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NOTE ON THE BRADLEY AND RAMANUJAN SUMMATION

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ABSTRACT. The hypergeometric series of Bradley and Ramanujan is evaluated by means of the binomial convolutions of Hagen and Rothe, which presents, alternatively, a short proof of the recent result of Bradley about Ramanujan's enigmatic claim.

For complex numbers α, β, γ and integer δ, define the sum of Ramanujan type by

$$S_\delta(\alpha,\beta,\gamma;z) = \gamma\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\frac{\Gamma(\beta+zk)\,\Gamma(k+\gamma+zk)}{\Gamma(\delta+k+\alpha+\beta+zk)\,\Gamma(1+\gamma+zk)}. \tag{1}$$

It reduces, under parameter replacements $\delta\to 0$, $\beta\to 1+\beta$ and $\gamma\to m$, to the sum of Bradley [2], who has recently presented a most plausible interpretation for Ramanujan's enigmatic claim, which may be restated in terms of S-sum as "the difference between $\Gamma(1+\beta-m)/\Gamma(1+\alpha+\beta-m)$ and $S_0(\alpha,1+\beta,m;z)$ $\cdots$ " (cf. Bradley [2]).

Theorem. *With the S-function defined as above, we have the following evaluations:*
A: Bradley [2, 1994]. *For $Re\,(\delta+\beta-\gamma)>0$,*

$$S_\delta(\alpha,\beta,\gamma;0) = \frac{\Gamma(\beta)\,\Gamma(\delta+\beta-\gamma)}{\Gamma(\delta+\beta)\,\Gamma(\delta+\alpha+\beta-\gamma)}. \tag{2a}$$

B. *For $Re\,(1-\alpha-\beta+\gamma)>0$,*

$$S_\delta(\alpha,\beta,\gamma;-1) = \frac{\Gamma(\beta)\,\Gamma(1-\beta)\,\Gamma(1-\alpha-\beta+\gamma)}{\Gamma(\delta+\alpha+\beta)\,\Gamma(1-\alpha-\beta)\,\Gamma(1-\beta+\gamma)}. \tag{2b}$$

C: Bradley [2, 1994]. *When α is a non-positive integer,*

$$S_0(\alpha,\beta,\gamma;z) = \frac{\Gamma(\beta-\gamma)}{\Gamma(\alpha+\beta-\gamma)}. \tag{2c}$$

D. *When α is a non-positive integer,*

$$S_1(\alpha,\beta,\gamma;z) = \frac{\alpha z-\beta+\gamma}{\alpha z-\beta}\frac{\Gamma(\beta-\gamma)}{\Gamma(1+\alpha+\beta-\gamma)}. \tag{2d}$$

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Proof. For $z = 0$ and -1, we can rewrite

$$S_\delta(\alpha,\beta,\gamma;0) = \frac{\Gamma(\beta)}{\Gamma(\delta+\alpha+\beta)} \times {}_2F_1\left[\begin{matrix}\alpha,\ \gamma\\ \delta+\alpha+\beta\end{matrix}\right],$$

$$S_\delta(\alpha,\beta,\gamma;-1) = \frac{\Gamma(\beta)}{\Gamma(\delta+\alpha+\beta)} \times {}_2F_1\left[\begin{matrix}\alpha, & -\gamma\\ & 1-\beta\end{matrix}\right],$$

which yield (2a) and (2b), respectively, in view of the Gauss theorem [1] (see also [3])

$$(3) \qquad {}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix}\right] = \frac{\Gamma(c-a)\,\Gamma(c-b)}{\Gamma(c)\,\Gamma(c-a-b)}, \qquad Re(c-a-b) > 0.$$

When $\alpha = -n$, a non-positive integer, the S-function defined in (1) may be reformulated as

$$\begin{aligned} S_\delta(-n,\beta,\gamma;z) &= \gamma\sum_{k=0}^{n}(-1)^{\delta+n}\binom{n}{k}\frac{(\gamma+zk)_k}{\gamma+zk}(1-\beta-zk)_{n-k-\delta} \\ &= \sum_{k=0}^{n}\frac{\gamma}{\gamma+zk}\binom{-\gamma-zk}{k}\frac{n!}{(\beta+zk)_\delta}\binom{\delta-1+\beta+zk}{n-k}, \end{aligned}$$

which reduce, respectively for $\delta = 0$ and 1, to

$$(4a) \qquad S_0(-n,\beta,\gamma;z) = n!\binom{\beta-\gamma-1}{n},$$

$$(4b) \qquad S_1(-n,\beta,\gamma;z) = \frac{n!}{\beta+zn}\,\frac{\beta-\gamma+zn}{\beta-\gamma}\binom{\beta-\gamma}{n},$$

by means of the Hagen-Rothe [5] (see also [3, 4]) formulae

$$(5a) \qquad \sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\binom{c-bk}{n-k} = \binom{a+c}{n},$$

$$(5b) \qquad \sum_{k=0}^{n}\frac{a}{a+bk}\binom{a+bk}{k}\frac{c-bn}{c-bk}\binom{c-bk}{n-k} = \frac{a+c-bn}{a+c}\binom{a+c}{n}.$$

It is obvious that (2c) and (2d) are respectively the reformulations of (4a) and (4b). □

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