

UNIVERSAL CO-ANALYTIC SETS

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ABSTRACT. There is a universal Π_1^1 equivalence relation. The existence of a Π_1^1 set universal for $\underline{\Pi}_1^1$ non-Borel is independent of the usual axioms of mathematics.

§0. INTRODUCTION

One of the fundamental results of descriptive set theory is the existence of a *universal co-analytic subset* of the plane. Recall that a set $A \subseteq X$, where X is Polish, is said to be *analytic*, or Σ_1^1 , if there is a Polish space Y , $B \subseteq Y$ Borel, such that A is a continuous image of B . A set $A \subseteq X$ is said to be *co-analytic*, or Π_1^1 , if $X - A$ is analytic. A *Polish space* is a separable topological space that admits a complete metric—examples include \mathbb{R} , \mathbb{C} , \mathbb{R}^2 , \mathbb{R}^3 , $[0, 1]$, 2^ω , ω^ω , and so on. A set $A \subseteq \mathbb{R} \times \mathbb{R}$ is *universal co-analytic* if it is co-analytic and for all $B \subseteq \mathbb{R}$, B is co-analytic if and only if there exists $z \in \mathbb{R}$ such that

$$B = A^z =_{\text{df}} \{x \in \mathbb{R} : (z, x) \in A\}.$$

This note investigates the possibility of Π_1^1 sets in the plane that are universal for specified subclasses of Π_1^1 . Theorem 1.7 answers a question raised by Kechris at 2.71 of [Ke]: there is a universal Π_1^1 equivalence relation. At 1.1 of [Mi] Miller recalls Mauldin's question regarding the existence of a universal Π_1^1 non-Borel set. 2.4 and 2.9 below show that this cannot be decided in ZFC, thereby providing the answer.

The results in this paper will in fact be stated for the Polish space ω^ω . Similar results can be proved for any other perfect Polish space—such as \mathbb{R} . Except when there are explicit warnings, the notation below will follow that of [Je]. ZFC* will denote a large, fixed fragment of ZFC. Lower case Greek letters will denote ordinals; lower case Roman letters will denote reals, while uppercase Roman letters will stand for sets of reals. General background can be found in [Je] or [Mo].

§1. UNIVERSAL Π_1^1 EQUIVALENCE RELATIONS

In what follows, we will need to not only make use of basic facts about Π_1^1 sets such as the existence of norms, as discussed in Chapter 40 of [Je], but also the finer analysis of Π_1^1 sets we obtain by considering the *effective* theory. For instance, if $z \in \omega^\omega$, $L_\delta[z]$ admissible, $A \in \Pi_1^1$, $\varphi : A \rightarrow \text{WO}$ a Π_1^1 norm, where WO is the collection of codes for wellorderings, and $B \subseteq A$ a $\Sigma_1^1(z)$ set, then the classical

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boundedness theorem tells us that there is some $\alpha < \omega_1^{\forall}$ such that $|\varphi(x)|$, the order type of $\varphi(x)$, is always less than α for any $x \in A$; a proof of this classical theorem, and a discussion of the definitions involved, can be found in §40 of [Je]. We will however need the sharper result that there is some $\alpha < \delta$ that provides the bound, and moreover some such α can be produced in $L_\delta[z]$ from the codes for A and B in a $\Sigma_1^{L_\delta[z]}$ fashion; here the reader can find a discussion of the sharper result at 4A.4 of [Mo]; [Ba] provides a reference for background information regarding admissible sets.

1.1. Definition. $E \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ is said to be a *universal* Π_1^1 equivalence relation if:

- (i) $E \in \Pi_1^1$;
- (ii) $F \subseteq \omega^\omega$ is a Π_1^1 equivalence relation if and only if

$$\exists z \in \omega^\omega (F = E^z =_{\text{df}} \{(x, y) : (z, x, y) \in E\}).$$

1.2. Definition. For $E, F \subseteq \omega^\omega \times \omega^\omega$ both equivalence relations, we write $E \sqsubseteq_c F$ if there exists a continuous injection $f : \omega^\omega \rightarrow \omega^\omega$ such that

$$\forall x, y \in \omega^\omega (xEy \Leftrightarrow f(x)Ff(y)).$$

For $x \in \omega^\omega$, ω_1^x denotes the least x -admissible.

1.3 Lemma. *If there is a universal Π_1^1 equivalence relation, then there is a Π_1^1 equivalence relation $E^* \subseteq \omega^\omega$ such that for any Π_1^1 equivalence F we have $F \sqsubseteq_c E^*$.*

Proof. Let E be the universal Π_1^1 equivalence relation. Let $\langle x_0, y_0 \rangle E^* \langle x_1, y_1 \rangle$ if $y_0 = y_1$ and $x_0 E^{y_0} x_1$. Then for any $F = E^z$, we can let $f(x) = \langle x, z \rangle$ witness the reduction of F to E^* . \square

1.4 Definition. Let $A \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be Π_1^1 , and let $\varphi_A : \omega^\omega \times \omega^\omega \times \omega^\omega \rightarrow \{x \in \omega^\omega : x \text{ codes a linear ordering}\}$ arise from the Kleene-Brouwer ordering, so that φ_A is Δ_1^1 , and $\varphi_A^{-1}(\{x \in \omega^\omega : x \text{ codes a wellordering}\}) = A$. Then for $z \in \omega^\omega$, $\alpha \in \omega_1$, let $A_\alpha^z = \{(y_0, y_1) \in \omega^\omega \times \omega^\omega : (z, y_0, y_1) \in A \wedge \varphi_A(z, y_0, y_1) < \alpha\}$.

1.5 Lemma. *Let φ_A and A be as in 1.4. Suppose A_δ^z is an equivalence relation for some $\delta \in \omega_1$. Suppose γ is z -admissible with $\gamma \leq \delta$. Then:*

$$\{\alpha < \gamma : A_\alpha^z \text{ is an equivalence relation}\}$$

is closed and unbounded in γ .

Proof. Fix $\alpha_0 < \gamma$. Choose a real u such that α_0 is recursive in u and $\mathbb{L}_\gamma[z, u]$ is admissible; for instance, if we let u arise by generically collapsing α_0 to ω over $\mathbb{L}_\gamma[z]$, as in 19.9 of [Je], we will still have that $\mathbb{L}_\gamma[z, u]$ satisfies KP; an alternative method of finding such a u is provided by [Sa]. Using the Σ_1^1 boundedness theorem over $\mathbb{L}_\gamma[z, u]$ we can find $\alpha_1 < \omega_1^{(z, u)}$, $\alpha_1 > \alpha_0$, such that

$$\begin{aligned} \forall x_0, x_1, x_2 \in \omega^\omega \\ (x_0, x_0) \in A_{\alpha_1}^z, \\ (x_0, x_1) \in A_{\alpha_0}^z \Rightarrow (x_1, x_0) \in A_{\alpha_1}^z, \\ (x_0, x_1) \in A_{\alpha_0}^z, (x_1, x_2) \in A_{\alpha_0}^z \Rightarrow (x_0, x_2) \in A_{\alpha_1}^z. \end{aligned}$$

The existence of such an α_1 , working for all x_0, x_1, x_2 in \mathbb{V} , follows as in the remarks preceding 1.1. Repeating the argument, and using the Σ_1 definability of α_{i+1} from α_i , we obtain a sequence $(\alpha_i)_{i \in \omega} \in \mathbb{L}_\gamma[z, u]$ such that for all $i \in \omega$, $\forall x_0, x_1, x_2 \in \omega^\omega$

$$\begin{aligned} \alpha_i &< \alpha_{i+1}, \\ (x_0, x_1) \in A_{\alpha_i}^z &\Rightarrow (x_1, x_0) \in A_{\alpha_{i+1}}^z, \\ (x_0, x_1) \in A_{\alpha_i}^z, (x_1, x_2) \in A_{\alpha_i}^z &\Rightarrow (x_0, x_2) \in A_{\alpha_{i+1}}^z. \end{aligned}$$

Then $\alpha = \bigcup_{i \in \omega} \alpha_i < \gamma$ is as required to show unboundedness.

The fact that $\{\alpha < \gamma : A_\alpha^z \text{ is an equivalence relation}\}$ is closed follows from the well-known observation that an increasing union of equivalence relations again forms an equivalence relation. \square

1.6 Lemma. *Let φ_A and A be as in 1.4. Then for $z \in \omega^\omega$, $\{\alpha \in \omega_1 : A_\alpha^z \text{ is an equivalence relation}\}$ is $\Pi_1^1(z)$ in the codes.*

Proof. A_α^z is uniformly $\Delta_1^1(z, \alpha)$, and thus the statement that it forms an equivalence relation is uniformly $\Pi_1^1(z, \alpha)$. \square

1.7 Theorem. *There exists a universal Π_1^1 equivalence relation.*

Proof. Let $G \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be universal Π_1^1 , so that $B \subseteq \omega^\omega \times \omega^\omega$ is Π_1^1 if and only if there exists $z \in \omega^\omega$ such that $B = G^z =_{\text{df}} \{(x, y) : (z, x, y) \in G\}$. Let $A = G \cup \{(z, x, x) : z, x \in \omega^\omega\}$, so that A^z is the minimal reflexive relation including G^z for all $z \in \omega^\omega$. Fix φ_A for A as in 1.4. Without loss of generality $\varphi_A(z, x, x) = 0$ for all x, z .

Now let $E = \{(z, x, y) \in A : \exists \alpha < \omega_1^{(x, y, z)} \text{ such that } A_\alpha^z \text{ is an equivalence relation, } \varphi_A(z, x, y) < \alpha\}$.

Claim. E is Π_1^1 .

This follows at once from Spector–Gandy in light of 1.6; further details can be found in 4F of [Mo]. $(\square \text{ claim})$

Claim. If G^z is an equivalence relation, then $E^z = G^z$.

This follows at once from the definitions and 1.5. $(\square \text{ claim})$

Claim. $\forall z \in \omega^\omega (E^z \text{ is an equivalence relation})$.

Since $E^z = \bigcup \{A_\alpha^z : \alpha \in \omega_1, A_\alpha^z \text{ is an equivalence relation}\}$ by 1.5. $(\square \text{ claim})$

The last three claims complete the proof. Thus, as a consequence of 1.3, we obtain a Π_1^1 equivalence relation F such that if E is any Π_1^1 equivalence relation, then $E \sqsubseteq_c F$.

§2. UNIVERSAL Π_1^1 NON-BOREL

2.1 Definition. $A \subseteq \omega^\omega \times \omega^\omega$ is said to be *universal for Π_1^1 non-Borel* if:

- (i) $A \in \Pi_1^1$;
- (ii) $B \subseteq \omega^\omega$ is Π_1^1 and non-Borel if and only if $\exists z \in \omega^\omega$ such that

$$B = A^z =_{\text{df}} \{x \in \omega^\omega : (z, x) \in A\}.$$

2.2 Definition. $A \subseteq \omega^\omega \times \omega^\omega$ is said to be *universal for uncountable Σ_2^1* if:

- (i) $A \in \Sigma_2^1$;

(ii) $B \subseteq \omega^\omega$ is an uncountable Σ_2^1 set if and only if $\exists z \in \omega^\omega$ such that

$$B = A^z.$$

2.3 Lemma. *If $\mathbb{V} = \mathbb{L}$, then there is no universal for uncountable Σ_2^1 set.*

Proof. Suppose then that we have $\Phi : \omega_1 \rightarrow \omega^\omega$ Δ_2^1 in the codes and $A \subseteq \omega^\omega$ Σ_2^1 with only uncountable sections. It suffices to find $B \in \Sigma_2^1$ such that $B \subseteq \omega^\omega$, $B \notin \{A^z : z \in \omega^\omega\}$ and B uncountable.

We now define two functions with domain ω_1 by simultaneous induction on $\alpha \in \omega_1$. Given $g(\beta)$ defined for all $\beta \leq \alpha$, we let $f(\alpha)$ be the least $\gamma \in \omega_1$ such that for $w = \Phi(\gamma)$ we have $w \notin \mathbb{L}_{g(\alpha)}$. We then let $g(\alpha + 1)$ be the least δ such that

- (i) $\mathbb{L}_\delta \models \text{ZFC}^*$, $\delta > \alpha$, $\delta > \sup\{g(\beta) : \beta \leq \alpha\}$;
- (ii) $\exists x, y \in \mathbb{L}_\delta (\Phi(\alpha) = x \wedge \forall \beta \leq \alpha (y \neq \Phi(f(\beta))) \wedge y \in A^x)$.

For λ a limit we set $g(\lambda) = \bigcup_{\alpha \in \lambda} g(\alpha)$.

It follows from the construction that f and g are Σ_2^1 in the codes. Thus $B = \{w \in \omega^\omega : \exists \beta \in \omega_1 (w = \Phi(f(\beta)))\}$ is Σ_2^1 , and is uncountable by the assumptions that Φ has uncountable image and A has uncountable sections. Moreover, it also follows from Φ being a surjection that $B \notin \{A^z : z \in \omega^\omega\}$. \square

2.4 Theorem. *If $\mathbb{V} = \mathbb{L}$, then there is no universal for Π_1^1 non-Borel set.*

Proof. Suppose instead that $A \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ is Π_1^1 and for $B \subseteq \omega^\omega \times \omega^\omega$ an arbitrary set in the plane, $B \in \Pi_1^1 - \Delta_1^1$ if and only if $\exists x \in \omega^\omega$ such that $B = A^x$. Using some Δ_1^1 pairing function $\langle \cdot, \cdot \rangle : \omega^\omega \cong \omega^\omega \times \omega^\omega$, this is seen to be equivalent to the existence of a set as in 2.1. It will suffice to demonstrate a contradiction by producing a universal for uncountable Σ_2^1 set.

Let $A_0 \subseteq A$ uniformize A in the third coordinate, so that A_0 is Π_1^1 and

$$\forall x, y \in \omega^\omega (\exists z ((x, y, z) \in A) \Leftrightarrow \exists! z ((x, y, z) \in A_0)).$$

Let $B \subseteq \omega^\omega \times \omega^\omega$ be given by $B = \{(x, y) : \exists z (x, y, z) \in A\} \cup \{(x, y) : \exists z_0, z_1 ((x, z_0, y) \in A_1 \wedge (x, z_1, y) \in A_0, z_0 \neq z_1)\}$. B is Σ_2^1 . Moreover, if $x \in \omega^\omega$, then A^x is non-Borel, hence uncountable, and thus $B^x = \{y : (x, y) \in B\}$ is again uncountable.

Let G be an uncountable Σ_2^1 set. Say $G = \{y : \exists z (y, z) \in G_0\}$ where G_0 is Π_1^1 . Let $G_1 = \{(y, z) : z \text{ codes a wellfounded model, } \mathbb{L}_\alpha \models \text{ZFC}^*, y \in \mathbb{L}_\alpha, \mathbb{L}_\alpha \models \text{“}\exists y_0 \text{ such that } (y, y_0) \in G_0\text{”}\}$. Now let $G_2 \subseteq G_1$ uniformize G_1 in the second coordinate with $G_2 \in \Pi_1^1$.

It follows from the definitions and $\mathbb{V} = \mathbb{L}$ that $G = \{y : \exists z (y, z) \in G_2\}$. Since $\mathbb{V} = \mathbb{L}$ and G is uncountable, G_2 is non-Borel. By assumption on A we can find x such that $G_2 = A^x$. But now since G_2 has been uniformized in the second coordinate $A^x = A_0^x$, and $B^x = G_2$, as required. \square

It will be convenient, during the remainder of the paper, for us to make liberal use of the definitions and theorems from *forcing*. For instance, if M is a structure, \mathbb{P} is a partial order in M , φ a sentence in the language of set theory, then the display

$$M^{\mathbb{P}} \models \varphi$$

is simply a shorthand expression for saying that whenever $G \subseteq \mathbb{P}$ is M -generic, G appearing in a generic extension of \mathbb{V} , then $M[G] \models \varphi$. It is a standard fact

that if $\varphi(\dot{x})$ is Σ_n in the Levy hierarchy, and M satisfies Δ_0 -comprehension and Σ_n -replacement, then the set of $a \in M$ such that

$$M^{\mathbb{P}} \models \varphi(a)$$

is uniformly $\Sigma_n(\mathbb{P})$ over M .

Chapter 3 of [Je] provides an excellent source for these and other basic facts about forcing. There is one respect in which I will go beyond the notation of [Je], and that is in using $\text{Coll}(\omega, \alpha)$ to denote the forcing notion that collapses α to ω ; this partial order, consisting of finite partial functions from ω to α , is discussed at 19.9 of [Je].

It is worth noting in advance that if $A \in \Pi_1^1(x)$, then there is a sense in which the issue of whether A is Borel is internal to $\mathbb{L}[x]$. Note that $A \in \Delta_1^1$ if and only if there is some $\alpha < \omega_1^{\mathbb{V}}$ such that for $\varphi_A : A \rightarrow \text{WO}$ a $\Pi_1^1(x)$ norm on A , $\varphi_A(y)$ has order type less than α for all $y \in A$. However, this statement is $\Pi_2^1(x, \alpha)$, and hence absolute to $\mathbb{L}[x]^{\text{Coll}(\omega, \alpha)}$. Therefore, we can determine whether A is Borel in \mathbb{V} by asking whether A is Borel in a suitably chosen generic extension of $\mathbb{L}[x]$.

We will also be making use of the notion of sharps. This large cardinal hypothesis states that for every $x \in \omega^\omega$ there exists a corresponding $x^\#$. This $x^\#$ codes a theory of indiscernibles for $\mathbb{L}[x]$. Provided we choose an appropriate coding system, we may assume $x^\# \in \omega^\omega$. $x^\#$ will be a $\Pi_2^1(x)$ singleton, and satisfy the key conditions of wellfoundedness and remarkability – as mentioned following 2.5 below. It is perhaps worth remarking that the existence of $x^\#$ implies the existence of a club class of indiscernibles for $\mathbb{L}[x]$; moreover, we will have that for any $n \in \omega$, φ a formula in set theory, $\mathbb{L}[x] \models \varphi(x, \omega_1^{\mathbb{V}}, \dots, \omega_n^{\mathbb{V}})$ if and only if $x^\#$ codes the formula asserting that $\varphi(x, c_1 \cdots c_n)$ holds, where c_1, \dots, c_n are names for the first n many indiscernibles. The reader can find a more complete discussion of sharps in §30 of [Je].

In order to complete the proof of the complementary result for 2.4 it will be necessary to introduce some unpleasant notation. Before giving this notation, it might be helpful to briefly describe the ideas that will underlie the proof. Let us start to assume $\forall x \in \omega^\omega (x^\# \exists)$.

We should begin by thinking of Π_1^1 sets as built up effectively in ω_1 many “stages” – just as in recursion theory we may think of a recursively enumerable set as being a countable union of recursive sets. Now fix A_0 universal Π_1^1 . We describe the sections of our universal Π_1^1 non-Borel set A . Take $z \in \omega^\omega$, and view it as a “guess” at $x^\#$, for some $x \in \omega^\omega$, where $\mathbb{L}[x]$ calculates that A_0^x is non-Borel. As long as we are at a stage where the guess is good, we place the reals in A_0^x into A^z ; as soon as the guess starts going bad, we add in the reals of some fixed Π_1^1 non-Borel set G . If this is all arranged correctly, then we will have A^z non-Borel for all $z \in \omega^\omega$, and for $z = x^\#$, A_0^x non-Borel, we will have $A^z = A_0^x$.

2.5 Notation. For z a theory of indiscernibles with definable Skolem functions, η a linear ordering, let $\Gamma(z, \eta)$ denote the model generated by applying z to the indiscernibles $(c_i)_{i \in \eta}$ and closing under the Skolem functions. (This is what §30 of [Je] would refer to as (Σ, η) , for $\Sigma = z$.)

As mentioned above, two crucial properties of $x^\#$ are remarkability and wellfoundedness. Remarkability asserts that $\Gamma(x^\#, \bar{\eta})$ end extends $\Gamma(x^\#, \eta)$ whenever η has limit type and is an initial segment of $\bar{\eta}$. Wellfoundedness asserts that $\Gamma(x^\#, \alpha)$

is wellfounded for any ordinal α . Remarkability is essentially a syntactical requirement on $x^\#$, while wellfoundedness is $\Pi_2^1(x)$. By the earlier remarks, for $A \in \Pi_1^1(x)$, assuming $x^\#$, A will be Borel if and only if

$$\mathbb{L}[x]^{\text{Coll}(\omega, <\omega_1^y)} \models \text{“}A \text{ Borel”}$$

if and only if

$$\mathbb{L}[x]^{\text{Coll}(\omega, \omega_1^y)} \models \text{“}A \text{ Borel,”}$$

since $\mathbb{L}_{\omega_1^y}(x) \prec \mathbb{L}[x]$; this is in turn equivalent to $x^\#$ coding the formula asserting that $\mathbb{L}[x]^{\text{Coll}(\omega, c_1)}$ satisfies that A is Δ_1^1 .

2.6 Notation. For the rest of the paper, fix $A_0 \subseteq \omega^\omega$ universal Π_1^1 in the usual sense, so that $B \subseteq \omega^\omega$ is Π_1^1 if and only if $\exists z \in \omega^\omega (B = A_0^z)$. Let φ_{A_0} be a Δ_1^1 norm for A_0 , so that $\forall x_0, x_1 \in \omega^\omega ((x_0, x_1) \in A_0 \Leftrightarrow \varphi_{A_0}(x_0, x_1) \in \text{WO})$. Fix $G \subseteq 2^\omega$ Π_1^1 and non-Borel. Fix φ_G a Δ_1^1 norm for G .

2.7 Notation. Let $D \subseteq \omega^\omega$ consist of those reals $z \in \omega^\omega$, such that for some real $x(z) \in \omega^\omega$ we have:

- (i) z codes a theory of indiscernibles for $\mathbb{L}[x(z)]$, and z satisfies the first order requirements of remarkability and indiscernibility;
- (ii) $\Gamma(z, \omega)$ is an ω -model;
- (iii) $x(z) = \langle x_0(z), x_1(z) \rangle$, where $x_0(z)$ codes a pair of continuous functions $f_0, f_1 : 2^\omega \rightarrow \omega^\omega$, such that for all $y \in 2^\omega$, $f_1(y)$ witnesses that $f_0(y) \notin A_0^{x_1(z)}$, and such that f_0 is injective;
- (iv) $\Gamma(z, \omega) \models \text{“}A_0^{x_1(z)} \text{ is non-Borel in } \mathbb{L}[x(z)]^{\text{Coll}(\omega, c_1)} \text{.”}$

It is routine to check that D is Δ_1^1 : As remarked earlier, (i) is essentially syntactical; (ii) is Borel, since $\Gamma(z, \omega)$ can be recovered from z in a Borel fashion; that (iii) is Borel follows from the compactness of 2^ω and the fact that $x(z)$ can be produced in a uniformly $\Delta_1^1(z)$ manner; and finally, (iv) is simply a statement about the theory coded by z , and is therefore arithmetical in z . As in the remarks following 2.5, if z satisfies (iv) and $z = x(z)^\#$, then we will indeed have that $A_0^{x_1(z)}$ is non-Borel. Note that for $z \in D$ it follows from (ii) and the requirement of remarkability that $\Gamma(z, \eta)$ is an ω -model for any linear ordering η .

2.8 Lemma. *For $z \in D$, $z = x(z)^\#$ if and only if for all $\alpha < \omega_1$ the wellfounded part of $\Gamma(z, \alpha + \omega)$ includes α .*

Proof. The condition on the right-hand side is necessary since $\Gamma(x(z)^\#, \alpha + \omega) \supseteq \alpha$. That it is sufficient follows by remarkability. \square

Note that if α is not included in the wellfounded part of $\Gamma(z, \alpha + \omega)$, then, by remarkability, α is not included in the wellfounded part of $\Gamma(z, \beta + \omega)$ for any $\beta > \alpha$.

2.9 Theorem. *If $\forall x \in \omega^\omega (x^\# \exists)$, then there is a universal Π_1^1 non-Borel set.*

Proof. We describe $A \subseteq \omega^\omega \times \omega^\omega$ fiber by fiber. Fix $z \in \omega^\omega$. First of all, if $z \notin D$, then let $A^z = i(G)$, where $i : 2^\omega \rightarrow \omega^\omega$ is the natural injection.

So now suppose $z \in D$. Let $x = \langle x_0, x_1 \rangle = x(z)$ where x_0 codes f_0, f_1 as above. Let $A^z = \{y \in \omega^\omega : y \in A_0^{x_1}, \text{ and for } \|\varphi_{A_0}(x_1, y)\| = \alpha, \Gamma(z, \alpha + \omega) \text{ includes } \alpha\} \cup \{y \in \omega^\omega : \exists w \in 2^\omega, y = f_0(w), w \in G, \text{ and for } \|\varphi_G(w)\| = \alpha, \Gamma(z, \alpha + \omega) \text{ does not include } \alpha\}$.

Claim. $\forall z \in \omega^\omega (A^z \text{ is } \Pi_1^1)$.

This is immediate if $z \notin D$, so suppose instead that $z \in D$. Then the calculation of $f_0^{-1}(y)$, assuming it exists, is uniformly $\Delta_1^1(z, y)$, since f_0 is injective. For $v \in \text{WO}$, the calculation $\Gamma(z, \|v\| + \omega) \supseteq \|v\|$ can be phrased in both a $\Sigma_1^1(z, v)$ and a $\Pi_1^1(z, v)$ fashion, thus showing that $\{y \in A_0^{x_1} : \|\varphi_{A_0}(x_1, y)\| = \alpha \Rightarrow \Gamma(z, \alpha + \omega) \supseteq \alpha\}$ and $\{w \in G : \|\varphi_G(w)\| = \alpha \Rightarrow \Gamma(z, \alpha + \omega) \not\supseteq \alpha\}$ are uniformly $\Pi_1^1(z)$. Since $f_0''[2^\omega]$ is compact, and $\Delta_1^1(x_0)$, this suffices. $(\square \text{ claim})$

Claim. $A \text{ is } \Pi_1^1$.

By the uniformity of the definition of A^z as z ranges over D and $\sim D$. $(\square \text{ claim})$

Claim. $\forall z \in \omega^\omega (A^z \text{ is non-Borel})$.

Again this is immediate if $z \notin D$, so suppose $z \in D$. Now if we in fact have $x^\# = z$, then there will be arbitrarily large $\gamma < \omega_1^V$ such that $\mathbb{L}[x]^{\text{Coll}(\omega, \gamma)} \models \text{“}\exists y \in A_0^{x_1} (\varphi_{A_0}(x_1, y) \geq \gamma)\text{”}$, by condition (iv) of 2.7, and thus $A_0^{x_1}$ must be non-Borel; since $A_0^{x_1} = A^z$ by construction, this suffices.

So, suppose instead that $z \in D$, $z \neq x^\#$, and α is least so that $\Gamma(z, \alpha + \omega) \not\supseteq \alpha$ (the existence of such an α follows by 2.8); note then that $\forall \beta > \alpha \Gamma(z, \beta + \omega) \not\supseteq \alpha$. Therefore A^z consists of the disjoint union of $\{y \in \omega^\omega : y \in A_0^{x_1}, \|\varphi_{A_0}(x_1, y)\| < \alpha\}$ and $\{f_0(w) : w \in G \|\varphi_G(w)\| \geq \alpha\}$. Since $f_0''[2^\omega]$ is disjoint from $\{y \in \omega^\omega : y \in A_0^{x_1}, \|\varphi_{A_0}(x_1, y)\| < \alpha\}$ and since $\{w \in G : \|\varphi_G(w)\| \geq \alpha\}$ is non-Borel, it follows that A^z is non-Borel, as required. $(\square \text{ claim})$

Claim. If $B \subseteq \omega^\omega$ is Π_1^1 non-Borel, then $\exists z \in \omega^\omega (A^z = B)$.

Since B is not Δ_1^1 , $\sim B$ is uncountable, and thus contains a perfect set. Let $B = \{y \in \omega^\omega : \forall x \in \omega^\omega (y, x) \notin B_0\}$ where B_0 is closed, and fix $f_0, f_1 : 2^\omega \rightarrow \omega^\omega$ uniformly continuous with $(f_0(w), f_1(w)) \in B_0$ for all $w \in 2^\omega$. Let x_0 code (f_0, f_1) . Choose x_1 so that $B = A_0^{x_1}$. Let $x = \langle x_0, x_1 \rangle$. It is immediate from the definitions that for $z = x^\#$ we have $A^z = B$. $(\square \text{ claim})$

The last three claims complete our argument by ensuring that A_1 will be our universal Π_1^1 non-Borel set. \square

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