A FINITENESS CRITERION FOR COHOMOLOGY OF FRÉCHET-MONTEL SHEAVES

V. D. GOLOVIN

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Abstract. The paper deals with a finiteness criterion for the cohomology with compact supports of calculable Fréchet-Montel sheaves over locally compact topological spaces.

1. Result: redaction and commentary

A sheaf of Abelian groups $\mathcal{F}$ over a topological space $X$ is called calculable (cf. [3]) if for every integer $k \geq 1$ and any open neighborhood $U$ of an arbitrary point $x \in X$ there exists an open neighborhood $V \subset U$ of $x$ such that the restriction map $H^k(U; \mathcal{F}) \rightarrow H^k(V; \mathcal{F})$ has zero image.

A sheaf of Abelian groups $\mathcal{F}$ over a locally compact topological space $X$ is called a Fréchet-Montel sheaf (cf. [4]) if the following conditions are satisfied:

(a) for every open set $U \subset X$ the group of sections $\Gamma(U; \mathcal{F})$ is a Fréchet topological vector space;

(b) for any two open sets $V \subset U$ in $X$ the restriction map $\Gamma(U; \mathcal{F}) \rightarrow \Gamma(V; \mathcal{F})$ is linear and continuous, and completely continuous when $V$ is relatively compact in $U$.

Grothendieck proved that for a calculable Fréchet-Montel sheaf $\mathcal{F}$ over a compact topological space $X$ the vector spaces $H^k(X; \mathcal{F})$ ($k = 0, 1, \ldots$) are finite-dimensional (cf. [3], p. 4). This implies the Cartan-Serre finiteness theorem [2].

On the other hand, Andreotti and Kas proved that for a coherent analytic sheaf $\mathcal{F}$ over a complex space $X$, countable at infinity, the vector space $H^k_c(X; \mathcal{F})$ is finite-dimensional if for some relatively compact open set $U \subset X$ the canonical map $H^k_c(U; \mathcal{F}) \rightarrow H^k_c(X; \mathcal{F})$ is surjective (cf. [1], p. 241). This assertion plays an important role in the study of $q$-pseudoconcave complex spaces.

We prove in the present paper the following theorem:

Theorem. Let $X$ be a locally compact topological space with a countable base of open sets, and let $\mathcal{F}$ be a calculable Fréchet-Montel sheaf over $X$. Then the vector space $H^k_c(X; \mathcal{F})$ is finite-dimensional if and only if for some relatively compact open set $U \subset X$ the canonical map $H^k_c(U; \mathcal{F}) \rightarrow H^k_c(X; \mathcal{F})$ is surjective.

This implies the Grothendieck theorem providing the space $X$ has a countable base of open sets and the Andreotti-Kas criterion both cited above.
2. Auxiliary notions and facts

Let $X$ be a locally compact topological space, countable at infinity, let $\mathcal{U} = (U_{i})_{i \in I}$ be a locally finite covering of $X$ by relatively compact open sets, let $J$ be a finite subset in $I$, and let $\mathcal{F}$ be an arbitrary sheaf of Abelian groups over $X$. We set $C_{j}^{k}(\mathcal{U}; \mathcal{F}) = \prod_{i \in I} \Gamma(U_{i}; \mathcal{F})$, where the direct sum is taken over all collections $(i_{0}, \ldots, i_{k})$ of indices among which at least one belongs to $J$. Then $\delta C_{j}^{k}(\mathcal{U}; \mathcal{F}) \subset C_{j}^{k+1}(\mathcal{U}; \mathcal{F})$ and, consequently, a subcomplex $C_{j}(\mathcal{U}; \mathcal{F})$ of the cochain complex $C(\mathcal{U}; \mathcal{F})$ is defined. We consider the cohomology groups $H_{j}^{k}(\mathcal{U}; \mathcal{F}) = H^{k}C_{j}(\mathcal{U}; \mathcal{F})$.

Lemma 1. Let $\mathcal{F}$ be a Godement sheaf over $X$, i.e., there exists a family of Abelian groups $G_{x}$ $(x \in X)$ such that $\Gamma(U; \mathcal{F}) = \prod_{x \in U} G_{x}$ for every open set $U \subset X$. Then $H_{j}^{k}(\mathcal{U}; \mathcal{F}) = 0$ for $k \neq 0$.

Proof. One can assume that $I = \{1, 2, \ldots\}$ and $J = \{1, \ldots, n\}$ for some $n \in I$. Then the sets $S_{i} = U_{i} \setminus \bigcup_{j > i} U_{j}$ $(i = 1, 2, \ldots)$ cover the space $X$ and do not intersect in pairs. Let $C_{0}(Z)$ be a Godement sheaf defined by the constant sheaf $Z$ over $X$, i.e., $\Gamma(U; C_{0}(Z)) = Z^{U}$ for every open set $U \subset X$. We set $\lambda_{i}(x) = 1$ for $x \in S_{i}$, and $\lambda_{i}(x) = 0$ for $x \in X \setminus S_{i}$. Then we get a sequence of the sections $\lambda_{i} \in \Gamma(X; C_{0}(Z))$ $(i = 1, 2, \ldots)$. Let $f = (f_{i_{0} \ldots i_{k}}) \in C_{j}^{k}(\mathcal{U}; \mathcal{F})$ be a cocycle of a degree $k \geq 1$, i.e., $\delta f = 0$. Since the sheaf $\mathcal{F}$ is a module over the sheaf of rings $C_{0}(Z)$, the sections $g_{i_{0} \ldots i_{k-1}} = \sum_{i \in I} \lambda_{i} f_{i_{0} \ldots i_{k-1}}$ of $\mathcal{F}$ over $U_{i_{0} \ldots i_{k-1}}$ are defined. Obviously, the cochain $g = (g_{i_{0} \ldots i_{k-1}})$ belongs to the group $C_{j+1}^{k-1}(\mathcal{U}; \mathcal{F})$. Moreover,

\[ (\delta g)_{i_{0} \ldots i_{k}} = \sum_{i=1}^{\infty} \lambda_{i} \sum_{s=0}^{k} (-1)^{s} f_{i_{0} \ldots \hat{i}_{s} \ldots i_{k}} = f_{i_{0} \ldots i_{k}}, \]

because

\[ f_{i_{0} \ldots i_{k}} - \sum_{s=0}^{k} (-1)^{s} f_{i_{0} \ldots \hat{i}_{s} \ldots i_{k}} = (\delta f)_{i_{0} \ldots i_{k}} = 0. \]

In other words, $\delta g = f$. The lemma is proved.

We set $S(J) = X \setminus \bigcup_{i \in J} U_{i}$. Then $S(J)$ is a compact set in $X$ and for any sheaf of Abelian groups $\mathcal{F}$ over $X$ there is a canonical isomorphism $H_{j}^{0}(\mathcal{U}; \mathcal{F}) = \Gamma_{S(J)}(X; \mathcal{F})$.

If $\mathcal{F}$ is a flabby sheaf of Abelian groups over $X$, then with the help of Lemma 1 one can show easily that for every $k = 0, 1, \ldots$ there is a canonical isomorphism $H_{j}^{k}(\mathcal{U}; \mathcal{F}) = H_{S(J)}^{k}(X; \mathcal{F})$ and, consequently, $H_{j}^{k}(\mathcal{U}; \mathcal{F}) = 0$ for $k \neq 0$.

Lemma 2. For any sheaf of Abelian groups $\mathcal{F}$ over $X$ there exists a canonical map $H_{j}^{k}(\mathcal{U}; \mathcal{F}) \rightarrow H_{S(J)}^{k}(X; \mathcal{F})$. 

Proof. We make use of the Godement canonical resolution $0 \to \mathcal{F} \to C^0(\mathcal{F}) \to C^1(\mathcal{F}) \to \cdots$ of the sheaf $\mathcal{F}$ and consider the commutative diagram

$$
\begin{array}{ccccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \Gamma_{S(J)}(X; \mathcal{F}) & C^0_j(\mathcal{U}; \mathcal{F}) & C^1_j(\mathcal{U}; \mathcal{F}) & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Gamma_{S(J)}(X; C^0(\mathcal{F})) & C^0_j(\mathcal{U}; C^0(\mathcal{F})) & C^1_j(\mathcal{U}; C^0(\mathcal{F})) & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Gamma_{S(J)}(X; C^1(\mathcal{F})) & C^0_j(\mathcal{U}; C^1(\mathcal{F})) & C^1_j(\mathcal{U}; C^1(\mathcal{F})) & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
$$

in which all rows starting with the second are exact by Lemma 1. From this in the obvious way by means of Weil’s diagram chasing (cf. [5]) we get the desired map.

**Lemma 3.** Let $\mathcal{F}$ be a calculable sheaf of Abelian groups over $X$, and let $S$ be a compact set in $X$. Then for every integer $k \geq 0$ and any sufficiently fine locally finite covering $\mathcal{U} = (U_i)_{i \in I}$ of $X$ by open sets there exists a finite set $J \subset I$ such that $S \subset S(J)$ and the image of the map $H^k_S(X; \mathcal{F}) \to H^k_{S(J)}(X; \mathcal{F})$ is contained in the image of the map $H^j_j(\mathcal{U}; \mathcal{F}) \to H^k_{S(J)}(X; \mathcal{F})$.

**Proof.** First let $\mathcal{U}$ be an arbitrary locally finite covering of the space $X$ by relatively compact open sets, and let $J$ be a finite subset in $I$ such that $S \subset S(J)$. We consider the commutative diagram used in the proof of Lemma 2 in which all rows starting with the second are exact. Fixing an integer $k \geq 1$ and substituting in succession the covering $\mathcal{U}$ by more and more fine ones, after a finite number of steps with the help of Weil’s diagram chasing (cf. [5]) we get the assertion of the lemma.

By a Fréchet complex we mean a complex $E^\cdot$ in which $E^k$ are Fréchet spaces and differentials $d : E^k \to E^{k+1}$ are continuous linear maps. By a morphism of Fréchet complexes $f : E^\cdot \to F^\cdot$ we mean a sequence of continuous linear maps $f : E^k \to F^k$ compatible with the differentials.

**Lemma 4.** Let $f_n : E^k_n \to E^k_{n+1}$ ($n = 0, 1, \ldots$) be morphisms of Fréchet complexes, $f_{n,m} = f_{m-1} \circ \cdots \circ f_n$ for $n < m$, and let $E^\cdot = \lim \varinjlim E^k_n$ be the corresponding inductive limit of complexes. Let us assume that for some $k$ the following conditions are satisfied:

(a) the map $f_0 : E^k_0 \to E^k_1$ is completely continuous;
(b) the canonical map $H^k E^\cdot_n \to H^k E^\cdot$ is surjective.

Then for every $n \geq 1$ there exists $m > n$ such that the quotient-space $H^k E^\cdot_n / \ker f_{n,m}$ (where $\ker f_{n,m}$ is the kernel of the map $H^k E^\cdot_n \to H^k E^\cdot_m$ induced by the morphism of complexes $f_{n,m}$) is finite-dimensional and separated; in particular, the vector space $H^k E^\cdot$ is finite-dimensional.

**Proof.** For fixed $n \geq 1$ and every $m > n$ we set $G_m = \{(x, y) \in Z^k E^\cdot_n \times E^\cdot_{m-1} : f_{n,m}x = dy\}$. Then there is an exact sequence $\lim \Gamma G_m \xrightarrow{p_1} Z^k E^\cdot_n \to H^k E^\cdot$, where $p_1$
is the projection of the product $Z^k E_n \times E_{m-1}^k$ onto the first factor. By condition (b) the continuous linear map $Z^k E_0 \oplus \lim_{\to} G_m \to Z^k E_n$ is surjective. Consequently, in view of a theorem of Banach and the Baire theorem, there exists $m > n$ such that the continuous linear map of Fréchet spaces $Z^k E_0 \oplus G_m \to Z^k E_n$ is surjective. By condition (a) and the Schwartz almost epimorphism theorem the quotient-space $Z^k E_n / \ker f_{n,m}$ is finite-dimensional and separated. In particular, since the canonical map $H^k E_n \to H^k E$ is surjective, the vector space $H^k E$ is finite-dimensional.

3. Proof of the Theorem

The necessity of the condition of the Theorem is obvious; we prove the sufficiency. We set $C^c(X; \mathcal{F}) = \lim_{\to} C^c(J; \mathcal{F})$, where $\mathcal{U}$ runs through all locally finite coverings of the space $X$ by relatively compact open sets and $J$ runs through finite subsets in corresponding sets of indices. Then $H^k_c(X; \mathcal{F}) = H^k C^c(X; \mathcal{F})$. Obviously, one can choose sequences of coverings $\mathcal{U}_n$ and finite sets $J_n$ ($n = 0, 1, \ldots$), respectively, such that $C^c(X; \mathcal{F}) = \lim_{\to} C^c(J_n; \mathcal{U}_n; \mathcal{F})$. Since $S = \overline{U}$ is a compact set in $X$, there is the commutative diagram

\[
\begin{array}{ccc}
H^k(U; \mathcal{F}) & \longrightarrow & H^k_c(X; \mathcal{F}) \\
\downarrow & & \downarrow \\
H^k_S(X; \mathcal{F}) & & 
\end{array}
\]

in which the right inclined arrow is surjective. By Lemma 3 there exist, respectively, a locally finite covering $\mathcal{U}$ of $X$ by relatively compact open sets and a finite set of indices $J$ such that the canonical map $H^k(J; \mathcal{F}) \to H^k_c(X; \mathcal{F})$ is surjective. One can assume that in the sequences chosen above $\mathcal{U}_0 = \mathcal{U}$ and $J_0 = J$. By Lemma 4 the vector space $H^k_c(X; \mathcal{F})$ is finite-dimensional. The Theorem is proved.

REFERENCES


Krasnoshkol’naya nab., 22, kv. 185, 310125, Kharkov, 125 Ukraine