NOTE ON THE BRADLEY AND RANNUJAN SUMMATION

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Abstract. The hypergeometric series of Bradley and Ramanujan is evaluated by means of the binomial convolutions of Hagen and Rothe, which presents, alternatively, a short proof of the recent result of Bradley about Ramanujan’s enigmatic claim.

For complex numbers $\alpha$, $\beta$, $\gamma$ and integer $\delta$, define the sum of Ramanujan type by

$$ S_\delta(\alpha,\beta,\gamma; z) = \gamma \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \frac{\Gamma(\beta + zk)}{\Gamma(\beta +zk)(k + \gamma + zk)} \frac{\Gamma(k + \gamma + zk)}{\Gamma(1 + \gamma + zk)}. $$

(1)

It reduces, under parameter replacements $\delta \to 0$, $\beta \to 1 + \beta$ and $\gamma \to m$, to the sum of Bradley [2], who has recently presented a most plausible interpretation for Ramanujan’s enigmatic claim, which may be restated in terms of $S$-sum as “the difference between $\Gamma(1 + \beta - m)/\Gamma(1 + \alpha + \beta - m)$ and $S_0(\alpha, 1 + \beta, m; z)$ …” (cf. Bradley [2]).

Theorem. With the $S$-function defined as above, we have the following evaluations:

A: Bradley [2, 1994]. For $\Re(\delta + \beta - \gamma) > 0$,

$$ S_\delta(\alpha,\beta,\gamma; 0) = \frac{\Gamma(\beta) \Gamma(\delta + \beta - \gamma)}{\Gamma(\delta + \beta) \Gamma(\delta + \alpha + \beta - \gamma)}. $$

(2a)

B. For $\Re(1 - \alpha - \beta + \gamma) > 0$,

$$ S_\delta(\alpha,\beta,\gamma; -1) = \frac{\Gamma(\beta) \Gamma(1 - \beta) \Gamma(1 - \alpha - \beta + \gamma)}{\Gamma(\delta + \alpha + \beta) \Gamma(1 - \alpha - \beta) \Gamma(1 - \beta + \gamma)}. $$

(2b)

C: Bradley [2, 1994]. When $\alpha$ is a non-positive integer,

$$ S_0(\alpha,\beta,\gamma; z) = \frac{\Gamma(\beta - \gamma)}{\Gamma(\alpha + \beta - \gamma)}. $$

(2c)

D. When $\alpha$ is a non-positive integer,

$$ S_1(\alpha,\beta,\gamma; z) = \frac{\alpha z - \beta + \gamma}{\alpha z - \beta} \frac{\Gamma(\beta - \gamma)}{\Gamma(1 + \alpha + \beta - \gamma)}. $$

(2d)
Proof. For $z = 0$ and $-1$, we can rewrite

$$S_\delta(\alpha, \beta, \gamma; 0) = \frac{\Gamma(\beta)}{\Gamma(\delta + \alpha + \beta)} \times 2F_1 \left[ \begin{array}{c} \alpha, \gamma \\ \delta + \alpha + \beta \end{array} \right],$$

$$S_\delta(\alpha, \beta, \gamma; -1) = \frac{\Gamma(\beta)}{\Gamma(\delta + \alpha + \beta)} \times 2F_1 \left[ \begin{array}{c} \alpha, -\gamma \\ 1 - \beta \end{array} \right],$$

which yield (2a) and (2b), respectively, in view of the Gauss theorem [1] (see also [3])

$$(3) \quad 2F_1 \left[ \begin{array}{c} a, b \\ c \end{array} \right] = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}, \quad \text{Re}(c - a - b) > 0.$$ 

When $\alpha = -n$, a non-positive integer, the $S$-function defined in (1) may be reformulated as

$$S_\delta(-n, \beta, \gamma; z) = \gamma \sum_{k=0}^{n} (-1)^{\delta+n} \binom{n}{k} \frac{(\gamma + zk)_k}{\gamma + zk} (1 - \beta - zk)_{n-k-\delta}$$

$$= \sum_{k=0}^{n} \frac{\gamma}{\gamma + zk} \binom{-\gamma - zk}{k} \frac{n!}{(\beta + zk)_{\delta}} \frac{(\delta - 1 + \beta + zk)}{n - k},$$

which reduce, respectively for $\delta = 0$ and 1, to

$$(4a) \quad S_0(-n, \beta, \gamma; z) = n! \left( \frac{\beta - \gamma - 1}{n} \right),$$

$$(4b) \quad S_1(-n, \beta, \gamma; z) = \frac{n!}{\beta + zn} \left( \frac{\beta - \gamma}{n} \right),$$

by means of the Hagen-Rothe [5] (see also [3, 4]) formulae

$$(5a) \quad \sum_{k=0}^{n} \frac{a}{a + bk} \binom{a + bk}{k} \binom{c - bk}{n - k} = \binom{a + c}{n},$$

$$(5b) \quad \sum_{k=0}^{n} \frac{a}{a + bk} \binom{a + bk}{k} \binom{c - bn}{c - bk} \binom{c - bk}{n - k} = \frac{a + c - bn}{a + c} \binom{a + c}{n}.$$ 

It is obvious that (2c) and (2d) are respectively the reformulations of (4a) and (4b).

REFERENCES


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