PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 124, Number 12, December 1996, Pages 3653–3661
S 0002-9939(96)03470-3

REES ALGEBRAS OF IDEALS WITH LOW CODIMENSION

SUSAN MOREY AND BERND ULRICH

(Communicated by Wolmer V. Vasconcelos)

Abstract. For certain grade two perfect ideals, there is an expected description of the equations of the Rees algebra. In this paper, the Cohen–Macaulayness of the Rees algebra, numerical invariants of the ideal, and a condition on the minors of a presentation matrix of the ideal are shown to be related to the equations having this form.

1. Introduction

Let R be a local Gorenstein ring with infinite residue field and let I be an R-ideal of grade $g > 0$. An important tool for studying the Rees algebra $\mathcal{R} = R[It]$ of I is provided by the concept of a minimal reduction ([18]). Recall that an ideal J contained in I is said to be a reduction of I if $I^{r+1} = JI^r$ for some $r \geq 0$, and that the smallest such r is denoted by $r_J(I)$. Every ideal has a reduction that is minimal with respect to inclusion, called a *minimal reduction*. Now the *analytic spread* $\ell(I)$ of I is the minimal number of generators of J, denoted $\mu(J)$, of some (and hence every) minimal reduction J of I, and the *reduction number* $r(I)$ of I is defined as $\min\{r_J(I)\}$, where J ranges over all minimal reductions. Finally we say that an ideal I with analytic spread ℓ has the *expected reduction number* if $r(I) = \ell - g + 1$, which is the smallest positive value $r(I)$ can take if I is strongly Cohen–Macaulay (for instance, perfect of grade 2 or perfect Gorenstein of grade 3) and satisfies G_ℓ ([22, the proof of 2.5]). Here the condition G_s, s an integer, means that $\mu(I_p) \leq \dim R_p$ for every prime ideal $p \in V(I)$ with $\dim R_p \leq s - 1$ ([3]).

In the context of grade 2 perfect ideals (satisfying G_ℓ), it turns out that the expected reduction number is exactly what characterizes the Cohen–Macaulayness of the Rees algebra $\mathcal{R}$ ([13, 3.5]). Furthermore, ideals having the expected reduction number can be described explicitly in terms of their Hilbert Burch matrix ([21, 5.4]):

Theorem 1.1 (cf. [21, 5.4])**.** *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect R-ideal of grade 2 with $\ell = \ell(I)$, let φ be an n by $n-1$ matrix presenting I, and let φ' be the $n - \ell$ by $n - 1$ matrix consisting of the last*

Received by the editors March 31, 1995 and, in revised form, June 21, 1995.
1991 *Mathematics Subject Classification.* Primary 13A30; Secondary 13H10, 13C14.
Key words and phrases. Analytic spread, Cohen–Macaulay ring, reduction number, Rees algebra.
The second author was partially supported by the NSF.

$n-\ell$ rows of φ. Assume that I satisfies G_ℓ. The following are equivalent:

(a) *$\mathcal{R}$ is Cohen–Macaulay.*
(b) *$r(I) < \ell$ (in which case $r(I) = 0$ or $r(I) = \ell - 1$).*
(c) *After elementary row operations on φ, $I_{n-\ell}(\varphi') = I_{n-\ell}(\varphi)$.*

Here $I_k(\varphi)$ is the ideal generated by the k by k minors of φ.

In this note, we wish to explore how the defining equations of the Rees algebra $\mathcal{R}$ relate to the above conditions. More specifically we fix generators $f_1, \dots, f_n$ of I corresponding to an $n \times m$ presentation matrix φ and choose elements $x_1, \dots, x_s$ generating $I_1(\varphi)$. There exists an $s \times m$ matrix $B(\varphi)$ (called a *Jacobian dual* of φ) having linear entries in the polynomial ring $R[T_1, \dots, T_n]$, so that the matrix equation

$$\underline{T} \cdot \varphi = \underline{x} \cdot B(\varphi) \tag{1}$$

is satisfied. Mapping the variables T_i to the elements $f_i t$ in $\mathcal{R}$, one obtains a presentation

$$\mathcal{R} = R[T_1, \dots, T_n]/Q, \tag{2}$$

where Q obviously contains the entries of the vector in (1). Thus $I_s(B(\varphi))$ annihilates the $\mathcal{R}$-ideal $(\underline{x})\mathcal{R}$, and hence must be contained in Q since $\operatorname{grade}(\underline{x})\mathcal{R} > 0$. Therefore

$$(\underline{x} \cdot B(\varphi), I_s(B(\varphi))) \subset Q, \tag{3}$$

and one says that the *defining ideal* of $\mathcal{R}$ has the *expected form* (in the sense of [23, 3.1]) if equality holds in (3).

In this note we wish to provide a characterization, similar to Theorem 1.1 for when the defining equations of $\mathcal{R}$ have the expected form.

Theorem 1.2. *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect R-ideal of grade 2 with $\ell = \ell(I)$ and $n = \mu(I)$, let φ be an n by $n-1$ matrix presenting I, and let φ' be the $n-\ell$ by $n-1$ matrix consisting of the last $n-\ell$ rows of φ. Assume that I satisfies G_ℓ and that $\mu(I_1(\varphi)) \le \ell$. Write $I_1(\varphi) = (x_1, \dots, x_\ell)$. The following are equivalent:*

(a) *$Q = (\underline{x} \cdot B(\varphi), I_\ell(B(\varphi)))$.*
(b) *$\mathcal{R}$ is Cohen–Macaulay and $I_{n-\ell}(\varphi) = I_1(\varphi)^{n-\ell}$.*
(c) *$r(I) < \ell$ and $I_{n-\ell}(\varphi) = I_1(\varphi)^{n-\ell}$.*
(d) *After elementary row operations on φ, $I_{n-\ell}(\varphi') = I_1(\varphi)^{n-\ell}$.*

Notice that if the presentation matrix φ has linear entries in $k[x_1, \dots, x_d]$, then $B(\varphi)$ can be chosen to have linear entries in $k[T_1, \dots, T_n]$, and thereby becomes uniquely determined. For such matrices we obtain the following generalization of the first author's work [16] as a consequence of Theorem 1.2.

Theorem 1.3. *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over an infinite field, let I be a perfect R-ideal of grade 2 with a linear presentation matrix φ, assume that $\mu(I) > d$ and that I satisfies G_d. Then $\ell(I) = d$, $r(I) = \ell(I) - 1$, $\mathcal{R}$ is Cohen–Macaulay, and $Q = (\underline{x} \cdot B(\varphi), I_d(B(\varphi)))$.*

We will also describe a method for constructing grade 2 perfect ideals with arbitrary analytic spread and arbitrary number of generators that have the expected reduction number (Construction 3.5 and Proposition 3.6). As a consequence, one can easily illustrate the independence of the various conditions in Theorem 1.2.

Finally we will turn to the case of grade 3 Gorenstein ideals with linear presentation matrix, where we will establish an analogue of Theorem 1.1 in a special case (Proposition 4.1).

2. Grade two perfect ideals

In this section we prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. One always has the inclusions

$$I_{n-\ell}(\varphi') \subset I_{n-\ell}(\varphi) \subset I_1(\varphi)^{n-\ell}.$$

Thus part (d) is equivalent to saying that $I_{n-\ell}(\varphi') = I_{n-\ell}(\varphi)$ and $I_{n-\ell}(\varphi) = I_1(\varphi)^{n-\ell}$. Now Theorem 1.1 implies the equivalence of (b), (c), and (d).

Thus it suffices to show that (a) and (b) are equivalent. Notice that if (a) holds, then the ideal $(\underline{x} \cdot B(\varphi), I_\ell(B(\varphi)))$ has grade $n-1$, and hence is Cohen–Macaulay (this is trivial if $n-1 < \ell$, and follows from [10, 3.4] or [4, 4.3] if $n-1 \geq \ell$). Thus in either case we may assume that $\mathcal{R}$ is Cohen–Macaulay. It remains to prove that under this condition, the equality $Q = (\underline{x} \cdot B(\varphi), I_\ell(B(\varphi)))$ holds if and only if $I_{n-\ell}(\varphi) = I_1(\varphi)^{n-\ell}$. To this end we may assume that $\ell < n$ ([8, 9.1]).

Write $d = \dim R$. Since I satisfies G_ℓ, one has $\operatorname{grade} I_1(\varphi) \geq \ell$ and hence $I_1(\varphi)$ is a complete intersection of grade ℓ. Thus, after a purely transcendental extension of the residue field if needed, one can easily construct an R-regular sequence $z_{\ell+1}, \ldots, z_d$ that is regular on the associated graded ring $gr_I(R)$ (since this ring is Cohen–Macaulay) and on $R/I_1(\varphi)^{n-\ell}$ (since $I_1(\varphi)^{n-\ell}$ is perfect of grade ℓ), so that the image of I in $R/(z_{\ell+1}, \ldots, z_d)$ still satisfies G_ℓ (since I satisfies G_ℓ) (see [19, the proof of 3.5] for details). Hence we do not change any of our assumptions and conclusions when passing to the ring $R/(z_{\ell+1}, \ldots, z_d)$, thus reducing to the case $\ell = d$.

To establish the equivalence of the two equalities $Q = (\underline{x} \cdot B(\varphi), I_d(B(\varphi)))$ and $I_{n-d}(\varphi) = I_1(\varphi)^{n-d}$, notice that $Q = (\underline{x} \cdot B(\varphi), I_d(B(\varphi)))$ holds if and only if $Q = (\underline{x} \cdot B(\varphi)) : (\underline{x})$. This follows since

$$(\underline{x} \cdot B(\varphi), I_d(B(\varphi))) \subset (\underline{x} \cdot B(\varphi)) : (\underline{x}),$$

so that in either case $\operatorname{grade}((\underline{x} \cdot B(\varphi)) : (\underline{x})) \geq n-1$, which implies the equality

$$(\underline{x} \cdot B(\varphi)) : (\underline{x}) = (\underline{x} \cdot B(\varphi), I_d(B(\varphi)))$$

([11, 1.8 and 1.5]). Writing $Q_1 = (\underline{x} \cdot B(\varphi))$, it remains to show that $Q = Q_1 : (I_1(\varphi))$ if and only if $I_{n-d}(\varphi) = I_1(\varphi)^{n-d}$.

Since $I_1(\varphi)\mathcal{R}$ has positive grade in $\mathcal{R}$, we have already seen that $Q_1 : (I_1(\varphi)) \subset Q$. Thus $Q = Q_1 : (I_1(\varphi))$ if and only if $I_1(\varphi)Q \subset Q_1$, or equivalently, $I_1(\varphi)\mathcal{A} = 0$, where $\mathcal{A}$ is the ideal of the symmetric algebra $S(I) = R[T_1, \ldots, T_n]/Q_1$ that fits into the exact sequence

$$0 \to \mathcal{A} \longrightarrow S(I) \longrightarrow \mathcal{R} \to 0.$$

After elementary row operations on φ, we may assume that the first d generators of I generate a minimal reduction J. Notice that I/J is presented by the matrix φ'. Furthermore $I_{n-d}(\varphi) = I_{n-d}(\varphi')$ by [21, 2.6] since $r(I) < d$.

Now [13, 4.10] shows that

$$\mathcal{A} = [\mathcal{A}]_d S(I) \quad \text{and} \quad [\mathcal{A}]_d \cong S_d(I/J).$$

Thus $I_1(\varphi)\mathcal{A}=0$ if and only if $I_1(\varphi)$ annihilates $S_d(I/J)=S_d(\text{coker }\varphi')$. But any minimal presentation matrix of $S_d(\text{coker }\varphi')$ has all its entries in the ideal $I_1(\varphi')$, which is contained in $I_1(\varphi)$. Thus there is a natural epimorphism

$$\Pi : S_d(\text{coker }\varphi') \longrightarrow \oplus^s R/I_1(\varphi)$$

with $s=\mu(S_d(\text{coker }\varphi'))$. Now $I_1(\varphi)$ annihilates $S_d(\text{coker }\varphi')$ if and only if Π is an isomorphism, which simply means that we have an equality of (finite) lengths

$$\lambda(S_d(\text{coker }\varphi'))=\lambda(\oplus^s R/I_1(\varphi)).$$

Write $A=R/I_{n-d}(\varphi')$ and $M=S_d(\text{coker }\varphi')$, and let B and N be the corresponding objects obtained when replacing the $n-d$ by $n-1$ matrix φ' by a generic matrix of the same size and localizing at the irrelevant maximal ideal. Since $d \le (n-1)-(n-d)+1$, one knows that N is a Cohen–Macaulay module of rank 1 over the Cohen–Macaulay ring B ([5, p.204] or [11, 3.11]). Let $\underline{z}$ be a B-regular sequence with $B/(\underline{z}) \cong A$. By the Cohen–Macaulayness of N and B one can compare lengths $\lambda(-)$ and multiplicities $e(\underline{z},-)$ to conclude that

$$\lambda(M)=\lambda(N\otimes_B A)=e(\underline{z},N)=(\text{rank}_B N)e(\underline{z},B)=e(\underline{z},B)=\lambda(B/(\underline{z}))=\lambda(A)$$

([17, Theorem 8, p. 311, and Proposition 11, p. 341]). Thus

$$\lambda(S_d(\text{coker }\varphi'))=\lambda(R/I_{n-d}(\varphi'))=\lambda(R/I_{n-d}(\varphi)).$$

On the other hand, since $I_1(\varphi)$ is a complete intersection of grade d,

$$\lambda(\oplus^s R/I_1(\varphi))=\binom{n-1}{d}\lambda(R/I_1(\varphi))=\lambda(R/I_1(\varphi)^{n-d}).$$

Thus

$$\lambda(S_d(\text{coker }\varphi'))=\lambda(\oplus^s R/I_1(\varphi))$$

holds if and only if

$$\lambda(R/I_{n-d}(\varphi))=\lambda(R/I_1(\varphi)^{n-d}),$$

which is equivalent to $I_{n-d}(\varphi)=I_1(\varphi)^{n-d}$. □

Remark 2.1. The above proof also shows that if the assumptions of Theorem 1.2 (a)–(d) hold and if $\ell < n$, then after suitable row operations, $I_{n-\ell}(\varphi')=I_{n-\ell}(\varphi)$ as well as $I_1(\varphi')=I_1(\varphi)$.

Remark 2.2. Under the same assumptions as in Theorem 1.2 (a)–(d), $\mathcal{R}$ is perfect over $R[T_1,\dots,T_n]$.

Proof. Since the ideal $Q=(\underline{x}\cdot B(\varphi), I_\ell(B(\varphi)))$ has grade $n-1$, it is perfect by [4, 4.3] (or [10, 3.4]). □

Remark 2.3. If in Theorem 1.2, $n=\ell+1$, then the condition $I_{n-\ell}(\varphi)=I_1(\varphi)^{n-\ell}$ is trivially satisfied. If in addition, $r(I)<\ell$, or equivalently, $\mathcal{R}$ is Cohen–Macaulay, then $I_1(\varphi)=I_1(\varphi')$ is automatically a complete intersection of grade ℓ (cf. Theorem 1.1 or [19, 4.5]).

We come now to the proof of the main application of Theorem 1.2.

Proof of Theorem 1.3. By [22, 4.3], $\ell(I)=d$, and thus after elementary row operations on φ, the first d generators of I generate a minimal reduction of I. Thus $I_{n-d}(\varphi')$ is primary to the irrelevant maximal ideal $\mathfrak{m}=(x_1,\dots,x_d)$. Since φ' is an $n-d$ by $n-1$ matrix with linear entries we conclude that $I_{n-d}(\varphi') = \mathfrak{m}^{n-d}$. Now the obvious containments

$$I_{n-d}(\varphi') \subset I_1(\varphi)^{n-d} \subset \mathfrak{m}^{n-d}$$

imply the equality $I_{n-d}(\varphi') = I_1(\varphi)^{n-d}$, and our assertions follow from Theorem 1.2 (and Theorem 1.1 (b)). □

Remark 2.4. In Theorem 1.3, the assumption that φ is linear can be weakened to the condition that the nonzero entries of φ are forms of the same degree and $I_1(\varphi)$ is a complete intersection.

Proof. The ideal $I_1(\varphi)$ has grade d, and hence is generated by a regular sequence $z_1, \ldots, z_d$ consisting of forms of the same degree. Thus R is flat over $k[z_1, \ldots, z_d]$, and φ is a matrix with linear entries in the polynomial ring $k[z_1, \ldots, z_d]$. □

3. Examples

In part for the purpose of illustrating the various conditions in Theorem 1.2, we are now going to present a construction which for each pair of integers (s, n), $2 \le s < n$, returns perfect grade 2 ideals I with $\ell(I) = s$ and $\mu(I) = n$ having the expected reduction number and an essentially predescribed Hilbert Burch matrix. This construction is based on [21], but was also inspired by [6].

First we need several lemmas.

Lemma 3.1. *Let R be a Noetherian ring, let $K = (h_1, \ldots, h_m)$ be an R-ideal, let $1 \le r < \min\{t, n\}$, let ψ be an r by t matrix with entries in R, and assume that* $\operatorname{grade} I_r(\psi) \ge \min\{t-r+1,\ \operatorname{grade} K\}$. *Over the polynomial ring*

$$S = R[\{X_{ijk} \mid 1 \le i \le n-r,\ 1 \le j \le t,\ 1 \le k \le m\}]$$

consider elements

$$a_{ij} = \sum_{k=1}^{m} X_{ijk} h_k \quad \text{for} \quad 1 \le i \le n-r,\ 1 \le j \le t,$$

and the n by t matrix

$$\Phi = \begin{pmatrix} a_{ij} \\ \hline \psi \end{pmatrix}.$$

Then $\operatorname{grade} I_{r+1}(\Phi) \ge \min\{t-r+1,\ (n-r)(t-r),\ \operatorname{grade} K\}$.

Proof. Let p be a prime ideal of S with $I_{r+1}(\Phi) \subset p$. We need to prove that

$$\operatorname{grade} I_{r+1}(\Phi) S_p \ge \min\{t-r+1,\ (n-r)(t-r),\ \operatorname{grade} K\}.$$

If $I_r(\psi) \subset p$, we are done by our assumption on ψ. If $I_r(\psi) \not\subset p$, then we may assume that $\Delta \notin p$, where Δ is the r by r minor of ψ involving the first r columns of ψ. Since $\Delta \notin p$, it suffices to show that

$$\operatorname{grade} I_{r+1}(\Phi) S_\Delta \ge \min\{(n-r)(t-r),\ \operatorname{grade} K\}.$$

Thus, replacing R by R_Δ, we may assume that Δ is invertible in R. But then after elementary row operations on ψ we can replace Φ by the matrix

$$\Phi = \left(\begin{array}{ccc|c} \multicolumn{4}{c}{a_{ij}} \\ \hline 1 & & 0 & \\ & \ddots & & b_{\nu\mu} \\ 0 & & 1 & \end{array} \right)$$

with $b_{\nu\mu} \in R$ for $1 \le \nu \le r,\ r+1 \le \mu \le t$. Now applying column operations on Φ we may assume that

$$\Phi = \left(\begin{array}{ccc|c} & a_{i\nu} & & a'_{i\mu} \\ \hline 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \end{array}\right)$$

where

$$a'_{i\mu} = a_{i\mu} - \sum_{\nu=1}^{r} b_{\nu\mu} a_{i\nu} \quad \text{for} \quad r+1 \le \mu \le t.$$

But $a'_{i\mu} = \sum_{k=1}^{m} X'_{i\mu k} h_k$ with $X'_{i\mu k} = X_{i\mu k} - \sum_{\nu=1}^{r} b_{\nu\mu} X_{i\nu k}$. Now there is a homogeneous R-algebra automorphism F on S given by $F(X_{i\nu k}) = X_{i\nu k}$ for $1 \le \nu \le r$, and $F(X_{i\mu k}) = X'_{i\mu k}$ for $r+1 \le \mu \le t$. Applying F^{-1} to Φ, we may assume that

$$\Phi = \left(\begin{array}{ccc|c} & & a_{ij} & \\ \hline 1 & & 0 & \\ & \ddots & & 0 \\ 0 & & 1 & \end{array}\right).$$

But then $I_{r+1}(\Phi)$ contains the S-ideal $(a_{i\mu} \mid 1 \le i \le n-r,\ r+1 \le \mu \le t)$. Finally, by the definition of the elements a_{ij}, the latter ideal has grade at least $\min\{(n-r)(t-r), \operatorname{grade} K\}$. ☐

Lemma 3.2. *With the assumptions of Lemma 3.1,*

$$\operatorname{grade} I_i(\Phi) \ge \min\{t-i+2, (n-i+1)(t-i+1), \operatorname{grade} K\}$$

for $r+1 \le i \le \min\{t, n\}$.

Proof. We induct on the difference $i - r \ge 1$, the case $i - r = 1$ being covered by Lemma 3.1. So let $2 \le i - r \le \min\{t-r, n-r\}$. Applying Lemma 3.1 to the $r+1$ by t matrix

$$\chi = \left(\begin{array}{ccc} a_{n-r,1} & \cdots & a_{n-r,t} \\ \hline & \psi & \end{array}\right)$$

in place of Φ, we conclude that

$$\operatorname{grade} I_{r+1}(\chi) \ge \min\{t-r+1, t-r, \operatorname{grade} K\} = \min\{t-r, \operatorname{grade} K\}.$$

But then our induction hypothesis with ψ replaced by χ yields the assertion for $I_i(\Phi)$. ☐

Lemma 3.3. *Let* $2 \le s < n$. *Use the assumptions of Lemma 3.1 with* $r = n-s$ *and* $t = n-1$ *and suppose in addition that* $\operatorname{grade} K \ge s$. *Then* $\operatorname{grade} I_i(\Phi) \ge n-i+1$ *for* $n-s+1 \le i \le n-1$.

Proof. This is an immediate consequence of Lemma 3.2. ☐

Lemma 3.4 (cf. [9, 2.1 and its proof]). *Let* $(R, \mathfrak{m})$ *be a local Cohen–Macaulay ring with infinite residue field* k, *let* $S = R[X_1, \ldots, X_N]$ *be a polynomial ring, and let* J *be an* S*-ideal contained in* $\mathfrak{m}S$. *If* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_N) \in R^N$, *write* $\overline{\underline{\alpha}} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_N) \in k^N$ *for the residue modulo* $\mathfrak{m}$ *and let* $\Pi_{\underline{\alpha}} : S \to R$ *stand for the* R*-algebra map specializing the variables* X_i *to* α_i.

Then there is a dense open subset U *of* k^N *such that* $\operatorname{grade} \Pi_{\underline{\alpha}}(J) \geq \operatorname{grade} J$ *for every* $\underline{\alpha}$ *with* $\overline{\underline{\alpha}} \in U$.

We are now ready to describe our examples.

Construction 3.5. *Let* R *be a local Gorenstein ring with infinite residue field* k, *let* $2 \leq s < n$, *let* ψ *be an* $n-s$ *by* $n-1$ *matrix with entries in* R *and* $\operatorname{grade} I_{n-s}(\psi) = s$. *For* $1 \leq i \leq s,\ 1 \leq j \leq n-1$, *let* b_{ij} *be general elements in* $I_{n-s}(\psi)$, *define an* n *by* $n-1$ *matrix*

$$\varphi = \left(\frac{b_{ij}}{\psi} \right),$$

and set $I = I_{n-1}(\varphi)$.

To be more precise, take Φ to be the n by $n-1$ matrix of Lemma 3.3 with $K = I_{n-s}(\psi)$. By that lemma, the ideals $I_i(\Phi)$ of the polynomial ring $S = R[X_1, \ldots, X_N]$ have grade at least $n-i+1$ for $n-s+1 \leq i \leq n-1$, and then by Lemma 3.4, there exists a dense open subset U of k^N such that $\operatorname{grade} I_i(\Pi_{\underline{\alpha}}(\Phi)) \geq n-i+1$ for $n-s+1 \leq i \leq n-1$ whenever $\overline{\underline{\alpha}} \in U$. We now take φ to be any of the matrices $\Pi_{\underline{\alpha}}(\Phi)$ with $\overline{\underline{\alpha}} \in U$.

Proposition 3.6. *With the notation of 3.5,* I *is a perfect* R*-ideal of grade* 2 *with presentation matrix* φ, I *satisfies* G_s, $\ell(I) = s$, *and* $r(I) = \ell(I) - 1$. *Furthermore,* $I_i(\varphi) = I_i(\psi)$ *for* $1 \leq i \leq n-s$.

Proof. By construction, $I_i(\varphi) = I_i(\psi)$ for $1 \leq i \leq n-s$. In particular, $I \subset I_{n-s}(\psi) \neq R$, and thus $I \neq R$. Therefore, since $\operatorname{grade} I_i(\varphi) \geq n-i+1$ for $n-s+1 \leq i \leq n-1$, it follows that I is a perfect ideal of grade 2 presented by φ and that I satisfies G_s. Now the equality $I_{n-s}(\varphi) = I_{n-s}(\psi)$ implies that $\ell(I) = s$ and $r(I) = \ell(I) - 1$ ([21, 5.3]). □

We now return to discussing the various conditions of Theorem 1.2. Let I be an ideal as in Proposition 3.6. If we take ψ to be a matrix so that $I_1(\psi)$ is a complete intersection of grade s and $I_{n-s}(\psi) \neq I_1(\psi)^{n-s}$ (such ψ exists whenever $n \neq s+1$), then by Proposition 3.6, all assumptions of Theorem 1.2 are satisfied and I has the expected reduction number, but the defining ideal of the Rees algebra $\mathcal{R}$ fails to have the expected form (as can be seen from Theorem 1.2). On the other hand, choosing ψ to be a matrix with $I_1(\psi)$ not a complete intersection (which is possible as long as $n \neq s+1$), one obtains a grade 2 perfect ideal satisfying G_s that has the expected reduction number, but whose content ideal $I_1(\varphi)$ is no longer a complete intersection (and in particular requires more than s generators). Finally, it has been known for some time that even for $n = s+1$ there exist ideals which satisfy all the assumptions of Theorem 1.2, but do not have the expected reduction number ([1, 8.4]).

4. Grade three Gorenstein ideals with linear presentation

Let R be a local Gorenstein ring with infinite residue field and let I be a grade 3 perfect Gorenstein ideal satisfying G_ℓ with $\ell = \ell(I) < n = \mu(I)$. It is known that if I has the expected reduction number $r(I) = \ell - 2$, then necessarily $n = \ell + 1$ ([21, 5.6]). On the other hand, if $n = \ell + 1$ and $\mathcal{R}$ is Cohen–Macaulay, then $r(I) = \ell - 2$ ([12, 3.8]). Numerous examples seem to suggest that the Cohen–Macaulayness of $\mathcal{R}$ alone suffices to imply $n = \ell + 1$ (and hence $r(I) = \ell - 2$) (cf. also [16, 1.2 and 4.5]). Our next result addresses this problem in a special case.

Proposition 4.1. *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field of characteristic zero, let I be a Gorenstein R-ideal of grade 3 with a linear presentation matrix, assume that $d < \mu(I) \leq 2d$ and that R/I is an isolated singularity. If $\mathcal{R}$ is Cohen–Macaulay, then $\mu(I) = \ell(I) + 1$.*

Proof. Let $f_1, \dots, f_n$ be homogeneous minimal generators of I, and let $K = k(Z_1, \dots, Z_n)$ be a purely transcendental extension of k. Consider the element $f = \sum_{i=1}^n Z_i f_i$ of $IK[x_1, \dots, x_d]$. Further write $\overline{R} = K[x_1, \dots, x_d]/(f)$ and $\overline{I} = I\overline{R}$.

We first prove that $\overline{R}$ is a rational singularity. But f is a homogeneous polynomial of degree

$$\frac{\mu(I) - 1}{2} < d,$$

and hence [7, 3.8] or [24, 2.2] implies that $\overline{R}$ is rational once we have shown that the punctured spectrum of this ring has only rational singularities. So write

$$S = k[x_1, \dots, x_d, Z_1, \dots, Z_n],$$

let q be a prime ideal of S containing f but not containing $(x_1, \dots, x_d)$, and set

$$p = q \cap k[x_1, \dots, x_d].$$

If $I \not\subset p$, then $(S/(f))_q$ is a localization of a polynomial ring over k. Thus, in showing that $(S/(f))_q$ is rational, we may assume that $I \subset p$. But then after a permutation of the generators $f_1, \dots, f_n$, $IR_p = (f_1, f_2, f_3)R_p$ and f_1, f_2, f_3 form part of a regular system of parameters of R_p. Now, applying a homogeneous R_p-automorphism of the polynomial ring $R_p[Z_1, \dots, Z_n]$, we may assume that $f = Z_1 f_1 + Z_2 f_2 + Z_3 f_3$. We denote by Q the image of $qR_p[Z_1, \dots, Z_n]$ under this automorphism and notice that Q still contracts to pR_p. Write $T = R_p[Z_1, \dots, Z_n]_Q$. If $(Z_1, Z_2, Z_3) \not\subset Q$, we may assume that $Z_1 \notin Q$, and hence $(f, f_2, f_3)T = (f_1, f_2, f_3)T$, which implies the regularity of $T/(f)$. If $(Z_1, Z_2, Z_3) \subset Q$, then the associated graded ring (of the maximal ideal) of $T/(f)$ is defined by a single quadric which is a sum of three monomials in disjoint sets of variables. Now [7, 3.9] implies that this associated graded ring has only rational singularities, and then [7, 3.5] gives that $T/(f)$ is a rational singularity. This concludes the proof of the rationality of $\overline{R}$.

Since $\mathcal{R}$ is Cohen–Macaulay and f is general, we know that the associated graded ring $G = gr_I(R) \otimes_k K$ is Cohen–Macaulay and that the leading form f' of f in G is G-regular with $G/(f') \cong gr_{\overline{I}}(\overline{R})$. In particular, $gr_{\overline{I}}(\overline{R})$ is Cohen–Macaulay. Now $\overline{R}$ being a rational singularity, [15, 5.1] implies that $\mathcal{R}(\overline{I})$ is Cohen–Macaulay. This means that the a-invariant $a(gr_{\overline{I}}(\overline{R}))$ is negative ([20, Theorem 1.1]), and hence

$$a(G) = a(G/(f')) - 1 \leq -2.$$

On the other hand,

$$a(G) = \max\{-\operatorname{grade} I, r(I) - \ell(I)\}$$

([19, 3.5] or [2, 4.8], cf. also [14]), which gives $r(I) \leq \ell(I) - 2$. Thus by [21, 5.6], $\mu(I) = \ell(I) + 1$. □

Notice that in the setting of Proposition 4.1, $\ell(I) = d$ ([22, 4.3]). Furthermore, the converse of this proposition holds in that the equality $\mu(I) = \ell(I) + 1$ implies the Cohen–Macaulayness of $\mathcal{R}$ for any grade 3 Gorenstein ideal having a linear presentation matrix and satisfying G_ℓ ([19, 4.11]). In this case one also knows the defining equations of $\mathcal{R}$ ([16, 4.6] or [12, 2.10]).

References

[1] I. M. Aberbach and C. Huneke, An improved Briançon-Skoda theorem with applications to the Cohen-Macaulayness of Rees algebras, Math. Ann. **297** (1993), 343–369. MR **95b:**13005

[2] I. M. Aberbach, C. Huneke, and N. V. Trung, Reduction numbers, Briançon-Skoda theorems and depth of Rees rings, Compositio Math. **97** (1995), 403–434. MR **96g:**13002

[3] M. Artin and M. Nagata, Residual intersections in Cohen–Macaulay rings, J. Math. Kyoto Univ. **12** (1972), 307–323. MR **46:**166

[4] W. Bruns, A. Kustin, and M. Miller, The resolution of the generic residual intersection of a complete intersection, J. Algebra **128** (1990), 214–239. MR **91c:**13009

[5] D. Buchsbaum and D. Eisenbud, Remarks on ideals and resolutions, Symp. Math. **11** (1973), 193–204. MR **49:**2715

[6] A. Corso, C. Polini, and W. V. Vasconcelos, Links of prime ideals, Math. Proc. Cambridge Phil. Soc. **115** (1994), 431–436. MR **95d:**13001

[7] H. Flenner, Rationale quasihomogene Singularitäten, Arch. Math. **36** (1981), 35–44. MR **82g:**14036

[8] J. Herzog, A. Simis, and W. V. Vasconcelos, Koszul homology and blowing–up rings, in *Commutative Algebra*, Proceedings: Trento 1981 (S. Greco and G. Valla, Eds.), Lecture Notes in Pure and Applied Math. **84**, Marcel Dekker, New York, 1983, 79–169. MR **84k:**13015

[9] C. Huneke and B. Ulrich, Algebraic linkage, Duke Math. J. **56** (1988), 415–429. MR **89e:**13023

[10] C. Huneke and B. Ulrich, Residual intersections, J. reine angew. Math. **390** (1988), 1–20. MR **89j:**13024

[11] C. Huneke and B. Ulrich, Generic residual intersections, in *Commutative Algebra*, Proceedings: Salvador 1988 (W. Bruns and A. Simis, Eds.), Lecture Notes in Math. **1430**, Springer, Berlin, 1990, 47–60. MR **91h:**13013

[12] M. Johnson, Second analytic deviation one ideals and their Rees algebras, Pure and Appl. Alg. (to appear)

[13] M. Johnson and B. Ulrich, Artin–Nagata properties and Cohen–Macaulay associated graded rings, Compositio Math., to appear.

[14] B. Johnston and D. Katz, Castelnuovo regularity and graded rings associated to an ideal, Proc. Amer. Math. Soc. **123** (1995), 727–734. MR **95d:**13005

[15] J. Lipman, Cohen–Macaulayness in graded algebras, Math. Research Letters **1** (1994), 149–157. MR **95d:**13006

[16] S. Morey, Equations of blowups of ideals of codimension two and three, J. Pure and Appl. Alg. **109** (1996), 197–211. CMP 96:12

[17] D. G. Northcott, *Lessons on Rings, Modules, and Multiplicities*, Cambridge University Press, London, 1968. MR **38:**144

[18] D. G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Camb. Phil. Soc. **50** (1954), 145–158. MR **15:**596

[19] A. Simis, B. Ulrich, and W. V. Vasconcelos, Cohen–Macaulay Rees algebras and degrees of polynomial equations, Math. Ann. **301** (1995), 421–444. CMP 95:10

[20] N. V. Trung and S. Ikeda, When is the Rees algebra Cohen–Macaulay? Comm. Algebra **17** (1989), 2893–2922. MR **91a:**13009

[21] B. Ulrich, Ideals having the expected reduction number, Amer. J. Math. **118** (1996), 17–38.

[22] B. Ulrich and W. V. Vasconcelos, The equations of Rees algebras of ideals with linear presentation, Math. Z. **214** (1993), 79-92. MR **94g:**13002

[23] W. V. Vasconcelos, On the equations of Rees algebras, J. reine angew. Math. **418** (1991), 189–218. MR **92c:**13006

[24] K. Watanabe, Rational singularities with k*-action, in *Commutative Algebra*, Proceedings: Trento 1981 (S. Greco and G. Valla, Eds.), Lecture Notes in Pure and Applied Math. **84**, Marcel Dekker, New York, 1983, 339–351. MR **84e:**14005

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903
Current address: Department of Mathematics, University of Texas, Austin, Texas 78712
E-mail address: `morey@math.utexas.edu`

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824
E-mail address: `ulrich@math.msu.edu`