ON VOICULESCU’S DOUBLE COMMUTANT THEOREM

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Abstract. For a separable infinite-dimensional Hilbert space $H$, we consider the full algebra of bounded linear transformations $B(H)$ and the unique non-trivial norm-closed two-sided ideal of compact operators $\mathcal{K}$. We also consider the quotient $C^*$-algebra $\mathcal{C} = B(H)/\mathcal{K}$ with quotient map

$$\pi: B(H) \to \mathcal{C}.$$ 

For $A$ any $C^*$-subalgebra of $\mathcal{C}$, the relative commutant is given by $A' = \{ C \in \mathcal{C} : CA = AC \text{ for all } A \in A \}$. It was shown by D. Voiculescu in [7] that, for $A$ any separable unital $C^*$-subalgebra of $\mathcal{C}$,

$$(\text{VDCT}) \quad A'' = A.$$ 

In this note, we exhibit a non-separable unital $C^*$-subalgebra $A_0$ of $\mathcal{C}$ for which (VDCT) fails.

1. Introduction

For a separable infinite-dimensional Hilbert space $H$, we consider the full algebra of bounded linear transformations $B(H)$ and the unique non-trivial norm-closed two-sided ideal of compact operators $\mathcal{K}$. We also consider the quotient $C^*$-algebra $\mathcal{C} = B(H)/\mathcal{K}$ with quotient map

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For $A$ any $C^*$-subalgebra of $\mathcal{C}$, the relative commutant is given by $A' = \{ C \in \mathcal{C} : CA = AC \text{ for all } A \in A \}$. It was shown by D. Voiculescu in [7] that, for $A$ any separable unital $C^*$-subalgebra of $\mathcal{C}$,

$$(\text{VDCT}) \quad A'' = A.$$ 

In this note, we exhibit a non-separable unital $C^*$-subalgebra $A_0$ of $\mathcal{C}$ for which (VDCT) fails.

The construction of $A_0$ involves Berezin-Toeplitz operators on the Segal-Bargmann Hilbert space of Gaussian square-integrable entire functions on the complex plane $\mathbb{C}$. The necessary analysis was done in [1, 2] and we simply put the pieces together here in order to construct the desired example.

The analytic preliminaries are discussed in §2. In §3, we exhibit the algebra $A_0$ and show that (VDCT) fails. In §4, we discuss examples of non-separable $C^*$-subalgebras of $\mathcal{C}$ for which (VDCT) holds. In §5, there are some additional remarks.

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2. Preliminary results

As customary, for $\mathcal{B}$ a subalgebra of $B(H)$, we write $\mathcal{B}' = \{ T \in B(H) : TB = BT \}$ for all $B$ in $\mathcal{B}$. We will consider the particular Hilbert space

$$H^2 = H^2(\mathbb{C}^n, d\mu)$$

where $d\mu(z) = (2\pi)^{-n}e^{-|z|^2/2} dv(z)$ is Gaussian measure ($dv(z)$ is ordinary Lebesgue measure on $\mathbb{C}^n$) and $H^2$ consists of all the $d\mu$ square-integrable entire functions. This space is a Bergman space, with reproducing kernel functions $e^{z \cdot \bar{a}/2}$. Here,

$$z \cdot a = z_1\bar{a}_1 + \cdots + z_n\bar{a}_n$$

for $z = (z_1, \ldots, z_n)$, $a = (a_1, \ldots, a_n)$ in $\mathbb{C}^n$ and

$$|z|^2 = z \cdot z.$$ 

The kernel functions have the property that

$$h(a) = \langle h, e^{z \cdot \bar{a}/2} \rangle \equiv \int h(z)e^{\bar{a} \cdot z/2} d\mu(z)$$

for all $h$ in $H^2$ and $a$ in $\mathbb{C}^n$. $H^2$ is a closed subspace of $L^2 = L^2(\mathbb{C}^n, d\mu)$ and the orthogonal projection from $L^2$ onto $H^2$ is given by

$$(Pg)(a) = \langle g, e^{z \cdot \bar{a}/2} \rangle$$

for all $g$ in $L^2$.

For $f$ in $L^\infty(\mathbb{C}^n)$, the full algebra of bounded measurable functions, we can define a bounded Berezin-Toeplitz operator on $H^2$ by

$$(T_f h)(a) = P(fh)(a) = \langle f(z)h(z), e^{z \cdot \bar{a}/2} \rangle.$$ 

In [1, 2], a detailed study of these operators was carried out. In this connection, two $C^*$-subalgebras of $L^\infty$ are especially noteworthy: the algebras $AP$ and $ESV$. $AP$ consists of uniform limits of finite linear combinations of characters

$$\chi_a(z) = e^{\text{Im}(z \cdot a)}$$

($\text{Im}(z \cdot a) = (z \cdot a - a \cdot z)/2i$) while $ESV$ consists of all $f$ in $L^\infty$ for which (ignoring sets of measure zero)

$$(*) \quad \lim_{R \to \infty} \sup_{\{ z : |z| \geq R \}} \sup_{\{ w : |z-w| \leq 1 \}} \{|f(z) - f(w)|\} = 0.$$

The condition $(*)$ is uniformly closed and says that the function $f$ is “slowly varying at infinity”. The algebras $AP$ and $ESV$ have only the constant functions in common. $ESV$ contains, for example, all functions

$$\hat{f}(z) = f(z/|z|), \quad z \neq 0,$$

where $f$ is continuous on the unit sphere $S^{2n-1}$.

We denote by $\tau(AP)$, $\tau(ESV)$, $\tau(L^\infty)$ the $C^*$-algebras on $H^2$ generated, respectively, by all Berezin-Toeplitz operators $T_f$ with $f$ in $AP$, $ESV$, $L^\infty$. The algebra $\tau\{AP(\mathbb{C}^n)\}$ was shown in [1] to be exactly the “canonical commutation relation” algebra $CCR(\mathbb{C}^n)$ described in [3, pp. 19–22]. It follows that $\tau(AP)$ is a simple $C^*$-algebra.

In the discussion which follows, we will need to recall that an element $A$ of $B(H)$ is Fredholm if and only if $\pi(A)$ is invertible in $\mathcal{C}$. Since $\tau(AP)$ is simple, $\pi$ restricted
to $\tau(\mathcal{A}P)$ must be a $*$-isomorphism and it follows easily that the only Fredholm elements in $\tau(\mathcal{A}P)$ must be invertible. For $\tau\{\text{ESV}(\mathbb{C})\}$, on the other hand, it was checked in [1] that, for

$$\theta(z) = \begin{cases} z, & |z| \leq 1, \\ z/|z|, & |z| \geq 1, \end{cases}$$

$T_\theta$ is Fredholm with

$$\text{index}(T_\theta) = -1 = \dim \ker(T_\theta) - \dim \text{coker}(T_\theta).$$

It follows from standard operator theory that $T_\theta$ is neither invertible nor, even, a compact perturbation of an invertible. In fact, $T_\theta + K$ is Fredholm with

$$\text{index}(T_\theta + K) = -1$$
for all $K$ in $\mathcal{K}$.

3. Main result

We can now demonstrate the failure of (VDCT) in the non-separable case.

**Theorem.** (VDCT) fails for $A_0 = \pi\tau\{\mathcal{A}P(\mathbb{C})\}$.

**Proof.** By [2, Theorem D],

$$\{\pi\tau(\mathcal{A}P)\}' = \pi\tau(\text{ESV}).$$

Moreover, by [2, Proposition A and Theorem B],

$$\pi\tau(L^\infty) \subset \{\pi\tau(\text{ESV})\}'. $$

It follows that

$$\pi\tau(\mathcal{A}P) \subset \pi\tau(L^\infty) \subset \{\pi\tau(\mathcal{A}P)\}''.$$ 

To show that (VDCT) fails, we need only check that

$$\pi\tau(\mathcal{A}P) \neq \pi\tau(L^\infty).$$

For $n = 1$ ($\mathbb{C}^n = \mathbb{C}$), this is easy. Suppose that

$$\pi\tau(\mathcal{A}P(\mathbb{C})) = \pi\tau(L^\infty(\mathbb{C})).$$

Then, since $\mathcal{K} \subset \tau\{L^\infty(\mathbb{C})\}$ [2, Theorem 16], we must have

$$\tau\{L^\infty(\mathbb{C})\} = \tau\{\mathcal{A}P(\mathbb{C})\} + \mathcal{K}.$$ 

But, by the discussion at the end of §2, $T_\theta$ is in $\tau\{L^\infty(\mathbb{C})\}$ and is Fredholm with

$$\text{index}(T_\theta) = -1$$
while

$$T_\theta = A_\theta + K_\theta$$
for some $A_\theta$ in $\tau\{\mathcal{A}P(\mathbb{C})\}$ and $K_\theta$ in $\mathcal{K}$. It follows that $A_\theta = T_\theta - K_\theta$ must be Fredholm with

$$\text{index}(A_\theta) = \text{index}(T_\theta - K_\theta) = -1.$$

This is contradicted by the observation that $A_\theta$ is invertible.

**Remark.** In fact, (VDCT) fails for $\pi\tau(\mathcal{A}P(\mathbb{C}^n))$ for all $n$. One needs to consider $n \times n$ systems as in [4] to show that an index obstruction yields

$$\tau\{L^\infty(\mathbb{C}^n)\} \otimes M_n \neq \tau\{\mathcal{A}P(\mathbb{C}^n)\} \otimes M_n + \mathcal{K} \otimes M_n.$$
4. Positive results

It turns out that (VDCT) holds for many non-separable $C^*$-subalgebras in $\mathcal{C}$. Following up earlier work of [5], it was shown in [6] that, for $\mathcal{B}$ any von Neumann algebra in $B(H)$,

$$\pi(\mathcal{B})' = \pi(\mathcal{B}')$$

with the evident corollary (since $\mathcal{B} = \mathcal{B}'$) that

$$\pi(\mathcal{B})'' = \pi(\mathcal{B}')' = \pi(\mathcal{B}'') = \pi(\mathcal{B}).$$

We do not know of any other large class of non-separable $C^*$-subalgebras of $\mathcal{C}$ for which (VDCT) holds.

5. Additional remarks

The proof of our main result provides some additional interesting information. Since $\pi\tau(L^\infty)$ is contained in $\{\pi\tau(ESV)\}'$ it is clear that $\pi\tau(ESV)$ is in $\{\pi\tau(L^\infty)\}'$.

Since $\pi\tau(AP)$ is contained in $\pi\tau(L^\infty)$, we have

$$\{\pi\tau(L^\infty)\}' \subset \{\pi\tau(AP)\}'.
$$

Recalling that $\{\pi\tau(AP)\}' = \pi\tau(ESV)$, we finally get

$$\{\pi\tau(L^\infty)\}' \subset \{\pi\tau(AP)\}' = \pi\tau(ESV) \subset \{\pi\tau(L^\infty)\}'$$

so that

$$\{\pi\tau(L^\infty)\}' = \{\pi\tau(AP)\}' = \pi\tau(ESV).$$

Thus, we have exhibited two distinct unital $C^*$-subalgebras of $\mathcal{C}$ with the same relative commutant.

Moreover, suppose $\{\pi\tau(AP)\}'' = \mathcal{C}$. Since $\mathcal{C}' = \mathcal{C}1$ by the remarks in §4, we must have

$$\pi\tau(ESV) = \{\pi\tau(AP)\}' = \mathcal{C}1.$$ 

This conclusion is clearly false by [2, Theorem E]. It follows, since

$$\pi\tau(L^\infty) \subset \{\pi\tau(ESV)\}' = \{\pi\tau(AP)\}''$$

that $\pi\{\tau(L^\infty)\} \neq \mathcal{C}$ and so $\tau(L^\infty) \neq B(H^2)$.

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