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THE BOUNDEDNESS OF RIESZ s–TRANSFORMS OF MEASURES IN $\mathbb{R}^n$

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ABSTRACT. Let μ be a finite nonzero Borel measure in $\mathbb{R}^n$ satisfying $0 < c^{-1}r^s \le \mu B(x,r) \le cr^s < \infty$ for all $x \in \operatorname{spt}\mu$ and $0 < r \le 1$ and some $c > 0$. If the Riesz s-transform

$$\mathcal{C}_{s,\mu}(x) = \int \frac{y-x}{|y-x|^{s+1}}\, d\mu y$$

is essentially bounded, then s is an integer. We also give a related result on the L^2-boundedness.

1. INTRODUCTION

The Cauchy transform of a finite nonnegative Borel measure μ on $\mathbb{C}$ is the analytic function $\mathcal{C}_\mu$ defined in the complement of the support of μ, $\operatorname{spt}\mu$, by

$$\mathcal{C}_\mu(z) = \int \frac{d\mu\zeta}{\zeta - z}.$$

Often problems in complex analysis can be reduced to the study of relations between $\mathcal{C}_\mu$ and the geometric properties of μ. In this paper we investigate the behavior of an s-variant $\mathcal{C}_{s,\mu}$ of the Riesz transform in $\mathbb{R}^n$,

$$\mathcal{C}_{s,\mu}(x) = \int \frac{y-x}{|y-x|^{s+1}}\, d\mu y,$$

where $0 < s < n$. This is defined for all $x \in \mathbb{R}^n$ for which

$$\int \frac{d\mu y}{|y-x|^s} < \infty.$$

This transform is called Riesz s-transform. In many cases $\mathcal{C}_{s,\mu}$ does not exist in the usual sense for $x \in \operatorname{spt}\mu$ but it may exist as a principal value

$$\mathcal{C}_{s,\mu}(x) = \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{y-x}{|y-x|^{s+1}}\, d\mu y,$$

where $B(x,\epsilon)$ is the x-centered closed ball with radius ϵ.

David and Semmes (see [DS1, DS2]) have studied singular integrals on integral dimensional surfaces and rectifiable sets. $\mathcal{C}_{s,\mu}$ defines a bounded operator on $L^2(\mu) \to L^2(\mu)$ if s is an integer and μ is a surface measure on a sufficiently nice s-dimensional surface, e.g. a Lipschitz graph. In this case the principal values $\mathcal{C}_{s,\mu}(x)$ also exist μ a.e.

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Mattila and Preiss (see [MP]) have shown that if for μ almost all $x \in \mathbb{R}^n$ $0 < \liminf r^{-s}\mu B(x,r) \le \limsup r^{-s}\mu B(x,r) < \infty$ and the principal values $\mathcal{C}_{s,\mu}(x)$ exist μ a.e., then s is an integer and μ is s-rectifiable, i.e. there exist s-dimensional C^1 submanifolds M_i such that $\mu(\mathbb{R}^n \setminus \cup_i M_i) = 0$. In this paper we show that the essential boundedness of $\mathcal{C}_{s,\mu}$ implies that s is an integer. As a consequence it follows that also the L^2-boundedness forces s to be an integer. It is not known whether μ is s-rectifiable. David and Semmes have shown in [DS1] that the L^2-boundedness of a large class of singular integral operators implies the rectifiability of μ, and it forces s to be an integer.

2. Preliminary lemmas

Let μ be a finite Borel measure on $\mathbb{R}^n$, $\epsilon > 0$ and $x \in \mathbb{R}^n$. The truncated Riesz s-transform is defined by

$$\mathcal{C}^{\epsilon}_{s,\mu}(x) = \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{y-x}{|y-x|^{s+1}}\, d\mu y$$

and the maximal truncated transform by

$$\mathcal{C}^{*}_{s,\mu}(x) = \sup_{\epsilon>0} |\mathcal{C}^{\epsilon}_{s,\mu}(x)|.$$

If μ is only locally finite, we shall use the modified maximal function

$$\tilde{\mathcal{C}}^{*}_{s,\mu}(x) = \sup_{0<r<1} \left| \int_{B(x,1)\setminus B(x,r)} \frac{y-x}{|y-x|^{s+1}}\, d\mu y \right|.$$

For a finite measure μ, $\tilde{\mathcal{C}}^{*}_{s,\mu}$ is bounded if and only if $\mathcal{C}^{*}_{s,\mu}$ is bounded. We also denote

$$\underline{D}^s(\mu,x) = \liminf_{r\downarrow 0} r^{-s}\mu B(x,r)$$

and

$$\overline{D}^s(\mu,x) = \limsup_{r\downarrow 0} r^{-s}\mu B(x,r).$$

The Riesz s-transform $\mathcal{C}_{s,\mu}$ has the following properties.

Lemma 1. *The Riesz s-transform $\mathcal{C}_{s,\mu}$ of a finite Borel measure μ is defined for $\mathcal{L}^n$ a.e. x and $\mathcal{C}_{s,\mu} \in L^p_{loc}$ for $1 \le p < n/s$.*

The proof is straightforward and we omit it.

The proof of the following lemma was suggested by Stephen Semmes.

Lemma 2. *Suppose that μ is a finite Borel measure in $\mathbb{R}^n$ satisfying $\mu B(x,r) \le r^s$ for all $x \in \mathbb{R}^n$ and $r > 0$. Then for all $x \in \mathbb{R}^n$*

$$\mathcal{C}^{*}_{s,\mu}(x) \le \|\mathcal{C}_{s,\mu}\|_\infty + C,$$

where C is a constant depending on n and s and $\|\cdot\|_\infty$ is the L^∞ norm.

Proof. This is a variant of a well-known inequality of Cotlar (see e.g. [J]).

We give here the main steps of the proof. If $\|\mathcal{C}_{s,\mu}\|_\infty = \infty$ the result is clear. Suppose that $\|\mathcal{C}_{s,\mu}\|_\infty < \infty$.

Let $\epsilon > 0$ and $x \in \mathbb{R}^n$. Consider the mean value integral

$$\frac{1}{\alpha_n(\epsilon/2)^n}\int_{B(x,\epsilon/2)}\int_{B(x,\epsilon)}\frac{d\mu y}{|y-z|^s}\,d\mathcal{L}^n z \le \frac{1}{\alpha_n(\epsilon/2)^n}\int_{B(x,\epsilon)}\int_{B(0,2\epsilon)}\frac{d\mathcal{L}^n z}{|z|^s}\,d\mu y \le \frac{n}{n-s}2^{2n-s},$$

where $\alpha_n = \mathcal{L}^n B(0,1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Hence there is a point $z \in B(x,\epsilon/2)$ such that

$$\int_{B(x,\epsilon)}\frac{d\mu y}{|y-z|^s} \le \frac{n}{n-s}2^{2n-s} \tag{1}$$

and $|\mathcal{C}_{s,\mu}(z)| \le \|\mathcal{C}_{s,\mu}\|_\infty$.

Now we can estimate

$$|\mathcal{C}^\epsilon_{s,\mu}(x) - \mathcal{C}_{s,\mu}(z)| \le \int_{\mathbb{R}^n\smallsetminus B(x,\epsilon)}\left|\frac{y-x}{|y-x|^{s+1}} - \frac{y-z}{|y-z|^{s+1}}\right| d\mu y + \int_{B(x,\epsilon)}\frac{d\mu y}{|y-z|^s}.$$

We estimate the first term on the right-hand side. Divide the set $\mathbb{R}^n \smallsetminus B(x,\epsilon)$ into annuli $B(x,2^{i+1}\epsilon)\smallsetminus B(x,2^i\epsilon)$, $i = 0,1,2,\dots$. Using the mean value theorem and the fact that $|D(x|x|^{-s-1})| \le C|x|^{-s-1}$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^n\smallsetminus B(x,\epsilon)}\left|\frac{y-x}{|y-x|^{s+1}} - \frac{y-z}{|y-z|^{s+1}}\right| d\mu y &\le \int_{\mathbb{R}^n\smallsetminus B(x,\epsilon)}\left|D\frac{\xi(y)}{|\xi(y)|^{s+1}}\right||x-z|\,d\mu y\\ &\le C\sum_{i=0}^{\infty}\int_{B(x,2^{i+1}\epsilon)\smallsetminus B(x,2^i\epsilon)}\frac{|x-z|}{|\xi(y)|^{s+1}}\,d\mu y\\ &\le C2^{2s+1},\end{aligned} \tag{2}$$

where $\xi(y)$ is a point on the line segment between the points $y-x$ and $y-z$. Using the estimates (1) and (2) we get

$$|\mathcal{C}^\epsilon_{s,\mu}(x)| \le C2^{2s+1} + \frac{n}{n-s}2^{2n-s} + \|\mathcal{C}_{s,\mu}\|_\infty$$

and so $\mathcal{C}^*_{s,\mu}(x) \le \|\mathcal{C}_{s,\mu}\|_\infty + C2^{2s+1} + \frac{n}{n-s}2^{2n-s}$. □

There is a useful variant of this lemma which states that under the assumptions on μ in Lemma 2 $\tilde{\mathcal{C}}^*_{s,\mu}$ is bounded everywhere if it is bounded on the support of μ.

Lemma 3. *Suppose that μ is a locally finite Borel measure in $\mathbb{R}^n$ satisfying $\mu B(x,r) \le r^s$ for all $x \in \mathbb{R}^n$ and $r > 0$. If $\tilde{\mathcal{C}}^*_{s,\mu}$ is bounded on the support of μ, then $\tilde{\mathcal{C}}^*_{s,\mu}$ is bounded on $\mathbb{R}^n$.*

Proof. The proof is similar to the proof of the preceding lemma.

Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Denote $d = \operatorname{dist}(x, \operatorname{spt}\mu) = |x - z|$, $z \in \operatorname{spt}\mu$. If $\epsilon \le d < 1/2$ we have

$$\begin{aligned}\left|\int_{B(x,1)\setminus B(x,\epsilon)} \frac{y-x}{|y-x|^{s+1}} d\mu y\right| \le & \left|\int_{B(z,2d)} \frac{y-x}{|y-x|^{s+1}} d\mu y\right| \\ &+ \left|\int_{(B(x,1)\setminus B(z,1))\cup(B(z,1)\setminus B(x,1))} \frac{y-x}{|y-x|^{s+1}} d\mu y\right| \\ &+ \int_{B(z,1)\setminus B(z,2d)} \left|\frac{y-x}{|y-x|^{s+1}} - \frac{y-z}{|y-z|^{s+1}}\right| d\mu y \\ &+ \left|\int_{B(z,1)\setminus B(z,2d)} \frac{y-z}{|y-z|^{s+1}}\right| \\ &\le 2^s + 4^s + C2^{2s+1} + 2\tilde{\mathcal{C}}^*_{s,\mu}(z),\end{aligned}$$

where the estimate for the first term is obvious, in the second we have used the fact that the domain of integration is contained in $B(x,2) \setminus B(x,1/2)$ and the third term is estimated by (2). If $d \ge 1/2$ or $\epsilon > 2d$, similar estimates give the result. □

3. Tangent measures and Riesz s-transforms

Let μ_i, $i = 1, 2, \ldots$, and μ be locally finite Borel measures in $\mathbb{R}^n$. We say that measures (μ_i) *converge weakly* to μ if $\int \varphi \, d\mu_i \to \int \varphi \, d\mu$ for every continuous compactly supported function $\varphi : \mathbb{R}^n \to \mathbb{R}$. This is denoted by $\mu_i \to \mu$.

Let μ be a locally finite Borel measure in $\mathbb{R}^n$. For any $a \in \mathbb{R}^n$ and $r > 0$ let $T_{a,r}\mu$ be the image of μ under the map $T_{a,r}$, $T_{a,r}(x) = (x - a)/r$, that is,

$$T_{a,r}\mu(A) = \mu(rA + a).$$

We say that ν is a *tangent measure* of μ at a if ν is a locally finite nonzero Borel measure in $\mathbb{R}^n$ and if there are positive numbers $r_i \downarrow 0$ such that

$$r_i^{-s} T_{a,r_i}\mu \to \nu.$$

We denote the set of all tangent measures of μ at a by $\operatorname{Tan}(\mu, a)$. The use of the particular normalization constants r_i^{-s} is motivated by the density assumptions we shall make on the measure μ below. Under them our definition corresponds to that of Preiss in [P].

Basic properties of tangent measures can be found in [P] and [M].

The following lemma tells about the usefulness of tangent measures in the investigation of the boundedness of maximal Riesz s-transforms.

Lemma 4. *Let μ be a locally finite Borel measure in $\mathbb{R}^n$. Suppose that $0 < \underline{D}^s(\mu, x) \le \overline{D}^s(\mu, x) < \infty$ for μ almost all $x \in \operatorname{spt}\mu$ and $\tilde{\mathcal{C}}^*_{s,\mu} < \infty$ μ almost everywhere. Then at μ almost all points $a \in \mathbb{R}^n$ for every $\nu \in \operatorname{Tan}(\mu, a)$ there exist constants $0 < c_1 \le c_2 < \infty$ such that*

$$c_1 r^s \le \nu B(x, r) \le c_2 r^s \tag{3}$$

for all $x \in \operatorname{spt}\nu$ and $0 < r < \infty$. Additionally, for μ almost all $a \in \mathbb{R}^n$ and for every $\nu \in \operatorname{Tan}(\mu, a)$

$$\tilde{\mathcal{C}}^*_{s,\nu} \text{ is bounded on } \mathbb{R}^n. \tag{4}$$

Proof. Let $\epsilon > 0$ and B be a set such that

$$B = \{x \in \operatorname{spt}\mu : 0 < c' r^s \le \mu B(x,r) \le c'' r^s < \infty \\ \text{for all } 0 < r < r_0 \text{ and } \tilde{\mathcal{C}}^*_{s,\mu}(x) < M\},$$

where c', c'', r_0, and M are positive constants. The set B can be chosen so that $\mu(\mathbb{R}^n \setminus B) < \epsilon$. It suffices to consider the μ density points of the set B, i.e. those points $a \in B$ where

$$\lim_{r \downarrow 0} \frac{\mu\big(B(a,r) \setminus B\big)}{\mu B(a,r)} = 0.$$

At these points $\operatorname{Tan}(\mu, a) = \operatorname{Tan}(\mu \llcorner B, a)$, where $\mu \llcorner B$ is the restriction of the measure μ to the set B.

Let $a \in B$ be a μ-density point of the set B and $\nu = \lim_{i\to\infty} r_i^{-s} T_{a,r_i}\mu \in \operatorname{Tan}(\mu, a)$. Now, for every $x \in \operatorname{spt}\nu$ and $r > 0$, by the lower semicontinuity of the weak convergence

$$\begin{aligned} 0 < \nu U(x,r) &\le \liminf_{i\to\infty} r_i^{-s} T_{a,r_i}\mu\big(B(x,r)\big) \\ &= \liminf_{i\to\infty} r_i^{-s} \mu\big(B(r_i x + a, r r_i) \cap B\big) \end{aligned}$$

holds for open balls $U(x,r)$. Thus for any $r > 0$ the set $B(r_i x + a, r r_i) \cap B \neq \emptyset$ for all sufficiently large i. Therefore we can choose a sequence of points $a_i \in B$ such that $x_i = (a_i - a)/r_i \to x$. Now for any $r < t$

$$\begin{aligned} \nu B(x,r) \le \nu U(x,t) &\le \liminf_{i\to\infty} r_i^{-s} \mu B(r_i x + a, t r_i) \\ &\le \liminf_{i\to\infty} r_i^{-s} \mu B(a_i, 2 t r_i) \\ &\le \liminf_{i\to\infty} r_i^{-s} c'' (2 t r_i)^s = c_2 t^s. \end{aligned}$$

As $t \downarrow r$ we get $\nu B(x,r) \le c_2 r^s$. The lower bound can be found similarly by using the fact that

$$\nu B(x,r) \ge \limsup_{i\to\infty} r_i^{-s} \mu B(r_i x + a, r r_i)$$

when $x \in \operatorname{spt}\nu$ and $r > 0$.

For the proof of the second statement let the set B be as before, $x \in \operatorname{spt}\nu$, $\nu \in \operatorname{Tan}(\mu, a)$, and $0 < r < R < \infty$. As before, we can choose a sequence of points $a_i \in B$ such that $x_i = (a_i - a)/r_i \to x$. If $\nu\big(\partial B(x,R)\big) = \nu\big(\partial B(x,r)\big) = 0$, we find that

$$\begin{aligned} &\left| \int_{B(x,R)\setminus B(x,r)} \frac{y-x}{|y-x|^{s+1}}\, d\nu y \right| = \lim_{i\to\infty} \left| \int_{B(x_i,R)\setminus B(x_i,r)} \frac{y-x_i}{|y-x_i|^{s+1}}\, d\nu y \right| \\ &= \lim_{i\to\infty} \left| r_i^{-s} \int_{B(x_i,R)\setminus B(x_i,r)} \frac{y-x_i}{|y-x_i|^{s+1}}\, dT_{a,r_i}\mu y \right| \\ &= \lim_{i\to\infty} \left| \int_{B(r_i x_i + a, R r_i)\setminus B(r_i x_i + a, r r_i)} \frac{y-(r_i x_i + a)}{|y-(r_i x_i + a)|^{s+1}}\, d\mu y \right| \\ &= \lim_{i\to\infty} \left| \int_{B(a_i, R r_i)\setminus B(a_i, r r_i)} \frac{y-a_i}{|y-a_i|^{s+1}}\, d\mu y \right| \le 2M. \end{aligned}$$

By approximation this holds for all $0 < r < R < \infty$. The conditions of Lemma 3 hold for the measure ν and thus $\tilde{\mathcal{C}}^*_{s,\nu}$ is bounded on $\mathbb{R}^n$. □

Theorem 5. *Let μ be a locally finite nonzero Borel measure in $\mathbb{R}^n$. Suppose that $0 < \underline{D}^s(\mu, x) \leq \overline{D}^s(\mu, x) < \infty$ for μ almost all $x \in \operatorname{spt}\mu$. If $\tilde{\mathcal{C}}^*_{s,\mu} < \infty$ μ almost everywhere, then s is an integer.*

Proof. If the theorem is false, $\tilde{\mathcal{C}}^*_{s,\mu} < \infty$ μ almost everywhere for some measure μ satisfying the assumptions of the theorem in some $\mathbb{R}^n$ for a non-integral s. Let n be the smallest possible dimension where this can happen. We want to show that this happens also in $\mathbb{R}^{n-1}$.

Let μ be a measure in $\mathbb{R}^n$ satisfying the above assumptions and $a \in \mathbb{R}^n$. We can find a tangent measure $\nu \in \operatorname{Tan}(\mu, a)$ such that the conditions (3) and (4) of Lemma 4 hold. Because $s < n$ the support of ν cannot be the whole space $\mathbb{R}^n$. This implies that there exist points $b \notin \operatorname{spt}\nu$ and $c \in \operatorname{spt}\nu$ and a number $r > 0$ such that

$$U(b, r) \cap \operatorname{spt}\nu = \emptyset \quad \text{and}$$
$$c \in \partial B(b, r) \cap \operatorname{spt}\nu.$$

Without loss of generality we may assume that $c = 0$ and b is on the negative x_1-axis. Because $0 \in \operatorname{spt}\nu$, the condition (3) holds for the measure ν at the point 0 and therefore both (3) and (4) hold for every tangent measure $\sigma \in \operatorname{Tan}(\nu, 0)$ as one can see from the proof of Lemma 4. Now $\operatorname{spt}\sigma \subset H = \{x \in \mathbb{R}^n : x_1 \geq 0\}$.

Because n was chosen to be minimal, $\operatorname{spt}\sigma \not\subset \partial H$. Denote $S_\delta = \{x \in \mathbb{R}^n : x_1 \geq \delta|x|\}$. Now there exists $\delta < 1$, and a sequence $x_i \in S_\delta \cap \operatorname{spt}\sigma \setminus \{0\}$ such that $x_i \to 0$. Otherwise we should have, for some $\lambda = \lim r_i^{-s} T_{0,r_i}\sigma$ with $r_i \downarrow 0$ and for $M > 0$, $\delta > 0$,

$$\begin{aligned}\lambda\Big(\operatorname{int}\big(S_\delta \cap U(0, M)\big)\Big) &\leq \liminf_{i\to\infty} r_i^{-s} T_{0,r_i}\sigma\Big(\operatorname{int}\big(S_\delta \cap U(0, M)\big)\Big)\\ &\leq \liminf_{i\to\infty} r_i^{-s}\sigma\big(S_\delta \cap B(r_i M)\big) = 0.\end{aligned}$$

Thus $\operatorname{spt}\lambda \subset \partial H$ which contradicts the minimality of n.

We may assume $|x_1| > |x_2| > \dots$ and by passing to a subsequence that the balls $B_i = B(x_i, \delta|x_i|/2)$ are disjoint and contained in the ball $B(0, 1)$. Then for any k

$$\begin{aligned}\int_{B(0,1)\setminus B(0,|x_k|/2)} \frac{y_1}{|y|^{s+1}}\, d\sigma y &\geq \sum_{i=1}^k \int_{B_i} \frac{y_1}{|y|^{s+1}}\, d\sigma y\\ &\geq \sum_{i=1}^k \frac{\delta}{2^{s+2}|x_i|^s}\sigma B_i\\ &\geq \sum_{i=1}^k c_1 \frac{\delta}{2^{s+2}|x_i|^s}\left(\frac{\delta|x_i|}{2}\right)^s\\ &= c_1 2^{-2s-2}\delta^{s+1}k\end{aligned}$$

and so

$$\begin{aligned}\left|\int_{B(0,1)\setminus B(0,|x_k|/2)} \frac{y}{|y|^{s+1}}\, d\sigma y\right| &\geq \left|\int_{B(0,1)\setminus B(0,|x_k|/2)} \frac{y_1}{|y|^{s+1}}\, d\sigma y\right|\\ &\geq c_1 2^{-2s-2}\delta^{s+1}k \underset{k\to\infty}{\longrightarrow} \infty.\end{aligned}$$

Thus

$$\sup_{\epsilon>0}\left|\int_{B(0,1)\setminus B(0,\epsilon)}\frac{y}{|y|}^{s+1}\,d\sigma y\right|=\infty.$$

This contradicts (4) in Lemma 4. Assumption that there would be a minimal n is false and the result follows.

Note that due to Lemma 2 the almost everywhere finiteness of $\tilde{\mathcal{C}}^*_{s,\mu}$ can be replaced by the essential boundedness of $\mathcal{C}_{s,\mu}$ if μ also satisfies the conditions of Lemma 2. □

4. L^2-boundedness of Riesz s-transforms

Next we shall study Riesz s-transforms as singular integral operators. Let μ be a positive Radon measure in $\mathbb{R}^n$ for which there exist constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 r^s \leq \mu B(x,r) \leq c_2 r^s \tag{5}$$

for all $x \in \operatorname{spt}\mu$ and $0 < r < \operatorname{diam}(\operatorname{spt}\mu)$. We define a singular integral operator $\mathcal{C}_{s,\mu}$ by

$$\langle \mathcal{C}_{s,\mu} f, g\rangle = \frac{1}{2}\iint \frac{y-x}{|y-x|^{s+1}}\big(f(y)g(x)-g(y)f(x)\big)\,d\mu y\,d\mu x$$

for all $f,g \in L^2(\mu)$ whose supports are separated by a positive distance (see [C] for more details). Saying that the operator $\mathcal{C}_{s,\mu}$ is bounded on L^2 means that there exists a constant $C < \infty$ such that

$$\|\mathcal{C}_{s,\mu} f\|_2 \leq C\|f\|_2$$

for $f \in L^2(\mu)$. We shall show that if there exists a nonzero measure μ such that it satisfies (5) and the singular integral operator $\mathcal{C}_{s,\mu}$ is bounded on L^2, then s is an integer.

Theorem 6. *Let μ be a nonzero Radon measure in $\mathbb{R}^n$ such that for $x \in \operatorname{spt}\mu$ and $0 < r < diam(\operatorname{spt}\mu)$*

$$0 < c_1 r^s \leq \mu B(x,r) \leq c_2 r^s < \infty.$$

If $\mathcal{C}_{s,\mu}$ is bounded on L^2, then s is an integer.

Proof. We first note that if the singular integral operator is bounded on L^2, then also the maximal truncated operator $\tilde{\mathcal{C}}^*_{s,\mu}$ is bounded on L^2, see [C] or [J]. Then also $\tilde{\mathcal{C}}^*_{s,\mu} < \infty$ μ almost everywhere and we can use Theorem 5 to get the result. □

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