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CLASSIFYING SPACES AND HOMOTOPY SETS OF AXES OF PAIRINGS

KENSHI ISHIGURO

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Dedicated to Professor Teiichi Kobayashi on his 60th birthday

Abstract. We consider the maps between classifying spaces of the form $BK \times BL \to BG$. The main theorem shows that if the restriction map on BL is a weak epimorphism, then the restriction on BK should factor through the classifying spaces of the center of the compact Lie group G. An application implies that BG is an H–space (*Hopf space*) if and only if G is abelian.

For a map $f : Y \to Z$, *the set of the homotopy classes of axes*, denoted by $f^{\perp}(X, Z)$, consists of all homotopy classes of maps $\alpha : X \to Z$ such that there is a map (called *a pairing*) $\mu : X \times Y \to Z$ with restrictions (*axes*) $\mu|_X \simeq \alpha$ and $\mu|_Y \simeq f$, [16]. If Z is an H–space, then $f^{\perp}(X, Z) = [X, Z]$: for the H–multiplication $m : Z \times Z \to Z$, a pairing of f and α is given by the composite map $m \circ (\alpha \times f)$. It follows that, for example, if G, K, and L are compact Lie groups, then $f^{\perp}(K, G) = [K, G]$ with $f : L \to G$. In this paper we will study the classifying space version, namely $f^{\perp}(BK, BG)$ with $f : BL \to BG$. As group theoretical analog indicates, our results will show that few maps in $[BK, BG]$ belong to $f^{\perp}(BK, BG)$, in general. Other forms of axial maps (H–pairing) are studied in [6].

The main theorem deals with the case that the map $f : BL \to BG$ is a weak epimorphism studied in [8]. (We recall the definition in §2.) An obvious example of a weak epimorphism is given by a map $f = B\rho$ induced by a group epimorphism ρ. The unstable Adams operations $\{\psi^k\}$ are also weak epimorphisms.

Theorem 1. *Let L and G be connected compact Lie groups and let K be a compact (not necessarily connected) Lie group. If $f : BL \to BG$ is a weak epimorphism, the following hold:*

1. *If $\alpha \in f^{\perp}(BK, BG)$, then the map α factors through $BZ(G)$ up to homotopy, where $Z(G)$ denotes the center of G.*
2. *Moreover, we have $f^{\perp}(BK, BG) = Hom(K, Z(G))$.*

Taking $L = G$ and $f = id$ (the identity map), our problem asks about possible BK–actions on BG. In fact, this work was motivated by the following result of G. Dula and D. Gottlieb.

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Theorem 2 ([2])**.** *Let $\alpha : X \to Z$ be a map, and X an H–space. Then the following are equivalent.*

(a) *There exists a space Y and a homotopy equivalence $X \times Y \to Z$ such that the orbit map $X \to X \times Y \to Z$ is homotopy equivalent to α.*
(b) *The map α is the orbit of an action of X on Z and $\alpha^{\#} : [Z, X] \to [X, X]$ is onto.*

For instance, let $X = BS^1$ and $Z = BU(n)$. A map $\alpha : BS^1 \to BU(n)$ is induced by a homomorphism $\rho : S^1 \to U(n)$ so that $\alpha = B\rho$. Since $BU(n)$ is indecomposable, [7], Theorem 2 implies that the BS^1–action on $BU(n)$ under the map α does not exist if α is induced by the inclusion $S^1 = U(1) \hookrightarrow U(n)$. Theorem 1 says that the BS^1–action on $BU(n)$ under the map α exists if and only if $\rho(S^1)$ is central in $U(n)$. Generally, a consequence of Theorem 1 shows that BK–actions on BG under α exist if and only if the map $\alpha : BK \to BG$ is induced by a homomorphism $\rho : K \to G$ such that $\rho(K)$ is central in G. In particular, if K is connected and G is semi–simple, there are no non-trivial BK–actions on BG.

Furthermore, if we take $K = L = G$ and $f = \alpha = id$, the problem now asks whether BG is an H–space. Corollary 2.4 implies that BG is an H–space if and only if G is abelian. (G need not be connected.) Related results were obtained in [12] and [1].

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1. Mapping spaces and centralizers of p–toral groups

We give a necessary and sufficient condition that a map $\alpha : X \to Z$ be contained in $f^{\perp}(X, Z)$ in terms of mapping spaces. Then a special case of classifying space version is discussed.

Proposition 1.1. *Suppose X, Y and Z are pointed connected spaces. For a map $f : Y \to Z$, a map $\alpha : X \to Z$ is contained in $f^{\perp}(X, Z)$ if and only if the map f factors through $map(X, Z)_\alpha$, the connected component of the mapping space containing α, under the evaluation map $map(X, Z)_\alpha \xrightarrow{ev} Z$.*

Remark. It is easy to see that $\alpha \in f^{\perp}(X, Z)$ if and only if $f \in \alpha^{\perp}(Y, Z)$. Consequently α must factor through $map(Y, Z)_f$.

Proof of Proposition 1.1. If $\alpha \in f^{\perp}(X, Z)$, there is a pairing $\mu : X \times Y \to Z$ with $\mu|_X \simeq \alpha$ and $\mu|_Y \simeq f$. Let $\overline{\mu} : Y \to map(X, Z)_\alpha$ be the adjoint map of μ. Then we see that the map f is expressed as the composite of the adjoint map and the evaluation map :

$$f : Y \xrightarrow{\overline{\mu}} map(X, Z)_\alpha \xrightarrow{ev} Z$$

since $ev \circ \overline{\mu}(y) = \overline{\mu}(y)(*) = \mu(*, y) = f(y)$.

Conversely, suppose f factors through $map(X, Z)_\alpha$ so that $f \simeq ev \circ \overline{\mu}$ for some map $\overline{\mu} : Y \to map(X, Z)_\alpha$. Let $\epsilon : X \times map(X, Z)_\alpha \to Z$ be the canonical map with $\epsilon(x, \beta) = \beta(x)$ for $x \in X$ and $\beta \in map(X, Z)_\alpha$. Then a pairing is constructed as the composite

$$\mu : X \times Y \xrightarrow{1_X \times \overline{\mu}} X \times map(X, Z)_\alpha \xrightarrow{\epsilon} Z,$$

where 1_X is the identity map of X , since $\mu(x,y) = \epsilon \circ (1_X \times \overline{\mu})(x,y) = \epsilon(x, \overline{\mu}(y)) = \overline{\mu}(y)(x)$ so that $\mu(x,*) = \overline{\mu}(*)(x) = \alpha(x)$ and $\mu(*,y) = \epsilon(*, \overline{\mu}(y)) = ev \circ \overline{\mu}(y) \simeq f(y)$. □

We next consider the "BG"–version. It is worth to recall some property of homomorphisms. Suppose $\rho : L \to G$ and $\alpha : K \to G$ are homomorphisms. If there is a *pairing* homomorphism $\mu : K \times L \to G$ with $\mu|_K = \alpha$ and $\mu|_L = \rho$, then the image $\rho(L)$ must be contained in the centralizer of α in G, denoted by $C_G(\alpha)$. The following is a "BG"–analog at a prime p. If a map $\alpha : BK \to BG$ is induced by a homomorphism, let $C_G(\alpha)$ denote the centralizer of the homomorphism. For a p–toral group K (a group extension of a torus by a finite p–group), it is known, [5] and [14], that any map $\alpha : BK \to BG$ (at p) has the form $\alpha = B\eta$ $(\alpha = (B\eta)^\wedge_p)$ for some homomorphism η. Let $BG^\wedge_p$ denote the p–completion of BG. Since $map(BK, BG)_\alpha$ is p–equivalent to $BC_G(\alpha)$, the following is immediate from Proposition 1.1.

Corollary 1.2. *Suppose K is p–toral. Then $\alpha \in f^\perp(BK^\wedge_p, BG^\wedge_p)$ if and only if the map f factors through $BC_G(\alpha)^\wedge_p$ up to homotopy, under the map $BC_G(\alpha)^\wedge_p \to BG^\wedge_p$ induced by the inclusion.*

As the above indicates, if the mapping space is computable, then the set of the homotopy classes of axes $f^\perp(X,Z)$ would be determined. It is, however, hard to compute $map(X,Z)_\alpha$ or $map(Y,Z)_f$ in general. Thus our work is to consider the problem, with or without calculation of mapping spaces.

2. Weak epimorphisms and the proof of Theorem 1

We will begin with some lemmas to prove Theorem 1.

Lemma 2.1. *Let R and S be rings of polynomials in n indeterminates over a field. If there is an epimorphsim $\varphi : R \to S$, then φ is an isomorphsim.*

Proof. Let $\mathfrak{p} = Ker\ \varphi$ so that $R/\mathfrak{p} \cong S$. We need to show that $\mathfrak{p} = 0$. The ideal $\mathfrak{p}$ is prime, since $R/\mathfrak{p}$ is an integral domain. Consequently, if $dim(R)$ denotes the Krull dimension of R, then

$$height(\mathfrak{p}) + dim(R/\mathfrak{p}) = dim(R).$$

It follows that

$$\begin{aligned} dim(S) &= dim(R/\mathfrak{p}) \\ &= dim(R) - height(\mathfrak{p}). \end{aligned}$$

Since $dim(S) = dim(R) = n$, we see that $height(\mathfrak{p}) = 0$. In a domain, this means $\mathfrak{p} = 0$. □

Recall that, for any compact connected Lie group G, there is a covering $\gamma_G \to \widetilde{G} \to G$ such that $\widetilde{G} = G_s \times T$ is a product of a simply–connected Lie group and a torus, and that γ_G is a finite central subgroup of $\widetilde{G}$. Such a covering is called *a universal finite covering*.

Lemma 2.2. *Suppose G is a connected compact Lie group and H is a connected closed subgroup with inclusion $\iota : H \to G$. If the p–completed map $(B\iota)^\wedge_p : (BH)^\wedge_p \to (BG)^\wedge_p$ is rationally equivalent, then $H = G$.*

Proof. Since H and G are connected, we can find universal finite coverings $\widetilde{H}$ and $\widetilde{G}$ so that we have the the commutative diagram

$$\begin{array}{ccc} \widetilde{H} & \xrightarrow{\ \widetilde{\iota}\ } & \widetilde{G} \\ \downarrow & & \downarrow \\ H & \xrightarrow{\ \iota\ } & G \end{array}$$

Note that $\widetilde{H}$ and $\widetilde{G}$ are products of simply-connected simple Lie groups and a torus, $\widetilde{H} = \prod_i \widetilde{H}_i$ and $\widetilde{G} = \prod_j \widetilde{G}_j$ respectively. Since the map $(B\iota)_p^\wedge : (BH)_p^\wedge \to (BG)_p^\wedge$ is rationally equivalent, for each $\widetilde{G}_j$, by [8], we can find $\widetilde{H}_i$ such that the restricted map $(B\widetilde{H}_i)_p^\wedge \to (B\widetilde{G}_j)_p^\wedge$ is a rational equivalence, and the map $(B\widetilde{\iota})_p^\wedge$ is diagonal. Using the classification of compact simply-connected simple Lie groups and their maximal rank subgroups, one can show that $\widetilde{\iota}(\widetilde{H}_i) = \widetilde{G}_j$. Hence $\widetilde{\iota} : \widetilde{H} \longrightarrow \widetilde{G}$ is onto. Consequently the inclusion $\iota : H \to G$ is an epimorphism, and therefore $H = G$. □

Lemma 2.3. *Let G be a compact Lie group with center $Z(G)$. Then the map induced by the inclusion $i : Z(G) \hookrightarrow G$*

$$(Bi)_\# : [BK, BZ(G)] \to [BK, BG]$$

is one–to–one for any compact Lie group K.

Proof. Notice that $[BK, BZ(G)] = Hom(K, Z(G))$, since $Z(G)$ is abelian, [11]. For $\rho_1, \rho_2 \in Hom(K, Z(G))$, suppose $(Bi)_\#(B\rho_1) = (Bi)_\#(B\rho_2)$. Thus, for each p–toral subgroup K_q of K, we have $\rho_1(x) = \rho_2(x)$ for any $x \in K_q$, since the images of ρ_1 and ρ_2 are contained in the center $Z(G)$. Define $\rho \in Hom(K, Z(G))$ by $\rho(k) = \rho_1(k)\rho_2(k)^{-1}$ for any $k \in K$ so that $\rho|_{K_q} = 0$ for any p–toral subgroup K_q. Since $[BK_p^\wedge, BG_p^\wedge] = [\underrightarrow{\text{holim}}(BK_q)_p^\wedge, BG_p^\wedge] = \underleftarrow{\text{holim}}[(BK_q)_p^\wedge, BG_p^\wedge]$, it follows that $\rho = 0$. Consequently $\rho_1 = \rho_2$, and therefore $B\rho_1 = B\rho_2$. □

Remark. For $\iota : H \hookrightarrow G$ the map $[BK, BH] \to [BK, BG]$ need not be one–to–one. A counterexample is given by $[B\mathbb{Z}/2, BT^2] \not\hookrightarrow [B\mathbb{Z}/2, BU(2)]$ where T^2 is a maximal torus of $U(2)$.

We consider $f^\perp(BK, BG)$ when $f : BL \to BG$ is a weak epimorphism, defined as follows. Suppose L and G are connected. A map $BL \to BG$ or $BL_p^\wedge \to BG_p^\wedge$ is called a *weak epimorphism*, [8], if there exists a fibration $Z \to BL \to BG$ or $Z \to BL_p^\wedge \to BG_p^\wedge$ such that $H^*(\Omega Z; \mathbb{Q})$ is a finite dimensional $\mathbb{Q}$–module or that $H^*(\Omega Z; \mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}_p^\wedge$–module, respectively. Some examples were given in the introduction.

Proof of Theorem 1. First we show Part (1). Fix a prime p. Suppose K_q is a p–toral subgroup of K and $\alpha_q = \alpha_p^\wedge|_{(BK_q)_p^\wedge}$. Corollary 1.2 shows that the p–completed map $f_p^\wedge : BL_p^\wedge \to BG_p^\wedge$ factors through $BC_G(\alpha_q)_p^\wedge$ up to homotopy. Since K_q is p–toral, $\pi_0(C_G(\alpha_q))$ is a finite p–group. Since $BL_p^\wedge$ is 1–connected, the map $f_p^\wedge$ factors through the p–completed classifying space of the connected component of the centralizer $C_G(\alpha_q)$. Since f is a weak epimorphism, the map $f_p^\wedge$ is also a weak epimorphism. Consequently there is a fibration $Z \to BL_p^\wedge \to BG_p^\wedge$ satisfying certain finiteness conditions mentioned above. Let H be the connected component of $C_G(\alpha_q)$ with inclusion $\iota : H \to G$. Then we have the commutative diagram

$$\begin{array}{ccccc} Y & \longrightarrow & BL_p^\wedge & \xrightarrow{\overline{f}_p^\wedge} & BH_p^\wedge \\ \downarrow & & \| & & \downarrow (B\iota)_p^\wedge \\ Z & \longrightarrow & BL_p^\wedge & \xrightarrow{f_p^\wedge} & BG_p^\wedge \end{array}$$

Here Y is a homotopy fibre. Since $f_p^\wedge$ is a weak epimorphism, we see that $H^*(BL_p^\wedge;\mathbb{Q}_p^\wedge) \cong H^*(BG_p^\wedge;\mathbb{Q}_p^\wedge) \otimes H^*(Z;\mathbb{Q}_p^\wedge)$. Thus, for the induced homomorphism $(f_p^\wedge)^* : H^*(BG_p^\wedge;\mathbb{Q}_p^\wedge) \to H^*(BL_p^\wedge;\mathbb{Q}_p^\wedge)$ there is a homomorphism (left inverse) $r : H^*(BL_p^\wedge;\mathbb{Q}_p^\wedge) \to H^*(BG_p^\wedge;\mathbb{Q}_p^\wedge)$ with $r \circ (f_p^\wedge)^* = 1$. Consequently $r \circ (\overline{f}_p^\wedge)^* \circ ((B\iota)_p^\wedge)^* = 1$. Let φ be the endomorphism $((B\iota)_p^\wedge)^* \circ r \circ (\overline{f}_p^\wedge)^*$ of the polynomial ring $H^*(BH_p^\wedge;\mathbb{Q}_p^\wedge)$. Since $r \circ (\overline{f}_p^\wedge)^*$ is onto and $((B\iota)_p^\wedge)^*$ is one–to–one, the Krull dimension of the image of φ is equal to $rank(G)$. Hence the endomorphism φ is an isomorphsim and therefore $((B\iota)_p^\wedge)^*$ is onto. Lemma 2.1 implies that $BH_p^\wedge$ is rationally equivalent to $BG_p^\wedge$. From Lemma 2.2, we see that $C_G(\alpha_q) = G$. This implies that the map α_q is induced by a homomorphism into the center $Z(G)$. Hence $\alpha|_{(BK_q)_p^\wedge}$ factors through $BZ(G)_p^\wedge$ for any p–toral subgroup K_q .

Since $BK_p^\wedge \simeq \underrightarrow{\text{holim}}(BK_q)_p^\wedge$, [9], the map $\alpha_p^\wedge : BK_p^\wedge \to BG_p^\wedge$ factors through $BZ(G)_p^\wedge$ for any p. Let K_0 denote the connected component of K containing the identity so that there is an exact sequence $K_0 \to K \xrightarrow{\pi} \pi_0 K$. Consider the canonical fibration $BZ(G) \to BG \xrightarrow{Bq} B(G/Z(G))$. Since $(Bq \cdot \alpha|_{BK_0})_p^\wedge$ is null homotopic, the map $Bq \cdot \alpha|_{BK_0}$ is rationally null homotopic. Since K_0 is connected, then $Bq \cdot \alpha|_{BK_0} = 0$. This induces a map $\overline{\alpha} : B\pi_0 K \to B(G/Z(G))$ with $Bq \cdot \alpha = \overline{\alpha} \cdot B\pi$. Since $(Bq \cdot \alpha)_p^\wedge = 0$ for any p and the homomorphism π is onto, the restriction of $\overline{\alpha}$ on each p–Sylow subgroup of the finite group $\pi_0 K$ is null homotopic. This implies $\overline{\alpha} = 0$. Consequently α factors through $BZ(G)$.

Next we show Part (2). The canonical map $(Bi)_\# : [BK, BZ(G)] \to [BK, BG]$ factors through $f^\perp(BK, BG)$. Let Φ be the map $[BK, BZ(G)] \to f^\perp(BK, BG)$ and let Ψ be the map $f^\perp(BK, BG) \to [BK, BG]$ so that $(Bi)_\# = \Psi \circ \Phi$. Part (1) shows Φ is onto. Lemma 2.3 shows $(Bi)_\#$ is one–to–one. This implies Φ is one–to–one. Thus Φ is bijective. Since $[BK, BZ(G)] = Hom(K, Z(G))$, this completes the proof. □

Remark. As shown in §1, there is a strong relationship between the set of the homotopy classes of axes $f^\perp(BK, BG)$ and $map(BL, BG)_f$. Theorem 1 seems to indicate that $map(BL, BG)_f$ can be homotopy equivalent to $BZ(G)$ when $f : BL \to BG$ is a weak epimorphism. One can show, however, that $map(BS^3, BS^3)_{\psi^k} \not\simeq BZ(S^3)$ (without p–completion), from the following result of Dwyer–Mislin [3] : The space of pointed maps $map_*(BS^3, BS^3)_{\psi^k}$ is homotopy equivalent to Sullivan's profinite completion $SO(3)^\wedge$. What to ask is the p–equivalence between $map(BL, BG)_f$ and $BZ(G)$. A result of Jackowski–McClure–Oliver [9] and Notbohm [15] shows that if $f : BG \to BG$ is a self–equivalence, the map $BZ(G) \to map(BG, BG)_f$ is a mod p equivalence for any prime p. A related result can be found in [4].

Corollary 2.4. *Suppose G is a compact (not necessarily connected) Lie group. If BG is an H–space, then G is abelian.*

Proof. If BG is an H–space, then $(id)^{\perp}(BG, BG) = [BG, BG]$. Suppose first that G is connected. Taking $\alpha = id$ in Theorem 1, we see that the identity map of BG factors through $BZ(G)$. Consequently $G = Z(G)$, and therefore G is a torus. In general, there is the exact sequence $G_0 \to G \to \pi_0 G$. If BG is an H–space, from the fibration $BG_0 \to BG \to B\pi_0 G$ it follows that both BG_0 and $B\pi_0 G$ are also H–spaces. This implies that the groups G_0 and $\pi_0 G$ are abelian, since $\pi_0 G$ is a finite group so that $B\pi_0 G = K(\pi_0 G, 1)$. Let $\pi = \pi_0 G$ and let π_p denote a p-Sylow subgroup of the finite group π. Suppose G_p is the subgroup of G over π_p:

$$\begin{array}{ccccc} G_0 & \longrightarrow & G & \longrightarrow & \pi_0 G = \pi \\ \| & & \uparrow & & \uparrow \\ G_0 & \longrightarrow & G_p & \longrightarrow & \pi_p \end{array}$$

Since BG is an H–space, the space BG_p is also an H–space. Notice that G_p is p–toral. Consequently the H–multiplication on BG_p is induced by a homomorphism $G_p \times G_p \to G_p$. This implies that G_p is abelian for any prime p, and therefore $G(= G_0 \times \pi_0 G)$ is abelian. □

Remark. The p–completed version of Corollary 2.4 does not hold. For instance, the symmetric group Σ_3 is non-abelian, but the 2–completed classifying space $(B\Sigma_3)^\wedge_2 = B\mathbb{Z}/2$ is an H–space.

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Department of Mathematics, Fukuoka University, Fukuoka 814-80, Japan
E-mail address: kenshi@ssat.fukuoka-u.ac.jp