PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 12, December 1996, Pages 3605–3612
S 0002-9939(96)03432-6

IMAGES OF BILINEAR MAPPINGS INTO $\mathbf{R}^3$

S. J. BERNAU AND PIOTR J. WOJCIECHOWSKI

(Communicated by Lance W. Small)

Abstract. It is well-known that the image of a multilinear mapping into a vector space need not be a subspace of its target space. It is, however, far from clear which subsets of the target space may be such images. For vector spaces over the real numbers we give a complete classification of the images of bilinear mappings into a three-dimensional vector space. In Theorem 2.8 we show that either the image of a bilinear mapping into a three-dimensional space is a subspace, or its complement is either the interior of a double elliptic cone, or a plane from which two lines intersecting at the origin have been removed. We also show (Theorem 2.2) that the image of any multilinear mapping into a two-dimensional space is necessarily a subspace. Our methods are elementary and free of tensor considerations.

1. Preliminaries

We assume throughout this paper that all scalars are real. We reserve the symbol V, with or without subscripts, to denote a real vector space.[1]

A mapping $F : V_1 \times V_2 \times \ldots \times V_n \to V$ is called *multilinear* (more precisely *n-linear*) if F is linear separately on every coordinate. In greater detail this means that for each $i = 1, \ldots, n$ and every choice of elements $v_j \in V_j$ with $1 \leq i \leq n$ and $j \neq i$, the mapping $F_i : V_i \to V$ defined by

$$F_i(v) = F(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n)$$

is linear. Every mapping F_i, defined in this way, will be called a *linear mapping associated with F*, or more briefly, an *associated linear mapping*. Similarly we can define associated k-linear mappings for $k = 2, 3, \ldots, n$.

We begin with two examples of bilinear mappings whose images are not subspaces of the target space. Taking into account the dimensions of the vector spaces involved, these are the simplest possible examples. We show later that these are essentially the only such examples with images in $\mathbf{R}^3$.

Example 1.1. Let $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ be given by the formula:

$$F((x_1, x_2), (y_1, y_2)) = (x_1y_1, x_1y_2, x_2y_1).$$

Received by the editors January 24, 1995 and, in revised form, May 22, 1995.
1991 *Mathematics Subject Classification.* Primary 15A69.

[1]This saves multiple repetitions of the clause "Let $V, V_1, V_2, \ldots, V_n$ denote real vector spaces."

We describe the image of the mapping. Consider the system of equations:

$$\begin{aligned} x_1y_1 &= x, \\ x_1y_2 &= y, \\ x_2y_1 &= z. \end{aligned}$$

If $x \neq 0$, the system is consistent (one solution being $x_1 = 1$, $y_1 = x$, $y_2 = y$, and $x_2 = z/x$). If $x = 0$ the system is consistent if and only if $yz = 0$. Consequently range F is all of $\mathbf{R}^3$ except for the four open quadrants of the plane $x = 0$. Another description of range F is that it consists of two lines through the origin of $\mathbf{R}^3$ together with the complement of the plane which contains them. Notice that we need not use mutually perpendicular axes in $\mathbf{R}^3$ to construct this example, so our second description of range F is a little more informative.

Example 1.2. Let $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ be given by the formula:

$$F((x_1, x_2), (y_1, y_2)) = (x_1y_1 + x_2y_2, x_1y_2, x_2y_1).$$

To describe range F consider the system of equations:

$$\begin{aligned} x_1y_1 + x_2y_2 &= x, \\ x_1y_2 &= y, \\ x_2y_1 &= z. \end{aligned}$$

If this system is consistent we have

$$x^2 - 4yz = (x_1y_1 - x_2y_2)^2 \geq 0.$$

Conversely, suppose $x^2 - 4yz \geq 0$ and write $\Delta = \sqrt{x^2 - 4yz}$. It is routine to check that $x_1 = 1$, $y_1 = (x + \Delta)/2$, $x_2 = (x - \Delta)/2y$, $y_2 = y$ yield a solution when $y \neq 0$; while $y = 0$ has an easier solution $x_1 = 0$, $y_1 = z$, $x_2 = 1$, $y_2 = x$. We conclude that

$$\operatorname{range} F = \{(x, y, z) \in \mathbf{R}^3 : x^2 \geq 4yz\},$$

which is the closed region exterior to the cone $x^2 = 4yz$. If we use the standard axes in $\mathbf{R}^3$ this complementary cone has a circular cross section. If we use oblique axes, or change the linear scales on the axes, we still have the complement of a cone, but the cross section is now elliptical.

Obviously, the image of every multilinear mapping into $\mathbf{R}^1$ is a subspace of $\mathbf{R}^1$. The same is true for multilinear maps into $\mathbf{R}^2$. We demonstrate this by proving a theorem about associated linear mappings which is interesting in its own right.

Theorem 1.3. *Let $F : V_1 \times V_2 \times \ldots \times V_n \to V$ be a multilinear mapping. If the image of every associated linear mapping has dimension at most one, then* range F *is a subspace of V, again with dimension no greater than one.*

Proof. Let $u_i, v_i \in V_i$; $i = 1, \ldots, n$, and suppose

$$F(u_1, \ldots, u_n) \neq 0 \quad \text{and} \quad F(v_1, \ldots, v_n) \neq 0.$$

By linearity, at most one of

$$F(u_1, u_2, \ldots, u_n), \qquad F(v_1, u_2, \ldots, u_n), \qquad F(u_1 + v_1, u_2, \ldots, u_n)$$

is zero and similarly at most one of

$$F(v_1, v_2, \ldots, v_n), \qquad F(u_1, v_2, \ldots, v_n), \qquad F(u_1 + v_1, v_2, \ldots, v_n)$$

is zero. Hence there exists $t_1 \in V_1$ such that

$$F(t_1, u_2, \dots, u_n) \neq 0 \quad \text{and} \quad F(t_1, v_2, \dots, v_n) \neq 0.$$

Since $F(V_1, u_2, \dots, u_n)$ and $F(V_1, v_2, \dots, v_n)$ are one-dimensional, there exist nonzero $\lambda_1, \mu_1 \in \mathbf{R}$ such that

$$F(u_1, u_2, \dots, u_n) = \lambda_1 F(t_1, u_2, \dots, u_n),$$

$$F(t_1, v_2, \dots, v_n) = \mu_1 F(v_1, v_2, \dots, v_n).$$

Continue inductively to find $t_i \in V_i$ and $\lambda_i, \mu_i \in \mathbf{R}$, for $i = 2, \dots, n$, such that

$$\begin{aligned} F(u_1, u_2, \dots, u_n) &= \lambda_1 F(t_1, u_2, \dots, u_n) \\ &= \lambda_1 \lambda_2 F(t_1, t_2, u_3, \dots, u_n) \\ &= \dots \\ &= \lambda_1 \cdots \lambda_n F(t_1, \dots, t_n) \end{aligned}$$

and similarly

$$F(t_1, \dots, t_n) = \mu_1 \cdots \mu_n F(v_1, \dots, v_n).$$

So

$$F(u_1, \dots, u_n) = \lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n F(v_1, \dots, v_n).$$

Thus $F(V_1, \dots, V_n)$ is one-dimensional. □

Now the result on multilinear mappings into $\mathbf{R}^2$ follows immediately.

Corollary 1.4. *If $F : V_1 \times \dots \times V_n \to \mathbf{R}^2$ is a multilinear mapping, then the image of F is a subspace of $\mathbf{R}^2$.*

Proof. The images of the associated linear mappings are subspaces of $\mathbf{R}^2$. If one of them is $\mathbf{R}^2$, then the image of F is $\mathbf{R}^2$. Otherwise, by Theorem 1.3 the image of F is a subspace of dimension at most 1. □

Application to vector lattices. An archimedean vector lattice is a lattice-ordered vector space satisfying the condition that if for $a, b \geq 0$ and for every positive integer n we have $na \leq b$, then $a = 0$. By a well-known structure theorem due to Hölder we know that every totally ordered archimedean vector lattice is isomorphic to a subspace of $\mathbf{R}$. Below we consider a multilinear mapping into a vector lattice with the property that each element of its range is comparable with 0. (For further discussion of this type of condition and the situations in which it arises we refer to [3].) Before stating the next theorem we remind the reader that the positive cone W^+ of a vector lattice W is defined by $W^+ = \{ w \in W : w \geq 0 \}$, and the negative cone W^- is defined analogously.

Theorem 1.5. *Suppose W is an archimedean vector lattice and $F : V_1 \times \dots \times V_n \to W$ is a multilinear mapping satisfying the condition that $F(V_1, \dots, V_n) \subseteq W^+ \cup W^-$, then the image of F is a totally ordered subspace of W which is at most one-dimensional.*

Proof. The image of every associated linear mapping is a totally-ordered, archimedean subspace of W. By Hölder's structure theorem each such image is isomorphic to a subspace of $\mathbf{R}$, and is therefore, at most, one-dimensional. By Theorem 1.3, it follows that the image of F is one-dimensional. □

We can obtain a result similar to Theorem 1.3 by considering the images of associated bilinear mappings.

Theorem 1.6. *Let $F : V_1 \times V_2 \times \ldots \times V_n \to V$ be a multilinear mapping. If the image of every associated bilinear mapping is a subspace of dimension at most two, then* $\operatorname{range} F$ *is a subspace of V, again with dimension no more than two.*

Proof. We proceed by induction on n. The theorem is trivially true for $n \leq 2$. So we assume that $n \geq 3$ and that the theorem is true for all $(n-1)$-linear mappings. If all associated linear mappings have rank at most 1, Theorem 1.3 tells us that $\operatorname{range} F$ is a subspace of V of dimension at most one. This leaves only the case when there is an associated linear map with rank 2.

Without loss of generality we may assume there exist $v_i \in V_i$ for $1 \leq i \leq n-1$ such that the associated linear map $F_n = F(v_1, \ldots, v_{n-1}, \cdot\,)$ has rank 2. Write $P = \operatorname{range} F_n$; then P is a two dimensional subspace of V. There exist v_n and w_n in V_n such that

$$P = \langle F(v_1, \ldots, v_{n-1}, v_n), F(v_1, \ldots, v_{n-1}, w_n)\rangle.$$

Consider the $(n-1)$-linear map $F(v_1, \cdot, \ldots, \cdot\,) : V_2 \times \ldots \times V_n \to V$ defined by

$$F(v_1, \cdot, \ldots, \cdot\,)(w_2, \ldots, w_n) = F(v_1, w_2, \ldots, w_n).$$

Each of its associated bilinear maps is an associated bilinear map of F, so is a subspace of V of dimension at most two. By the induction hypothesis its range is a subspace of V of dimension at most 2. This range clearly contains the vectors, $F(v_1, \ldots, v_{n-1}, v_n)$ and $F(v_1, \ldots, v_{n-1}, w_n)$ which span P. It follows that $\operatorname{range} F(v_1, \cdot, \ldots, \cdot\,) = P$, and similarly $\operatorname{range} F(\,\cdot\,, v_2, \cdot, \ldots, \cdot\,) = P$. Let $w_1 \in V_1$. Then the four elements

$$\begin{aligned} p_1 &= F(v_1, v_2, \ldots, v_n), & p_2 &= F(v_1, v_2, \ldots, w_n), \\ p_3 &= F(w_1, v_2, \ldots, v_n), & p_4 &= F(w_1, v_2, \ldots, w_n) \end{aligned}$$

are all in P, and p_1 and p_2 form a basis for P. By considering the eigenvalues of the linear map A of P into itself determined by the conditions $Ap_1 = p_3$ and $Ap_2 = p_4$ we can find $\lambda \in \mathbf{R}$ such that $\lambda p_1 + p_3$ and $\lambda p_2 + p_4$ are independent, and hence span P (in fact this fails for at most two values of λ). Note that

$$\begin{aligned} \lambda p_1 + p_3 = F(\lambda v_1 + w_1, v_2 \ldots, v_n) &\in \operatorname{range} F(\lambda v_1 + w_1, \cdot, \ldots, \cdot\,), \\ \lambda p_2 + p_4 = F(\lambda v_1 + w_1, v_2 \ldots, w_n) &\in \operatorname{range} F(\lambda v_1 + w_1, \cdot, \ldots, \cdot\,). \end{aligned}$$

It follows that $\operatorname{range} F(\lambda v_1 + w_1, \cdot, \ldots, \cdot\,) = P$ by the same dimension argument used above. Now choose $x_i \in V_i$ for $i = 2, \ldots, n$. We have

$$F(w_1, x_2, \ldots, x_n) = F(\lambda v_1 + w_1, x_2, \ldots, x_n) - \lambda F(v_1, x_2, \ldots, x_n) \in P - P = P.$$

Since w_1 is arbitrary in V_1 we have $\operatorname{range} F = P$, a subspace of V of dimension two. □

2. Bilinear mappings into $\mathbf{R}^3$

In this section we will use the following notational convention. Given the bilinear mapping $F : U \times V \to W$, we will denote the associated linear mapping $F(u, \cdot\,)$ simply by u. Likewise we write v for the linear mapping $F(\,\cdot\,, v)$. Thus we have $u(v) = v(u) = F(u, v)$.

We first prove a variant of Theorem 1.3 for the case of bilinear mappings. It concerns only the associated linear mappings of the form $F(u, \cdot\,)$.

Theorem 2.1. *If $F : U \times V \to W$ is bilinear, and u_1 and u_2 are elements of U whose ranks are both 1, then* $\operatorname{range} u_1 + \operatorname{range} u_2 \subseteq \operatorname{range} F$. *In particular, if for every $u \in U$ the associated linear mapping u has rank at most one, then* $\operatorname{range} F$ *is a subspace of W.*

Proof. Suppose u_1, u_2 are rank 1 elements of U and v_1, $v_2 \in V$. We must show that $u_1(v_1) + u_2(v_2) \in \operatorname{range} F$. If either of these elements is zero the result is trivial, so we assume that neither is zero. Since $\operatorname{rank} u_1 = 1$, there exists α such that $u_1(v_2) = \alpha u_1(v_1)$. Similarly there exists β such that $u_2(v_1) = \beta u_2(v_2)$. Now we have

$$\operatorname{range} F \ni (xu_1 + yu_2)(\lambda v_1 + v_2) = x(\lambda + \alpha)u_1(v_1) + y(\lambda\beta + 1)u_2(v_2).$$

We may choose λ so that $(\lambda + \alpha)(\lambda\beta + 1) \neq 0$, and then choose x and y such that the last term above is $u_1(v_1) + u_2(v_2)$. We are done. □

Our next goal is to show that the image of an arbitrary bilinear mapping from $U \times V$ into $\mathbf{R}^3$ is either a subspace of $\mathbf{R}^3$ or is the image of its restriction to $U_1 \times V_1$, where U_1 and V_1 are two dimensional subspaces of U and V respectively. After accomplishing this it will be enough to examine the two by two case in order to obtain a complete classification of ranges of bilinear mappings into $\mathbf{R}^3$.

Theorem 2.2. *If $F : U \times V \to W$ is bilinear and* $\operatorname{range} F$ *is not a subspace, then there are subspaces $U_1 \subseteq U$ and $V_1 \subseteq V$ such that* $\dim U_1 = \dim V_1 = 2$ *and $F(U_1 \times V_1)$ is not a subspace of W. Additionally we may choose bases $\{u_1, u_2\}$ for U_1 and $\{v_1, v_2\}$ for V_1 such that* $\operatorname{rank} u_1 = 2 = \operatorname{rank} v_1$, *and* $\operatorname{range} u_1 \neq \operatorname{range} u_2$, $\operatorname{range} v_1 \neq \operatorname{range} v_2$.

Proof. Since $\operatorname{range} F$ is not a subspace, there are vectors $u_1, u_2 \in U$ and $v_1, v_2 \in V$ such that $u_1(v_1) + u_2(v_2) \notin \operatorname{range} F$. Define $U_1 = \langle u_1, u_2 \rangle$ and $V_1 = \langle v_1, v_2 \rangle$. It is clear that $F(U_1, V_1)$ is not a subspace of $\mathbf{R}^3$, and that $\operatorname{range} u_1 \neq \operatorname{range} u_2$, $\operatorname{range} v_1 \neq \operatorname{range} v_2$. If $\operatorname{rank} u_1 = \operatorname{rank} u_2 = 1$, Theorem 2.1 provides the contradiction $\operatorname{range} u_1 + \operatorname{range} u_2 \subseteq \operatorname{range} F$. Thus one of these ranks is 2, and we may renumber, if necessary, to obtain $\operatorname{rank} u_1 = 2$, and similarly to obtain $\operatorname{rank} v_1 = 2$. □

Remark 2.3. We can fine tune Theorem 2.2 by noting that if u_1 and u_2 are arbitrary elements of U such that $\operatorname{range} u_1 + \operatorname{range} u_2 = \mathbf{R}^3$, then at least one of them has rank 2, their ranges are different, and there exist v_1, $v_2 \in V$ such that $u_1(v_1) + u_2(v_2) \notin \operatorname{range} F$. Also if $\operatorname{rank} u_2 = 1$ and $\operatorname{rank} u_1 = 2$, we may replace u_2 with $u_2 + u_1$ to gain $\operatorname{rank} u_2 = 2 = \operatorname{rank} u_1$, without losing $\operatorname{range} u_1 \neq \operatorname{range} u_2$.

Our next result shows that for the range of a bilinear mapping into $\mathbf{R}^3$ to not be a subspace, most of its action takes place in a limited part of its domain.

Lemma 2.4. *Suppose $F : U \times V \to \mathbf{R}^3$ is a bilinear mapping whose range is not a subspace of $\mathbf{R}^3$. Let u_1, $u_2 \in U$, with* $\operatorname{rank} u_1 = 2$ *and* $\operatorname{range} u_1 \neq \operatorname{range} u_2$, *and let v_1, v_2 be elements of V such that $u_1(v_1) + u_2(v_2) \notin \operatorname{range} F$. Write $V_1 = \langle v_1, v_2 \rangle$ and $N = \ker u_1$. We have the following.*

(i) $V = V_1 \oplus N$.
(ii) $F(U, N) = \{0\}$.
(iii) $F(U, V) = \operatorname{range} F = F(U, V_1)$.

Proof. We will first prove that $N \subseteq \ker u_2$. Consider $u_1 - u_2$. If $\ker(u_1 - u_2) \subseteq N$ we have equality because $\ker(u_1 - u_2)$ has codimension at most 2 ($\operatorname{rank}(u_1 - u_2)$

cannot be 3) and N has codimension 2. Then for $n \in N$ we calculate $u_2(n) = (u_1)(n) - (u_1 - u_2)(n) = 0$, as required. Now suppose $v_0 \in \ker(u_1 - u_2) \setminus N$. There are two possibilities.

The first is that $v_0 \in N + \ker u_2$, in which case we write $v_0 = n + m$ with $n \in N \setminus \ker u_2$ and $m \in \ker u_2 \setminus N$, and note that $u_1(m) = u_2(n)$. If $V = N + \ker u_2$, we may assume $v_1 \in \ker u_2$ and $v_2 \in N$ and obtain the contradiction

$$u_1(v_1) + u_2(v_2) = (u_1 + u_2)(v_1 + v_2) \in \operatorname{range} F.$$

Thus we may choose $v \in V \setminus (N + \ker u_2)$. It is standard to prove that $u_1(v)$ and $u_1(m)$ are linearly independent elements of $\operatorname{range} u_1$, and hence span its range. Similarly $u_2(v)$ and $u_2(n)$ span the range of u_2. Because $u_1(m) = u_2(n)$ we see that

$$\langle u_1(v), u_1(m) = u_2(n), u_2(v) \rangle = \operatorname{range} u_1 + \operatorname{range} u_2 = \mathbf{R}^3,$$

so the three elements listed form a basis for $\mathbf{R}^3$. Now we have

$$\begin{aligned}(xu_1 + yu_2)(\alpha v + \beta m + \gamma n) &= \alpha x u_1(v) + \alpha y u_2(v) + (\beta x + \gamma y) u_1(m) \\ &= a u_1(v) + b u_2(v) + c u_1(m)\end{aligned}$$

for $x = a$, $y = b$, $\alpha = 1$, $\beta = ca/(a^2 + b^2)$ and $\gamma = cb/(a^2 + b^2)$ when a and b are not both zero. This shows that $\operatorname{range} F = \mathbf{R}^3$, a contradiction.

The second is that $v_0 \notin N + \ker u_2$. If we can find $n \in N \setminus \ker u_2$ and $m \in \ker u_2 \setminus N$, we will see as above that $u_1(v_0) = u_2(v_0)$, $u_1(m)$, $u_2(n)$ form a basis for $\mathbf{R}^3$. Then we have

$$(u_1 + u_2)((c/2)v_0 + am + bn) = a u_1(m) + b u_2(n) + c u_1(v_0),$$

which again provides the contradiction $\operatorname{range} F = \mathbf{R}^3$. We conclude that N and $\ker u_2$ are comparable. Since N has codimension 2, and $\ker u_2$ has codimension at most 2, it follows that $N \subseteq \ker u_2$, as claimed.

To prove (i) suppose $\lambda v_1 + \mu v_2 \in N$; then $\lambda u_1(v_1) + \mu u_1(v_2) = 0$. If $\lambda \neq 0$ we have $u_1(v_1) \in \operatorname{range} v_2$, which contradicts $u_1(v_1) + u_2(v_2) \notin \operatorname{range} F$. Thus $\lambda = 0$ and $\mu v_2 \in N \subseteq \ker u_2$, which forces $\mu = 0$. Since N has codimension 2, (i) follows.

For (ii) we choose $u \in U$. If $\operatorname{range} u \not\subseteq \operatorname{range} u_1$, we have $N \subseteq \ker u$, as above. If $\operatorname{range} u \subseteq \operatorname{range} u_1$, we check easily that $\operatorname{range}(u + u_2) \not\subseteq \operatorname{range} u_1$, which gives us $N \subseteq \ker(u + u_2)$. Then, for $n \in N$ we have $u(n) = (u + u_2)(n) - u_2(n) = 0$.

Part (iii) is now obvious from parts (i) and (ii). □

Theorem 2.5. *Suppose $F : U \times V \to \mathbf{R}^3$ is a bilinear mapping whose range is not a subspace of $\mathbf{R}^3$. Then there are subspaces U_1 and M of U and V_1 and N of V such that:*

(i) $U = U_1 \oplus M$, $V = V_1 \oplus N$;
(ii) $\dim U_1 = 2 = \dim V_1$;
(iii) $F(U, N) = \{0\} = F(M, V)$;
(iv) $\operatorname{range} F = F(U_1, V_1)$.

Proof. The first three items follow from Lemma 2.4 applied first to U, and then, symmetrically, to V. Item (iv) is then immediate.

It remains to consider the case when $\dim U = \dim V = 2$. □

Theorem 2.6. *Let $\dim U = \dim V = 2$, and let $F : U \times V \to \mathbf{R}^3$ be bilinear. Then the image of F is one of the following:*

(i) *A subspace of dimension at most two;*

(ii) *Two lines through the origin in* $\mathbf{R}^3$*, together with the complement of the plane which contains them;*
(iii) *The exterior and boundary of an elliptic cone centered at the origin.*

Proof. If all associated linear maps have rank at most 1, Theorem 1.3 tells us that $\operatorname{range} F$ is a subspace of dimension at most 1.

Suppose there is an associated linear map with rank 2. Without loss of generality we may take this to be $u_1 \in U$. If $\operatorname{range} u_2 \subseteq \operatorname{range} u_1$ for all $u_2 \in U$, we have $\operatorname{range} F = \operatorname{range} u_1$, which is a subspace of $\mathbf{R}^3$ of rank 2.

There remains the possibility that there exists $u_2 \in U$ such that $\operatorname{range} u_2 \not\subseteq \operatorname{range} u_1$. This possibility breaks into two cases.

Case 1. There exists $u_2 \in U$ such that $\operatorname{rank} u_2 = 1$ and $\operatorname{range} u_2 \not\subseteq \operatorname{range} u_1$. Choose $v_1 \notin \ker u_2$ and $v_2 \in \ker u_2$ such that $u_1(v_2) \neq 0$. An obvious dimension count shows that $\operatorname{range}(u_1) \cap \operatorname{range}(u_2) = \{0\}$. It follows that the vectors $u_1(v_1), u_1(v_2)$ and $u_2(v_1)$ are linearly independent, and therefore a basis for $\mathbf{R}^3$. Clearly u_1 and u_2 are linearly independent and so are v_1 and v_2. Thus

$$F(x_1u_1 + x_2u_2, y_1v_1 + y_2v_2) = x_1y_1u_1(v_1) + x_1y_2u_1(v_2) + x_2y_1u_2(v_1).$$

This is precisely the situation of Example 1.1, so $\operatorname{range} F$ consists of two lines through the origin of $\mathbf{R}^3$ together with the complement of the plane which they determine.

Case 2. There exists $u_2 \in U$ with $\operatorname{range} u_2 \not\subseteq \operatorname{range} u_1$, and all such u_2 have rank 2. Choose $v_1 \in V$ such that $0 \neq u_1(v_1) \in (\operatorname{range} u_1) \cap (\operatorname{range} u_2)$, and choose $v_2 \in V$ such that $u_2(v_2) = u_1(v_1)$. If $v_2 = \alpha v_1$ we have $(u_1 - \alpha u_2)(v_1) = 0$ and $\operatorname{rank}(u_1 - \alpha u_2) \leq 1$. Since $\alpha \neq 0$, $\operatorname{range}(u_1 - \alpha u_2) \not\subseteq \operatorname{range} u_1$, which would put us in the case 1 situation, a contradiction. Hence v_1, and v_2 are linearly independent elements of V, from which we conclude that the three vectors $u_1(v_1) = u_2(v_2)$, $u_1(v_2)$ and $u_2(v_1)$ are linearly independent, and a basis for $\mathbf{R}^3$. We have

$$F(x_1u_1 + x_2u_2, y_1v_1 + y_2v_2) = (x_1y_1 + x_2y_2)u_1(v_1) + x_1y_2u_1(v_2) + x_2y_1u_2(v_1),$$

which is, this time, the situation of Example 1.2. Thus $\operatorname{range} F$ is the exterior and boundary of an elliptic cone centered at the origin.

We have exhausted all possibilities, so our proof is complete. In particular, it is never true that $\operatorname{range} F = \mathbf{R}^3$. □

Remark 2.7. It is interesting to consider Theorem 2.6 from the perspective of the tensor product $\mathbf{R}^2 \bigotimes \mathbf{R}^2$. Our bilinear map $F : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ factors through the (four dimensional) tensor product $\mathbf{R}^2 \bigotimes \mathbf{R}^2$ (which we identify with $\mathbf{R}^4$) by way of the canonical bilinear map of $\mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}^2 \bigotimes \mathbf{R}^2$ given by

$$(a, b) \times (c, d) \mapsto (a, b) \otimes (c, d) = (ac, ad, bc, bd),$$

followed by a linear map into $\mathbf{R}^3$. It is easy to check that the image B of $\mathbf{R}^2 \times \mathbf{R}^2$ in $\mathbf{R}^2 \bigotimes \mathbf{R}^2$ is given by

$$B = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : x_1x_4 = x_2x_3\}.$$

It is then interesting to observe that the linear projection map of $\mathbf{R}^4$ into $\mathbf{R}^3$ given by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3)$ maps B to two lines through the origin in $\mathbf{R}^3$, together with the complement of the plane which they span; while projections of the form $(x_1, x_2, x_3, x_4) \mapsto (\lambda x_1 + \mu x_4, x_2, x_3)$ with $\lambda\mu \neq 0$ map B to the exterior and boundary of an elliptic cone. In other words, bilinear maps of $\mathbf{R}^2 \times \mathbf{R}^2$ into

$\mathbf{R}^3$ whose images are not subspaces are obtained by following the natural map of $\mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}^2 \bigotimes \mathbf{R}^2$ first with an orthogonal projection onto a three dimensional subspace of $\mathbf{R}^2 \bigotimes \mathbf{R}^2$ which contains three or two of the coordinate axes, and then with a linear isomorphism of the range of the projection.

We now combine Theorems 2.5, 2.6 and the obvious fact that the image of a bilinear mapping may be all of $\mathbf{R}^3$ to obtain our general classification.

Theorem 2.8. *Let $F : U \times V \to \mathbf{R}^3$ be bilinear. Then the image of F is one of the following:*

(i) *A subspace of $\mathbf{R}^3$;*
(ii) *Two lines through the origin of $\mathbf{R}^3$ together with the complement of the plane which contains them;*
(iii) *The exterior and boundary of an elliptic cone centered at the origin.*

This result combined with Theorem 1.6 enables us to say more about the images of multilinear mappings into $\mathbf{R}^3$.

Theorem 2.9. *The image of a multilinear mapping F into $\mathbf{R}^3$ either is a subspace, or it contains one of the sets (ii) or (iii) described in Theorem* 2.8.

Proof. If the image of one of the associated bilinear mappings is not a subspace of $\mathbf{R}^3$, then by Theorem 2.8 it is one the specified sets, and is contained in range F. Otherwise, the images of all the associated bilinear mappings are subspaces. If one of them is $\mathbf{R}^3$, then range $F = \mathbf{R}^3$. In the remaining case the ranges are at most two-dimensional, which by Theorem 1.6 forces range F to be a subspace. □

Added May 21, 1995. Theorem 2.8 does does not describe all possible images of arbitrary multilinear mappings. We have examples, for all $n = 3, 4, \ldots$ of n-linear maps into $\mathbf{R}^3$ whose images, by analogy with Theorem 2.8 (ii), consist of k coplanar lines ($2 \leq k \leq n$) through the origin of $\mathbf{R}^3$, together with the complement of the plane which contains them.

Added in Proof. We also have examples of trilinear maps from $\mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}^3$ whose images comprise all of $\mathbf{R}^3$ except for the nonzero points on a single line through the origin.

References

1. Greub, W., *Multilinear Algebra*, 2nd ed., Springer-Verlag New York Inc., 1978. MR **80c:**15017
2. Marcus, M., *Finite Dimensional Multilinear Algebra*. Part 1, Marcel Dekker, Inc., 1973. MR **50:**4599
3. Wojciechowski, P. J., *Lattice-ordered algebras with polynomial inequalities*, Forum Math. 7 (1995), 317–330. MR **96a:**03036

Department of Mathematical Sciences, The University of Texas at El Paso, El Paso, Texas 79968-0514

Current address: S. J. Bernau, College of Science, California State Polytechnic University, 3801 W. Temple Avenue, Pomona, California 91768-4031

E-mail address: S.J. Bernau: `sjbernau@csupomona.edu`

E-mail address: P.J. Wojciechowski: `piotr@math.utep.edu`