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A COEFFICIENT OF AN ASYMPTOTIC EXPANSION OF LOGARITHMS OF DETERMINANTS FOR CLASSICAL ELLIPTIC PSEUDODIFFERENTIAL OPERATORS WITH PARAMETERS

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Abstract. For classical elliptic pseudodifferential operators $A(\lambda)$ of order $m > 0$ with parameter λ of weight $\chi > 0$, it is known that $\log \det_\theta A(\lambda)$ admits an asymptotic expansion as $\lambda \to +\infty$. In this paper we show, with some assumptions, that the coefficient of $\lambda^{-1/\chi}$ can be expressed by the value of a zeta function at 0 for some elliptic ψDO on $M \times S^1$ multiplied by $\frac{m}{2}$.

I. Introduction

Let M be a compact oriented Riemannian manifold of dimension d and let $E \overset{\pi}{\to} \to M$ be a vector bundle of rank k. Let $A(\lambda)\colon\ C^\infty(E) \to C^\infty(E)$ be a classical elliptic pseudodifferential operator of order $m > 0$ with parameter λ of weight $\chi > 0$, where λ is a nonnegative real number. That is, the symbol of $A(\lambda)$ has an asymptotic expansion as in (2.1).

We assume that there is an angle θ such that the principal symbol $a_m(x, \xi, \lambda)$ does not have any eigenvalues on the ray $\{z \in \mathbf{C} | z = \rho e^{i\theta}, \rho \geq 0\}$ for $|\xi| + |\lambda|^{\frac{1}{\chi}} \neq 0$ and that $A(\lambda)$ does not have any eigenvalues in a sector $L_{[\theta-\varepsilon,\theta+\varepsilon]} = \{z \in \mathbf{C} | \theta - \varepsilon \leq \arg z \leq \theta + \varepsilon\}$ for some small $\varepsilon > 0$. We call this θ an Agmon angle. In fact from the compactness of the set $\{(x, \xi, \lambda) | x \in M, |\xi|^2 + |\lambda|^{\frac{2}{\chi}} = 1\}$, we know that $a_m(x, \xi, \lambda)$ does not have any eigenvalues in a sector $L_{[\theta-\delta,\theta+\delta]}$ for sufficiently small $\delta > 0$.

It is shown in [BFK] that as $\lambda \to +\infty$, $\log \det_\theta A(\lambda)$ admits an asymptotic expansion of the form

$$\log \det_\theta A(\lambda) \sim \pi_{-d} \lambda^{\frac{d}{\chi}} + \pi_{-d+1} \lambda^{\frac{d-1}{\chi}} + \cdots + \pi_0 + \pi_1 \lambda^{-\frac{1}{\chi}} + \cdots + \sum_{j=0}^{d} q_j \lambda^{\frac{j}{\chi}} \log \lambda$$

as $\lambda \to +\infty$. Here each coefficient π_i and q_j is computable in terms of the asymptotic symbol of $A(\lambda)$.

Let $\sum_{j=0}^{\infty} a_{m-j}(x, \xi, \lambda)$ be an asymptotic symbol of $A(\lambda)$ for some local coordinate U. We also assume that for each j, the function $\tilde{a}_{m-j}\colon\ U \times \mathbf{R}^d \times \mathbf{R} \to \{k \times k \text{ matrices}\}$ defined by $\tilde{a}_{m-j}(x, \xi, \lambda) = a_{m-j}(x, \xi, |\lambda|^\chi)$ is smooth. Then from $A(\lambda)$ we can construct a classical elliptic pseudodifferential operator P with the

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same Agmon angle θ on $M \times S^1$, where P is uniquely determined up to smoothing operators. Let $\zeta_P(s)$ be the zeta function constructed from the eigenvalues of P.

The purpose of this note is to show that $\pi_1 = \frac{m}{2}\zeta_P(0)$. So far we are unable to give a similar interpretation of other coefficients π_i $(i \neq 1)$ and q_j.

II. Construction of P

Suppose that in a local coordinate system U, the asymptotic symbol of $A(\lambda)$ is $\sigma(A(\lambda)) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi, \lambda)$. Then

$$a_{m-j}(x,\xi,\lambda)\colon\ U \times \mathbf{R}^d \times [0,\infty) \to \{k \times k \text{ matrices}\} \tag{2.1}$$

with

$$a_{m-j}(x, t\xi, t^{\chi}\lambda) = t^{m-j} a_{m-j}(x,\xi,\lambda) \quad \text{for } t > 0.$$

Now for each j, we extend a_{m-j} to

$$\tilde{a}_{m-j}\colon\ U \times S^1 \times \mathbf{R}^d \times \mathbf{R} \to \{k \times k \text{ matrices}\} \quad \text{by}$$
$$(x,t,\xi,\lambda) \mapsto a_{m-j}(x,\xi,|\lambda|^{\chi}).$$

Then $\tilde{a}_{m-j}$ is smooth by the assumption of the smoothness of $a_{m-j}(x,\xi,|\lambda|^{\chi})$. Note that $\tilde{a}_{m-j}(x,t,\xi,\lambda)$ is a homogeneous function of degree $m-j$ with respect to ξ, λ.

Consider a diagram

$$\begin{array}{ccc} p^*E & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \pi \\ M \times S^1 & \xrightarrow{\ p\ } & M \end{array}$$

where p is the natural projection.

Choose an elliptic ψDO $\widetilde{A}\colon\ C^{\infty}(p^*E) \to C^{\infty}(p^*E)$ whose asymptotic symbol is $\sigma(\widetilde{A}) \sim \sum_{j=0}^{\infty} \tilde{a}_{m-j}(x,t,\xi,\lambda)$ in a local coordinate $U \times S^1$. Since the principal symbol $a_m(x,\xi,|\lambda|^{\chi})$ of $\widetilde{A}$ does not have any eigenvalues in the sector $L_{[\theta-\delta,\theta+\delta]}$ for $|\xi| + |\lambda|^{\frac{1}{\chi}} \neq 0$, there exists $R > 0$ such that $\text{Spectrum}(\widetilde{A}) \cap \{z \mid |z| \geq R, \theta - \delta \leq \arg z \leq \theta + \delta\}$ is empty (see [Sh] for details). Since the spectrum of $\widetilde{A}$ is discrete, there are only finitely many eigenvalues of $\widetilde{A}$ in $\{z \mid |z| < R, \theta - \delta \leq \arg z \leq \theta + \delta\}$. Note that each eigenspace corresponding to an eigenvalue is a finite-dimensional vector space by the ellipticity of $\widetilde{A}$ of order $m > 0$. Let Q be the span of {eigenvectors of $\widetilde{A}$ whose eigenvalues are $\rho e^{i\theta}$ for some $\rho, 0 \leq \rho \leq R$}. Then Q is a finite-dimensional vector space. Define $\phi\colon\ C^{\infty}(p^*E) \to C^{\infty}(p^*E)$ be the natural projection onto Q.

Define $P = \widetilde{A} - Re^{i\theta}\phi$. Then P is injective with an Agmon angle θ and the asymptotic symbol of P is exactly the same as the asymptotic symbol of $\widetilde{A}$. Define $\zeta_P(s) = \sum_{\lambda_i} \lambda_i^{-s}$, where λ_i's are the eigenvalues of P. Then by [Se] (also see [Wo]), $\zeta_P(s)$ is regular at 0 with

$$\zeta_P(0) = \frac{e^{i\theta}}{m(2\pi)^{d+1}} \cdot \int_{M \times S^1} d\ \mathrm{vol}(x,t) \int_{|\xi|^2 + \lambda^2 = 1} d(\xi,\lambda)' \int_0^{\infty} \mathrm{tr}\ \tilde{r}_{-m-d-1}(x,t,\xi,\lambda,e^{i\theta}\mu)\, d\mu,$$

where $\tilde{r}_{-m-d-1}$ is the homogeneous part of degree $-m-d-1$ in the asymptotic symbol of the resolvent $(P-\mu I)^{-1}$. Note that $\tilde{r}_{-m-d-1}(x,t,\xi,\lambda,e^{i\theta}\mu)$ does not depend on t in this case.

Recall that $\log \det_\theta A(\lambda) \sim \pi_{-d}\lambda^{\frac{d}{\chi}} + \cdots + \pi_0 + \pi_1\lambda^{-\frac{1}{\chi}} + \cdots + \sum_{j=0}^{d} q_j \lambda^{\frac{j}{\chi}} \log\lambda$ as $\lambda \to +\infty$. Then our goal is to prove the following theorem.

Theorem. $\pi_1 = \frac{m}{2}\zeta_P(0)$.

III. Proof of the Theorem

From the Appendix of [BFK] we can derive the local formula

$$\pi_1 = \frac{-e^{i\theta}}{(2\pi)^d}\int_M \int_{\mathbf{R}^d}\int_0^\infty \operatorname{tr}\, r_{-m-d-1}(x,\xi,1,e^{i\theta}\mu)\, d\mu d\xi d \operatorname{vol}(x),$$

where r_{-m-d-1} is the homogeneous part of degree $-m-d-1$ in the asymptotic symbol of the resolvent $(A(\lambda)-\mu I)^{-1}$.

Note the relation $\tilde{r}_{-m-d-1}(x,t,\xi,\lambda,e^{i\theta}\mu) = r_{-m-d-1}(x,\xi,|\lambda|^\chi,e^{i\theta}\mu)$. Then

$$\zeta_P(0) = -\frac{2e^{i\theta}\cdot 2\pi}{m(2\pi)^{d+1}} \cdot \int_M d \operatorname{vol}(x) \int_{\substack{|\xi|^2+\lambda^2=1\\ \lambda>0}} d(\xi,\lambda)' \int_0^\infty \operatorname{tr}\, r_{-m-d-1}(x,\xi,|\lambda|^\chi,e^{i\theta}\mu)\, d\mu$$

since the integrand is even in λ.

Set

$$(\mathrm{I}) = \int_{\substack{|\xi|^2+\lambda^2=1\\ \lambda>0}} d(\xi,\lambda)' \int_0^\infty \operatorname{tr}\, r_{-m-d-1}(x,\xi,|\lambda|^\chi,e^{i\theta}\mu)\, d\mu.$$

Then using the projection from the upper hemisphere to $\{\xi \in \mathbf{R}^d \| \xi| < 1\}$, we obtain

$$\begin{aligned}(\mathrm{I}) &= \int_{|\xi|<1}\int_0^\infty \operatorname{tr}\, r_{-m-d-1}(x,\xi,(\sqrt{1-|\xi|^2})^\chi,e^{i\theta}\mu)(1-|\xi|^2)^{-\frac{1}{2}}\, d\mu d\xi \\ &= \int_{|\xi|<1}\int_0^\infty (1-|\xi|^2)^{-\frac{m+d+2}{2}} \operatorname{tr}\, r_{-m-d-1} \\ &\quad \cdot \left(x, \frac{\xi_1}{\sqrt{1-|\xi|^2}},\ldots,\frac{\xi_d}{\sqrt{1-|\xi|^2}},1,e^{i\theta}\frac{\mu}{\sqrt{1-|\xi|^2}^m}\right) d\mu d\xi,\end{aligned}$$

since the weight of μ is m.

Consider a map $\Phi\colon\ \mathbf{R}^d \times (0,\infty) \to \{\xi \in \mathbf{R}^d \| \xi| < 1\} \times (0,\infty)$ defined by

$$(\eta_1,\ldots,\eta_d,\nu) \mapsto \left(\frac{\eta_1}{\sqrt{1+|\eta|^2}},\ldots,\frac{\eta_d}{\sqrt{1+|\eta|^2}},\frac{\nu}{\sqrt{1+|\eta|^2}^m}\right) = (\xi_1,\ldots,\xi_d,\mu).$$

Then Φ is a diffeomorphism with $\det(J(\Phi)) = (1+|\eta|^2)^{-\frac{m+d+2}{2}} = (1-|\xi|^2)^{\frac{m+d+2}{2}}$. Hence we can get $(\mathrm{I}) = \int_{\mathbf{R}^d}\int_0^\infty \operatorname{tr}\, r_{-m-d-1}(x,\eta,1,e^{i\theta}\nu) d\nu d\eta$. □

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