for $\pi\tau\{AP(\mathbf{C}^n)\}$ for all n. One needs to consider $n \times n$ systems as in [4] to show that an index obstruction yields

$$\tau\{L^\infty(\mathbf{C}^n)\} \otimes M_n \neq \tau\{AP(\mathbf{C}^n)\} \otimes M_n + \mathcal{K} \otimes M_n.$$

4. Positive results

It turns out that (VDCT) holds for many non-separable C^*-subalgebras in $\mathcal{C}$. Following up earlier work of [5], it was shown in [6] that, for $\mathcal{B}$ *any* von Neumann algebra in $B(H)$,

$$\pi(\mathcal{B})' = \pi(\mathcal{B}')$$

with the evident corollary (since $\mathcal{B} = \mathcal{B}''$) that

$$\pi(\mathcal{B})'' = \pi(\mathcal{B}')' = \pi(\mathcal{B}'') = \pi(\mathcal{B}).$$

We do not know of any other large class of non-separable C^*-subalgebras of $\mathcal{C}$ for which (VDCT) holds.

5. Additional remarks

The proof of our main result provides some additional interesting information. Since $\pi\tau(L^\infty)$ is contained in $\{\pi\tau(ESV)\}'$ it is clear that $\pi\tau(ESV)$ is in $\{\pi\tau(L^\infty)\}'$. Since $\pi\tau(AP)$ is contained in $\pi\tau(L^\infty)$, we have

$$\{\pi\tau(L^\infty)\}' \subset \{\pi\tau(AP)\}'.$$

Recalling that $\{\pi\tau(AP)\}' = \pi\tau(ESV)$, we finally get

$$\{\pi\tau(L^\infty)\}' \subset \{\pi\tau(AP)\}' = \pi\tau(ESV) \subset \{\pi\tau(L^\infty)\}'$$

so that

$$\{\pi\tau(L^\infty)\}' = \{\pi\tau(AP)\}' = \pi\tau(ESV).$$

Thus, we have exhibited two *distinct* unital C^*-subalgebras of $\mathcal{C}$ with the same relative commutant.

Moreover, suppose $\{\pi\tau(AP)\}'' = \mathcal{C}$. Since $\mathcal{C}' = \mathbf{C}1$ by the remarks in §4, we must have

$$\pi\tau(ESV) = \{\pi\tau(AP)\}' = \mathbf{C}1.$$

This conclusion is clearly false by [2, Theorem E]. It follows, since

$$\pi\tau(L^\infty) \subset \{\pi\tau(ESV)\}' = \{\pi\tau(AP)\}'',$$

that $\pi\{\tau(L^\infty)\} \neq \mathcal{C}$ and so $\tau(L^\infty) \neq B(H^2)$.

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