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SELF-SIMILARITY IN INVERSE LIMIT SPACES OF THE TENT FAMILY

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ABSTRACT. Taking inverse limits of the one-parameter family of tent maps of the interval generates a one-parameter family of inverse limit spaces. We prove that, for a dense set of parameters, these spaces are locally, at most points, the product of a Cantor set and an arc. On the other hand, we show that there is a dense G_δ set of parameters for which the corresponding space has the property that each neighborhood in the space contains homeomorphic copies of every inverse limit of a tent map.

In 1967, R. F. Williams ([10]) proved that hyperbolic one-dimensional attractors are inverse limits of maps on branched one-manifolds. These attractors have the solenoid-like property of being everywhere locally homeomorphic with the product of a Cantor set and an arc. Also, for dissipation parameter near zero, most of the full attracting sets for maps in the Hénon family are homeomorphic with inverse limits of unimodal maps of the interval ([1]). Except at finitely many points (the points of a stable periodic orbit), these sets are locally homeomorphic with the product of a Cantor set and an arc (see the comment following Theorem 1).

Computer-generated pictures, at first glance, suggest that other one-dimensional (but non-hyperbolic) attractors might have a similar local structure. In particular, the transitive Hénon attractors appear to be, at most points, locally the product of a Cantor set and an arc. However, 'blowing up' computer pictures of these attractors usually indicates the presence of 'hooks' in the midst of regions that, under less scrutiny, look like a Cantor set of nearly parallel arcs.

In this paper we consider the local topological properties of a one-parameter family of conceptual models for the Hénon attractors, inverse limits of tent maps. We find the following: for a dense set of parameters, the inverse limit space is, except at finitely many points, the product of a Cantor set and an arc (Theorem 1). However, for a dense G_δ set of parameters, the inverse limit space is nowhere locally homeomorphic with the product of a Cantor set and an arc. In this second case, the inverse limit spaces display a remarkable form of self-similarity and local recapitulation of the entire family: not only does every open set in each space contain a homeomorphic copy of the entire space, each open set also contains a homeomorphic copy of every other inverse limit space appearing in the tent family (Corollary 6). In a forthcoming paper, we prove that the set of parameters for which this holds has full measure.

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We introduce terminology, notation, and preliminary results as needed for the main results.

Suppose that $\{X_i\}_{i=1}^{\infty}$ is a collection of compact metric spaces and that for each i, $f_i : X_{i+1} \to X_i$ is a continuous map. The *inverse limit space* of $\{X_i, f_i\}_{i=1}^{\infty}$ (or simply, of $\{f_i\}_{i=1}^{\infty}$) is

$$\{\underline{x} = (x_0, x_1, \dots) \mid \underline{x} \in \prod_{i=1}^{\infty} X_i, f_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$$

and has metric $\underline{d}$ given by

$$\underline{d}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

where for each i, d_i is a metric for X_i bounded by 1. For each i, Π_i will denote the restriction of the usual projection map from $\prod_{i=1}^{\infty} X_i$ into X_i to the inverse limit space defined above. If $X_i = X$ and $f_i = f$ for all i, the inverse limit space is denoted by (X, f), and the map $\hat{f} : (X, f) \to (X, f)$ defined by $\hat{f}((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots)$ is called the *induced homeomorphism.*

In the following, I will denote the unit interval $[0, 1]$. For $\lambda \in [1, 2]$, define the following families of maps: the family of tent maps $T_\lambda : I \to I$ is defined by

$$T_\lambda(x) = \begin{cases} \lambda x, & 0 \ \le x \le 0.5, \\ \lambda(1 - x), & 0.5 \ \le x \le 1; \end{cases}$$

if T_λ restricted to $[0, T_\lambda(0.5)]$ is rescaled to $h_\lambda : I \to I$, then h_λ is given by

$$h_\lambda(x) = \begin{cases} \lambda x, & 0 \ \le x \le \frac{1}{\lambda}, \\ 2 - \lambda x, & \frac{1}{\lambda} \ \le x \le 1; \end{cases}$$

and the rescaling of h_λ restricted to $[h_\lambda(1), 1]$ to $f_\lambda : I \to I$ (or the *core* of the corresponding tent map T_λ) is given by

$$f_\lambda(x) = \begin{cases} \lambda x + (2 - \lambda), & 0 \ \le x \le \frac{\lambda-1}{\lambda}, \\ -\lambda x + \lambda, & \frac{\lambda-1}{\lambda} \ \le x \le 1. \end{cases}$$

In the following, c_λ will denote the critical point $\frac{\lambda-1}{\lambda}$ of f_λ. A parameter value $\lambda \in [1, 2]$ is *periodic* if c_λ is periodic under f_λ, and *prefixed* if c_λ maps to a fixed point under some iterate of f_λ. The periodic parameters are dense in $[1, 2]$, and the prefixed parameters are dense in $[\sqrt{2}, 2]$ (see, for example, [7] or [5]).

Since $(I, f) = (J, f|_J)$ where $J = \bigcap_{n\ge0} f^n(I)$, (I, T_λ) is homeomorphic with (I, h_λ), while (I, h_λ) consists exactly of a homeomorphic copy of (I, f_λ) and an infinite ray entwined with (I, f_λ).

The first result indicates that the inverse limit space (I, f_λ) is locally well-behaved if the orbit of the critical point c_λ is finite (thus for a dense set of parameter values).

Theorem 1. *Let $\sqrt{2} \le \lambda \le 2$, and suppose that $\{f_\lambda^n(c_\lambda)\}_{n\in\mathbb{N}}$ is finite. Then all but finitely many points of (I, f_λ) have a neighbourhood that is homeomorphic with the product of a Cantor set and an arc.*

Proof. Let I_i, $1 \le i \le k$, be the closures of the components of $I \setminus \{f_\lambda^n(c_\lambda)\}_{n\in\mathbb{N}}$. The I_i's form a Markov partition for f_λ. That is, for each i, $f_\lambda|_{I_i}$ is one-to-one and $f_\lambda(I_i)$ is a union of a subcollection of $\{I_j\}_{j\le k}$.

For each $i \leq k$, let $f_{\lambda,i}^{-1}$ denote the inverse of $f_\lambda|_{I_i}$. Suppose $\underline{x} = (x_0, x_1, \dots) \in (I, f_\lambda)$ is such that $x_N \notin \{f_\lambda^n(c_\lambda)\}_{n\in\mathbb{N}}$ for some $N \in \mathbb{N}$. Then $x_N \in int(I_j)$ for some j. Let $\Sigma = \{(a_1, a_2, \dots) : a_l \leq k$ for each $l \in \mathbb{N}$, $f_\lambda(I_{a_1}) \supseteq I_j, f_\lambda(I_{a_{l+1}}) \supseteq I_{a_l}$ for each $l\}$. Since f is transitive for $\lambda \geq \sqrt{2}$, Σ, with the product topology, is a Cantor set. The set $U = \Pi_N^{-1}(int(I_j))$ is a neighbourhood of $\underline{x}$ and $\phi : \Sigma \times int(I_j) \mapsto U$ defined by $\phi((a_1, a_2, \dots), t) = (f_\lambda^N(t), \dots, f_\lambda(t), t, f_{\lambda,a_1}^{-1}(t), f_{\lambda,a_2}^{-1}(f_{\lambda,a_1}^{-1}(t)), \dots)$ is a homeomorphism. □

Most of the full attracting sets for maps in the Hénon family are homeomorphic to inverse limits of associated quadratic maps where the critical point has a finite orbit ([1]). According to [8], if the *kneading sequence* (defined following the next result) of the quadratic map appears as a kneading sequence for a map in the tent family, the inverse limit spaces for these two maps are homeomorphic, thus the full attracting set for the Hénon map has the structure described by Theorem 1. For a large number of the quadratic maps of interest, the kneading sequence does not appear in the tent family; in this case, the techniques of [1] can be used to prove that the full attracting sets also have this local structure.

The following lemma is a consequence of [3, Theorem 3].

Lemma 2. *For $\{\lambda_n\}_{n=1}^\infty \subseteq [1, 2]$, there is a sequence $\epsilon_n > 0, n \in \mathbb{N}$, such that if $|\eta_n - \lambda_n| < \epsilon_n$, for $n \in \mathbb{N}$, then the inverse limit of*

$$[0,1] \xleftarrow{h_{\lambda_1}} [0,1] \xleftarrow{h_{\lambda_2}} [0,1] \xleftarrow{h_{\lambda_3}} \dots$$

is homeomorphic with the inverse limit of

$$[0,1] \xleftarrow{h_{\eta_1}} [0,1] \xleftarrow{h_{\eta_2}} [0,1] \xleftarrow{h_{\eta_3}} \dots \quad .$$

We need the language of unimodal maps and kneading theory for Lemma 3. A map $f : I \to I$ is *unimodal* if there is $c \in (0, 1)$ such that f is strictly increasing on $[0, c)$ and strictly decreasing on $(c, 1]$; the point c is the *critical point* of f. We recall some basic notions of kneading theory for unimodal maps; for more details see [6]. For each $x \in [0, 1]$, the *itinerary* of x under the unimodal map f is given by $I(x) = b_0b_1b_2\dots$, where $b_i = R$ if $f^i(x) > c$, $b_i = L$ if $f^i(c) < c$, and $b_i = C$ if $f^i(x) = c$, with the usual convention that the itinerary stops after the first C. The *kneading sequence* of the map f, denoted $K(f)$, is defined to be the itinerary of $f(c)$. The *parity-lexicographical ordering* is put on the set of itineraries as follows. Set $L < C < R$. Let $W = w_1w_2\dots$ and $V = v_1v_2\dots$ be two distinct itineraries and let k be the first index where the itineraries differ. If $k = 1$, then $W < V$ iff $w_1 < v_1$. If $k > 1$ and $w_1 \dots w_{k-1} = v_1 \dots v_{k-1}$ has an even number of R's, i.e., has *even parity*, then $W < V$ iff $w_k < v_k$; if $w_1 \dots w_{k-1}$ has an odd number of R's, then $W < V$ iff $v_k < w_k$. It is an elementary fact that the map $x \mapsto I(x)$ is monotone, i.e., $x < y$ implies that $I(x) \leq I(y)$ [6, Lemma II.1.3]. As $f(c)$ is the maximum value of the function, it follows that if $A = a_0a_1\dots = K(f)$, then any shift of A ($a_ja_{j+1}\dots$ for $j \geq 0$) is less than or equal to the kneading sequence itself in the parity-lexicographical order, that is, A is *shift maximal.* If $\lambda \in [1, 2]$ and f_λ is as defined above, the map $\lambda \mapsto K(f_\lambda)$ is strictly increasing (see, for example, the appendix of [4]).

Recall that c_λ is the critical point of f_λ. Define $A_{a,\delta} = \{\lambda \in [\sqrt{2}, 2] :$ there exist n and $0 < a_\lambda < b_\lambda$ such that $f_\lambda^n(0) \in (a-\delta, a+\delta), f_\lambda^n(a_\lambda) = 0, f_\lambda^n(b_\lambda) = c_\lambda$, and f_λ^n is monotone on each of $[0, a_\lambda], [a_\lambda, b_\lambda]\}$. In the proof of the next result about $A_{a,\delta}$, we make use of the notion of *skeleton maps*. For $n \geq 1$ and $\lambda \in [\sqrt{2}, 2]$, the n^{th} *skeleton map* φ_n is defined by $\varphi_n(\lambda) = T_\lambda^n(c)$, where $c = 0.5$ is the critical point of T_λ. According to [7, §5], each φ_n is continuous, piecewise polynomial, and differentiable except at periodic parameters with period less than n. Also, φ_n' is never zero, so φ_n is strictly monotone on intervals containing no parameters with period less than n.

Lemma 3. *For $a \in [0,1]$ and $\delta > 0$, $A_{a,\delta}$ contains a dense open subset of $[\sqrt{2}, 2]$.*

Proof. Since the periodic parameters are dense in $[1, 2]$, given a parameter value λ and $\delta > 0$, there are n_0 and $\lambda_0 > \lambda$ such that c_{λ_0} is periodic under f_{λ_0} of period $k_0 \geq n_0$ and if f_γ has kneading sequence agreeing with that of f_{λ_0} to $n_0 - 1$ places, then $|\lambda - \gamma| < \delta$. It follows from the fact that the prefixed parameters are dense in $[\sqrt{2}, 2]$ that λ_0 can be chosen so that $k(f_{\lambda_0}) = WC$ has even parity and the sequences WR^4C and WR^3C are shift maximal. Let γ_0 (γ_1, respectively) denote the parameter value for which the associated tent (or core) map has kneading sequence WR^4C (WR^3C, respectively). Note that $\lambda_0 < \gamma_0 < \gamma_1 < \lambda_0 + \delta$ since $WC < WR^4C < WR^3C$. Also, there is no periodic kneading sequence between WR^4C and WR^3C, which are of periods k_0+4 and k_0+3 respectively, with period less than $k_0 + 4$. That is, if $k = k_0 + 4$, then $T_{\gamma_0}^k(c) = c$, $T_{\gamma_1}^k(c) = T_{\gamma_1}(c_{\gamma_1})$, and φ_n is monotonically increasing between γ_0 and γ_1. In the language of the core map f_γ, $f_{\gamma_0}^k(c_{\gamma_0}) = c_{\gamma_0}$, $f_{\gamma_1}^k(c_{\gamma_1}) = 1$, and $f_\gamma^k(c_\gamma)$ increases from c_{γ_0} to 1 as γ increases from γ_0 to γ_1.

For each $\gamma \in [\gamma_0, \gamma_1]$, the critical point d_γ for f_γ^k closest to and larger than c_γ has itinerary $RW'C$ where W' is the initial segment of $k(f_{\gamma_0})$ of length $n_0 + 1$. That is, $f_\gamma^k(d_\gamma) = 0$, f_γ^k is monotone decreasing on $[c_\gamma, d_\gamma]$, and there is $e_\gamma \in [c_\gamma, d_\gamma)$ such that $f_\gamma^k(e_\gamma) = c_\gamma$. Let x_γ denote the nonzero fixed point for f_γ. Since $k(f_\gamma) = RLW' \ldots > RLR^\infty$, $f_\gamma(0)$ with itinerary $W' \ldots$ is less than x_γ with itinerary R^∞. Then there is $y_\gamma \in [e_\gamma, d_\gamma]$ such that $f_\gamma^{k+1}(y_\gamma) = x_\gamma$. It follows that as γ increases from γ_0, $f_\gamma^{k+2}(c_\gamma)$ increases from 0 to 1, $f_\gamma^{k+2}(e_\gamma) = 0$, $f_\gamma^{k+2}(y_\gamma) = x_\gamma$, and f_γ^{k+2} is monotone on each of $[c_\gamma, e_\gamma]$ and $[e_\gamma, y_\gamma]$. For some $\gamma\prime \in [\gamma_0, \gamma_1]$, $f_\gamma^{k+2}(c_\gamma) \in (a-\delta, a+\delta)$. Let $a_\gamma = f_\lambda^2(e_\gamma)$ and $z_\gamma = f_\lambda^2(y_\gamma)$. Then $f_{\gamma\prime}^k(0) \in (a-\delta, a+\delta)$, $f_{\gamma\prime}^k(a_\gamma) = 0$, $f_{\gamma\prime}^k(z_\gamma) = x_\gamma$, and f is monotone on each of $[0, a_\lambda], [a_\lambda, z_\lambda]$. $\square$

If $A = \bigcap_{a,\delta \in Q \cap [0,1]} A_{a,\delta}$, then A contains a dense G_δ. If $\lambda \in A$, then for any $a \in [0,1]$ and $\delta > 0$, there are n and $0 < a_\lambda < b_\lambda$ such that f_λ^n is monotone on each of $[0, a_\lambda], [a_\lambda, b_\lambda]$, $f_\lambda^n(0) \in (a-\delta, a+\delta)$, $f_\lambda^n(a_\lambda) = 0$, and $f_\lambda^n(b_\lambda) = c_\lambda$. In particular, if $\eta \in A$, $\{f_\eta^k(c)\}_{k=1}^\infty$ is dense in $[0,1]$.

For $1 < \eta \leq \sqrt{2}$, let $J_0 = [f_\eta^2(1), 1]$ and $J_1 = f_\eta(J_0)$. It is easily checked that the maps $f_\eta^2|_{J_i}$, $i = 0, 1$ are conjugate (via affine homeomorphisms) with f_{η^2}. It follows that if $1 < \eta \leq \sqrt{2}$ and $\eta^{2^k} \leq 2$, then $f_\eta^{2^k}|_{J_i}$ is conjugate with $f_{\eta^{2^k}}$, where $J_0 = [f_{\eta^{2^k}}(1), 1]$ and $J_i = f_\eta^i(J_0)$, $1 \leq i \leq 2^{k-1}$. Thus, for such η, (I, f_η) contains 2^k homeomorphic copies of $(I, f_{\eta^{2^k}})$. In light of this, the set A in Theorems 4, 5, and 9 and Corollary 6 can be replaced by $A' = \{\eta \in [1,2] : \eta^{2^k} \in A$ for some $k \in \mathbb{N}\}$. Note that A' contains a dense G_δ subset of $[1,2]$.

Theorem 4. *Suppose that $\eta \in A$, and let $\lambda \in (1,2]$. Given any subcontinuum C of (I, f_λ), there is a subcontinuum of (I, f_η) homeomorphic with C.*

Proof. Let $\epsilon_n > 0, n \in \mathbb{N}$, be as in Lemma 2 with $\lambda_n = \lambda$ for all $n \in \mathbb{N}$. Let $b_1 = c_\lambda$, and choose $a_1 \in (0, b_1)$ such that $|1 + \frac{a_1}{b_1} - \lambda| < \epsilon_1$. There is $\delta_1 > 0$ such that $a_1 + \delta_1 < b_1$ and if $|a - a_1| < \delta_1$, then $|1 + \frac{a}{b_1} - \lambda| < \epsilon_1$. Let n_1 and b_2 be such that $|f_\eta^{n_1}(0) - a_1| < \delta_1$, $f_\eta^{n_1}(b_2) = b_1$ and $f_\eta^{n_1}$ has precisely one critical point in $(0, b_2)$ which maps to 0 under $f_\eta^{n_1}$. Now let $a_2 \in (0, b_2)$ be such that $|1 + \frac{a_2}{b_2} - \lambda| < \epsilon_2$, and choose δ_2 small enough so that $a_2 + \delta_2 < b_2$ and if $|a - a_2| < \delta_2$, then $|1 + \frac{a}{b_2} - \lambda| < \epsilon_2$. Continuing, n_2 and b_3 can be chosen such that $|f_\eta^{n_2}(0) - a_2| < \delta_2$, $f_\eta^{n_2}(b_3) = b_2$ and $f_\eta^{n_2}$ has precisely one critical point in $(0, b_3)$ which maps to 0 under $f_\eta^{n_2}$. We prove that if b_k and n_k are defined in this manner, then the inverse limit of

$$[0, b_1] \xleftarrow{f_\eta^{n_1}} [0, b_2] \xleftarrow{f_\eta^{n_2}} [0, b_3] \xleftarrow{f_\eta^{n_3}} \cdots$$

is homeomorphic with (I, h_λ).

For $b > 0$, define $\psi_b : [0,b] \to [0,1]$ by $\psi_b(x) = 1 - x/b$. For $k \in \mathbb{N}$, let $g_k = \psi_{b_k} \circ f_\eta^{n_k} \circ \psi_{b_{k+1}}^{-1}$. The commuting diagram

$$\begin{tikzcd}
{[0,1]} & {[0,1]} \arrow[l, "g_1"'] & {[0,1]} \arrow[l, "g_2"'] & \cdots \arrow[l, "g_3"'] \\
{[0,b_1]} \arrow[u, "\psi_{b_1}"] & {[0,b_2]} \arrow[l, "f_\eta^{n_1}"'] \arrow[u, "\psi_{b_2}"] & {[0,b_3]} \arrow[l, "f_\eta^{n_2}"'] \arrow[u, "\psi_{b_3}"] & \cdots \arrow[l, "f_\eta^{n_3}"']
\end{tikzcd}$$

shows that the inverse limit of $\{g_k\}_{k=1}^\infty$ is homeomorphic to the inverse limit of $\{f_\eta^{n_k}|_{[0,b_{k+1}]}\}_{k=1}^\infty$.

Claim. For each k there is λ_k such that $g_k = h_{\lambda_k}$ and $|\lambda_k - \lambda| < \epsilon_k$.

Proof of claim. It is easy to check that g_k is a piecewise linear unimodal map with the critical point mapping to a maximum at 1 and the magnitude of the slopes of the linear pieces equal, say, to λ_k. Also, $g(0) = 0$, so $g_k = h_{\lambda_k}$. Let $m = \eta^{n_k}$ (the magnitude of the slopes of the linear pieces of $f_\eta^{n_k}$) and $a = f_\eta^{n_k}(0)$. Then $m = \frac{a+b_k}{b_{k+1}}$, so $\lambda_k = b_{k+1}(m)\frac{1}{b_k} = \frac{a+b_k}{b_k} = 1 + \frac{a}{b_k}$. Since $|a - a_k| < \delta_k$, $|\lambda_k - \lambda| = |1 + \frac{a}{b_k} - \lambda| < \epsilon_k$. The claim is proved.

It follows that the inverse limit of $\{f_\eta^{n_k}|_{[0,b_{k+1}]}\}_{k=1}^\infty$ is homeomorphic with the inverse limit of $\{h_{\lambda_k}\}_{k=1}^\infty$, which is homeomorphic with (I, h_λ) by Lemma 2.

Define X to be the following subcontinuum of (I, f_η):

$$\underline{x} \in X \leftrightarrow \Pi_{N_i}(\underline{x}) \in [0, b_{i+1}], i = 0, 1, 2, \ldots,$$

where $N_0 = 0$, $N_i = \Sigma_{k=1}^i n_k$ for $i \in \mathbb{N}$. Then X is homeomorphic with the inverse limit of $\{f_\eta^{n_k}|_{[0,b_{k+1}]}\}_{k=1}^\infty$, hence with (I, h_λ). Since every subcontinuum of (I, f_λ) is homeomorphic with a subcontinuum of (I, h_λ), the theorem is proved. $\square$

In fact, the bonding map can be varied in Theorem 4.

Theorem 5. *Given $\eta \in A$, $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq [1,2]$ and any subcontinuum C of the inverse limit of $\{f_{\lambda_n}\}_{n=1}^\infty$, there is a subcontinuum of (I, f_η) homeomorphic with C.*

Proof. First we show that given $\{\lambda_n\}_{n=1}^{\infty} \subseteq [1,2]$, there is $\{\eta_n\}_{n=1}^{\infty} \subseteq (1,2]$ such that the inverse limit of $\{f_{\lambda_n}\}_{n=1}^{\infty}$ is homeomorphic with a subcontinuum of the inverse limit of $\{h_{\eta_n}\}_{n=1}^{\infty}$. According to Lemma 2, we can assume without loss of generality that $\lambda_n \in (1,2]$ for each n. Given $a \in [0,1)$, define $g_a : [a,1] \to [0,1]$ by $g_a(x) = \frac{1}{1-a}(x-a)$. Then $g_a^{-1}(x) = (1-a)x + a$.

Claim. Given $\lambda \in (1,2]$, $a_1 \in [0,1)$ and $\eta = 2 - a_1$, there is $a_2 \in [0,1)$ such that

$$\begin{tikzcd}
{[0,1]} & {[0,1]} \arrow[l, "f_\lambda"'] \\
{[a_1,1]} \arrow[u, "g_{a_1}"] & {[a_2,1]} \arrow[l, "h_\eta"'] \arrow[u, "g_{a_2}"']
\end{tikzcd}$$

is a commutative diagram of surjections.

Proof of claim. Let $a_2 = 1 - \frac{1-a_1}{2-a_1}\lambda$. Since $a_1 \in [0,1)$, $0 < \frac{1-a_1}{2-a_1} \leq 1/2$, so that $a_2 \in [0,1)$. Thus $g_{a_1}^{-1} \circ f_\lambda \circ g_{a_2}$ is a piecewise linear unimodal map whose slope in absolute value is constant and equal to $(1-a_1)(\lambda)(\frac{1}{1-a_2}) = 2 - a_1 = \eta$. Since $h_\eta(1) = a_1 = g_{a_1}^{-1} \circ f_\lambda \circ g_{a_2}(1)$, the diagram must commute, with $h_\eta([a_2,1]) = g_{a_1}^{-1} \circ f_\lambda \circ g_{a_2}([a_2,1]) = [a_1,1]$. The claim is established.

Given the sequence $\{\lambda_n\}_{n=1}^{\infty}$, define $a_1 = 2 - \eta_1$ and $\eta_1 = \lambda_1$. Then $\eta_1 \in (1,2]$ and $a_1 \in [0,1)$. Assume that $a_n \in [0,1)$ and $\eta_n = 2 - a_n \in (1,2]$ have been defined. Let $a_{n+1} = 1 - \frac{1-a_n}{2-a_n}\lambda_n$ and $\eta_{n+1} = 2 - a_{n+1}$. Then, by the claim, $a_{n+1} \in [0,1)$ and $\eta_{n+1} \in (1,2]$. The commutative diagram

$$\begin{tikzcd}
{[0,1]} & {[0,1]} \arrow[l, "f_{\lambda_1}"'] & {[0,1]} \arrow[l, "f_{\lambda_2}"'] & \arrow[l, "f_{\lambda_3}"'] & \cdots \\
{[a_1,1]} \arrow[u, "g_{a_1}"] & {[a_2,1]} \arrow[l, "h_{\eta_1}"'] \arrow[u, "g_{a_2}"] & {[a_3,1]} \arrow[l, "h_{\eta_2}"'] \arrow[u, "g_{a_3}"] & \arrow[l, "h_{\eta_3}"'] &
\end{tikzcd}$$

induces a homeomorphism between the inverse limit of $\{f_{\lambda_n}\}_{n=1}^{\infty}$ and the inverse limit of $\{h_{\eta_n}|_{[a_{n+1},1]}\}_{n=1}^{\infty}$, a subcontinuum of the inverse limit of $\{h_{\eta_n}\}_{n=1}^{\infty}$.

Given the sequence η_n, the proof of Theorem 4 can be modified slightly to construct a subcontinuum of (I, f_η) homeomorphic to the inverse limit of $\{h_{\eta_n}\}_{n=1}^{\infty}$. Specifically, given ϵ_k, η_k and b_k, there is a_k such that $|1 + \frac{a_k}{b_k} - \eta_k| < \epsilon_k$. Define δ_k, n_k and b_{k+1} as before. Then maps $g_k = h_{\gamma_k}$ can be defined so that $|\gamma_k - \eta_k| < \epsilon_k$, hence the inverse limit of $\{h_{\gamma_k}\}_{k=1}^{\infty}$ is homeomorphic to the inverse limit of $\{h_{\eta_k}\}_{k=1}^{\infty}$, and the inverse limit of $\{g_k\}_{k=1}^{\infty}$ (that is, the inverse limit of $\{h_{\gamma_k}\}_{k=1}^{\infty}$) is homeomorphic to the inverse limit of $\{f_\eta^{n_k}|_{[0,b_{k+1}]}\}_{k=1}^{\infty}$. $\square$

Corollary 6. *Let $\eta \in A$, $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq [1,2]$ and U an open subset of (I, f_η). For any subcontinuum C of the inverse limit of $\{f_{\lambda_n}\}_{n=1}^{\infty}$, there is a subcontinuum D of (I, f_η) homeomorphic with C with $D \subseteq U$.*

Proof. Without loss of generality, U is of the form $[U_1 \times U_2 \times \cdots \times U_n \times I \times \cdots] \cap (I, f_\eta)$. It is enough to show that there is a subcontinuum D' of (I, f_η) such that D' is homeomorphic to (I, f_η) and $\Pi_0(D') \subseteq U_n$, since then $\Pi_n(\hat{f}_\eta^n(D')) \subseteq U_n$ and $\hat{f}_\eta^n(D') \subseteq U$ contains a subcontinuum homeomorphic with C.

Since $\eta \in A$, $\{f_\eta^k(c_\eta)\}_{k=1}^\infty$ is dense in $[0,1]$. Choose $k \geq 2$ such that $f_\eta^k(c_\eta) = f_\eta^{k-2}(0) \in U_n$. There is $v \in [0,1]$ such that $f_\eta^{k-2}[0,v] \subseteq U_n$. Let $b_1 = v$ in the proof of Theorem 4, and construct $\{b_m\}_{m=1}^\infty \subseteq [0,1], \{n_m\}_{m=1}^\infty \subseteq \mathbb{N}$ such that the inverse limit space D'' of $\{f_\eta^{n_k}|_{[0,b_{m+1}]}\}_{m=1}^\infty$ is homeomorphic to (I, f_η). Then $\Pi_0(\hat{f}_\eta^{k-2}(D'')) = f_\eta^{k-2}(\Pi_0(D'')) \subseteq f_\eta^{k-2}([0,v]) \subseteq U_n$. □

Finally, we prove that if $\eta \in A$, then (I, f_η) has uncountably many endpoints. A subcontinuum T of a space X is an *end continuum in* X if whenever $T \subseteq A, T \subseteq B$ for continua $A, B \subseteq X$, then either $A \subseteq B$ or $B \subseteq A$. The point $x \in X$ is an *endpoint of* X if $\{x\}$ is an end continuum in X.

Lemma 7. *Suppose that T is a subcontinuum of (I, f) for a map $f : I \to I$, and that $0 \in \Pi_n(T)$ for infinitely many $n \in \mathbb{N}$. Then T is an end continuum.*

Proof. Let A, B be subcontinua of (I, f) with $T \subseteq A \cap B$. Then $0 \in \Pi_n(A)$ and $0 \in \Pi_n(B)$ for infinitely many n. It follows that either $\Pi_n(A) \subseteq \Pi_n(B)$ or $\Pi_n(B) \subseteq \Pi_n(A)$ infinitely often, hence for all n. □

Lemma 8. *Suppose that $(I, f) \supseteq T_1 \supseteq T_2 \supseteq \cdots$ where T_i is an end continuum of (I, f) for each i, and that $\bigcap_{i=1}^\infty T_i = \{\underline{x}\}$. Then $\underline{x}$ is an endpoint.*

Proof. Let A, B be subcontinua of (I, f) with $\underline{x} \in A \cap B$. For each i, $T_i \cap A \cap B \neq \emptyset$. Since (I, f) is atriodic, one of T_i, A, B is contained in the union of the other two. If $T_i \subseteq A \cup B$ for any i, then, since T_i is an end continuum, either $A \subseteq B$ or $B \subseteq A$, and the lemma is proved. If not, then without loss of generality, $A \subseteq T_i \cup B$ for infinitely many i, and $A \subseteq \bigcap_{i=1}^\infty (T_i \cup B) = B$. □

If $\eta \in A$, then c_η is recurrent, since $\{f_\eta^k(c_\eta)\}_{k=1}^\infty$ is dense in $[0,1]$. It follows from [2] that (I, f_η) has an infinite number of endpoints. We can say more.

Theorem 9. *For $\eta \in A$, (I, f_η) has uncountably many endpoints.*

Proof. Choose disjoint copies T_0, T_1 of (I, f_η) contained in (I, f_η). According to Corollary 6, there are disjoint copies T_{00}, T_{01} and T_{10}, T_{11} of (I, f_η) contained in T_0 and T_1 respectively so that $diam(T_{s_0 s_1}) < \frac{1}{2} diam(T_{s_0})$ for $s_i \in \{0,1\}$. Continuing in this manner, we can assign to each sequence $s_0 s_1 s_2 \ldots$ of zeros and ones an infinite nested sequence of subcontinua $\{T_{s_0 s_1 \ldots s_n}\}_{n \in \mathbb{N}}$ of (I, f_η) such that $T_{s_0 s_1 \ldots s_n}$ is homeomorphic to (I, f_η) for each n and $diam(T_{s_0 s_1 \ldots s_n}) \to 0$ as $n \to \infty$. For each n, $T_{s_0 s_1 \ldots s_n}$ is not an arc, hence $0 \in \Pi_k(T_{s_0 s_1 \ldots s_n})$ for infinitely many k, and $T_{s_0 s_1 \ldots s_n}$ is an end continuum. Since $diam(T_{s_0 s_1 \ldots s_n}) \to 0$, for each sequence $s_0 s_1 s_2 \ldots$, $|\bigcap_{n \in \mathbb{N}} \{T_{s_0 s_1 \ldots s_n}\}| = 1$. Finally, if $s_0 s_1 s_2 \ldots \neq t_0 t_1 t_2 \ldots$, $(\bigcap_{n \in \mathbb{N}} \{T_{s_0 s_1 \ldots s_n}\}) \cap (\bigcap_{n \in \mathbb{N}} \{T_{t_0 t_1 \ldots t_n}\}) = \emptyset$, thus there are an uncountable number of distinct endpoints. □

We do not know whether the set of endpoints of (I, f_λ) can be countably infinite.

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