THE CENTRAL INTERTWINING LIFTING
AND STRICT CONTRACTIONS

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ABSTRACT. In this paper we give a necessary and sufficient condition for the central intertwining lifting of a strict contraction to be strictly contractive. As an application, we obtain a factorization of $D_{A_c}^{-2}$ when the central intertwining lifting $A_c$ of $A$ is a strict contraction.

Let $H, H'$ be (complex, separable) Hilbert spaces, $T$ and $T'$ contractions in $L(H)$ and $L(H')$, respectively, having minimal isometric dilations $U$ and $U'$ in $L(K)$ and $L(K')$. If $A$ is a contraction operator in $L(H, H')$ such that $AT = T'A$, the celebrated Sz.-Nagy–Foias commutant lifting theorem ([3, 7, 8]) ensures the existence of an intertwining lifting of $A$, i.e. the existence of a contraction $B$ in $L(K, K')$ such that $BU = U'B$ and $B^*|_{H'} = A^*$. Recently, a particular intertwining lifting of $A$, called the central intertwining lifting of $A$ (see [4, 5, 6]) has played an important role in solving certain $H^2$-$H^\infty$ optimization problems appearing in control theory. Moreover, an example of a strictly contractive Hankel operator whose central intertwining lifting has norm one was given in [2]. The purpose of this note is to prove a necessary and sufficient condition (Theorem 1 below) for the central intertwining lifting $A_c$ of a strict contraction to be strictly contractive.

Necessary and sufficient conditions for the central intertwining lifting to be strictly contractive were also obtained in [5] when $T$ is a unilateral shift. In this case, the central intertwining lifting is strictly contractive if and only if the outer function $(I - A^*A)^{-1}_{|_{kerT^*}}$ is invertible (see [5]).

For the convenience of the reader we begin by reviewing some notation and terminology (see [3, 4, 5, 6, 8]). Throughout this note all Hilbert spaces are understood to be complex and separable. If $D, D'$ are Hilbert spaces and $C \in L(D, D')$ is a contraction (i.e. $\|C\| \leq 1$), we denote as usual the defect operator of $C$ by $D_C := (I - C^*C)^{1/2}$, the defect space of $C$ by $D_C := D_C^*$, and $C$ is said to be a strict contraction if $\|C\| < 1$. Let $D$ denote the open unit disc in $C$, and $T=\partial D$. The spaces $L^2(D)$ and $H^2(D)$ are the usual Lebesgue and Hardy spaces of $D$-valued functions on the unit circle $T$ relative to normalized Lebesgue measure $m$ on $T$, and $S \in L( H^2(D))$ is the multiplication by the variable on $H^2(D)$. Since the minimal isometric dilation of $T'$ is unique (up to an isomorphism (see [3, 8])), we may and do assume that $U' \in L(K')$, the minimal isometric dilation of $T'$ is the Sz.-Nagy–Schäffer minimal isometric dilation of $T'$, i.e. $K' = H' \oplus H^2(D_T)$ and $U'$...
on $\mathcal{H}' \oplus \mathbb{H}^2(D_{T'})$ is given by

$$U' = \begin{bmatrix} T' & 0 \\ D_{T'} & S' \end{bmatrix}: \mathcal{H}' \oplus \mathbb{H}^2(D_{T'}) \to \mathcal{H}' \oplus \mathbb{H}^2(D_{T'})$$

where $S' \in \mathcal{L}(\mathbb{H}^2(D_{T'}))$ is the multiplication by $z$ on $\mathbb{H}^2(D_{T'})$, (see [3, 8]). If $A$ is a strict contraction, then the central intertwining lifting $A_c$ of $A$ is the contraction in $\mathcal{L}(\mathcal{K}, \mathcal{H}' \oplus \mathbb{H}^2(D_{T'}))$ given by

$$(1) \quad A_c = \begin{bmatrix} \widetilde{A} & - \widetilde{A}^*T_A^*(I - zT_A^*)^{-1} \\ D_{T'} & - \widetilde{A}^*T_A^* \end{bmatrix}: \mathcal{K} \to \mathcal{H}' \oplus \mathbb{H}^2(D_{T'})$$

where $\widetilde{A} := AP$, $T_A := (I - \widetilde{A}^*\widetilde{A})U(I - U^*\widetilde{A}^*\widetilde{A})^{-1}$, and $P$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$ (see [4, 5]). Moreover, let $N_* \in \mathcal{L}(D_{T'})$, $L \in \mathcal{L}(\mathcal{H})$ be the operators defined by

$$N_* = I_{D_{T'}} + D_{T'}AD_A^{-2}A^*D_{T'}, \quad L = D_A^{-2} - (I - T^*A^*AT)^{-1}.$$  

Moreover, for any $k \in \mathbb{K}$ the function

$$(3) \quad z \to k_A(z) := D_{T'}AD_A^{-2}P(I - zU^*)^{-1}k$$

on $D$ will be denoted by $k_A$. Now we may state the main result of this note.

**Theorem 1.** Let $T$ and $T'$ be contractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$, respectively, having minimal isometric dilations $U \in \mathcal{L}(\mathcal{K})$ and $U' \in \mathcal{L}(\mathcal{H}' \oplus \mathbb{H}^2(D_{T'}))$. If $A$ is a strict contraction in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $AT = T'A$, then the central intertwining lifting $A_c$ of $A$ is a strict contraction if and only if for any $k \in \mathbb{K}$ the function $k_A$ is in $\mathcal{H}^2(D_{T'})$. Moreover, if $T$ is the multiplication by the variable on $\mathbb{H}^2(D)$, then the central intertwining lifting $A_c$ of $A$ is a strict contraction if and only if the function $\Theta_A: D \to \mathcal{L}(D_{T'}, D)$ given by

$$\Theta_A(z)d_{T'} = (D_A^{-2}A^*D_{T'}d_{T'})(z), \quad z \in D, \ d_{T'} \in D_{T'}$$

is bounded.

**Proof.** Without loss of generality we may assume (see [3, 8]) that $T$ is an isometry. In this case $U = T$, $P = I_{\mathcal{K}}$, and for any nonnegative integer $n$ let $i_n : D_{T'} \to \mathcal{H}' \oplus \mathbb{H}^2(D_{T'})$ be defined by $i_n d_{T'} = 0 \oplus e^{int}d_{T'}$, $d_{T'} \in D_{T'}$, and let $\mathcal{H}_n' = \mathcal{H}' \oplus \oplus_{k=0}^{n} e^{ikt}D_{T'}$ ($\subset \mathcal{H}' \oplus \mathbb{H}^2(D_{T'})$). Furthermore, let $P_n'$ denote the orthogonal projection from $\mathcal{H}' \oplus \mathbb{H}^2(D_{T'})$ onto $\mathcal{H}_n'$, $A_n := P_n'A_c$ and $T_n' := P_n'U_{|\mathcal{H}_n'}$. Then (see [3, 8]), for any nonnegative integer $n$

$$(4) \quad P_n' = U'P_n'_{n-1}U' + i_{n-1}^*i_{n-1}, \quad A_nT = T_n'A_n,$$

and according to (1) and Lemma V.1.1 of [3] ($A_{n+1}$ is a one-step intertwining lifting of $A_n$), for any nonnegative integer $n$

$$(5) \quad i_{n}^*A_n = i_{n-1}^*A_{n-1}T_{A_{n-1}}^*.$$
We will show that for any nonnegative integer \( n \)

\[
D_{A_n}^{-2} = D_{A_n}^{-2} + \sum_{k=1}^{n+1} T^k L T^k.
\]

Once we show this the proof can be completed as follows. It is easy to see that

\[
L = D_{A_n}^{-2} A^* D_{T}^{-1} N_{e}^{-1} D_{T}^{-1} AD_{A_n}^{-2},
\]

so by (6) and (7), \( \|A_c\| < 1 \) if and only if there exists a positive constant \( M \) such that for any \( h \) in \( \mathcal{H} \) (\( = \mathcal{K} \))

\[
\sum_{n=0}^{\infty} \|L^{1/2} T^n h\|^2 = \sum_{n=0}^{\infty} \|N_{e}^{-1/2} D_{T}^{-2} AD_{A_n}^{-2} T^n h\|^2 \leq M \|h\|^2.
\]

Taking into account (3), it follows that for any vector \( h \) in \( \mathcal{H} \)

\[
\|h_A\|^2_{H^2(D_{T}')} = \sum_{n=0}^{\infty} \|D_{T}^{-} AD_{A_n}^{-2} T^n h\|^2,
\]

so by (8) and (9) it follows that \( \|A_c\| < 1 \) if and only if for any vector \( h \) in \( \mathcal{H} \), \( h_A \) is in \( H^2(D_{T}') \) and the proof of the first part of the theorem is complete. Assume now that \( T \) is the multiplication by the variable on \( H^2(D) \) and \( \|A_c\| < 1 \). Then the operator \( M_A : H^2(D) \to H^2(D_{T}') \) defined by \( M_A := h_A, h \in H^2(D) \) is bounded. It is easy to check that \( M'_A \) is an analytic Toeplitz operator whose symbol is \( \Theta_A \) (i.e. \( M'_A h' = \Theta_A h' \), \( h' \in H^2(D_{T}') \)), so \( \Theta_A \) is bounded. Conversely, if \( \Theta_A \) is bounded, then \( M_A \) is bounded, hence for any function \( h \) in \( H^2(D) \), \( h_A \in H^2(D_{T}') \) and \( \|A_c\| < 1 \).

Now we show that (6) holds for any nonnegative integer \( n \). From (4) and (5) we obtain that for any nonnegative integer \( n \)

\[
D_{A_n}^2 = D_{A_{n-1}}^2 - T_{A_{n-1}} A_{n-1}^* D_{T}^{-1} A_{n-1} T_{A_{n-1}},
\]

\[
D_{A_{n-1}}^2 = I - T^* A_{n}^* A_{n} T,
\]

and we will show that

\[
D_{A_n}^{-2} = D_{A_{n-1}}^{-2} + T [D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}] T^*,
\]

from which (6) follows at once. In order to prove (11) we need to check that

\[
D_{A_{n-1}}^2 T [D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}] T^* - T_{A_{n-1}} A_{n-1}^* D_{T}^{-1} A_{n-1} T_{A_{n-1}} D_{A_{n-1}}^{-2}
\]

\[
- T_{A_{n-1}} A_{n-1}^* D_{T}^{-1} A_{n-1} [D_{A_{n-2}}^{-2} - D_{A_{n-2}}^{-2}] T^* = 0,
\]

or equivalently, using (1), (4) and (10),

\[
D_{A_{n-1}}^2 - D_{A_{n-2}}^2 - D_{A_{n-2}}^2 A_{n-1}^* D_{T}^{-1} A_{n-1} D_{A_{n-1}}^{-2}
\]

\[
- D_{A_{n-1}}^2 A_{n-1}^* D_{T}^{-1} A_{n-1} [D_{A_{n-2}}^{-2} - D_{A_{n-2}}^{-2}] = D_{A_{n-1}}^2 - D_{A_{n-2}}^2 - D_{A_{n-2}}^2 A_{n-1}^* D_{T}^{-1} A_{n-1} D_{A_{n-1}}^{-2}
\]

\[
= D_{A_{n-1}}^2 [D_{A_{n-1}}^2 - D_{A_{n-2}}^2 - A_{n-1}^* D_{T}^{-1} A_{n-1}] D_{A_{n-1}}^2 = 0,
\]
which follows easily since by (4) and (10)
\[ D_{A_{n-1}}^2 - D_{A_{n-2}}^2 - A_{n-1}^*D_{A_{n-1}}^2 + A_{n-1} \]
\[ = (I - T^*A_{n-1}^*A_{n-1}T) - D_{A_{n-2}}^2 - A_{n-1}^*D_{A_{n-1}}^2 + A_{n-1} \]
\[ = (I - A_{n-1}^*T_{n-1}^*T_{n-1}A_{n-1}) - D_{A_{n-2}}^2 - A_{n-1}^*D_{A_{n-1}}^2 + A_{n-1} = 0. \]

So we have proved (6) and the proof of the theorem is complete. \(\square\)

A direct application of (6) yields the following.

**Corollary 2.** Let \( T \) and \( T' \) be contractions in \( \mathcal{L}(\mathcal{H}) \) and \( \mathcal{L}(\mathcal{H}') \), respectively, having minimal isometric dilations \( U \) in \( \mathcal{L}(\mathcal{K}) \) and \( U' \) in \( \mathcal{L}(\mathcal{H} \oplus \mathcal{H}^2(\mathcal{D}_{T'})) \). Let \( A \) in \( \mathcal{L}(\mathcal{H}, \mathcal{H}') \) be a strict contraction such that \( AT = T'A \), and set \( \tilde{A} = AP \) (\( P \) is the orthogonal projection from \( \mathcal{K} \) onto \( \mathcal{H} \)). If the central intertwining lifting \( A_c \) of \( A \) is a strict contraction, then
\[ D_{A_c}^{-2} = \Psi^*\Psi, \]
where \( \Psi : \mathcal{K} \to \mathcal{K} \oplus \mathcal{H}^2(\mathcal{D}_{T'}) \) is given by
\[ \Psi := \begin{bmatrix} \tilde{D}_{A}^{-2} \\ N_s^{-1/2}D_{T'}\tilde{A}D_{A}^{-2}(I - zU^*)^{-1}U \end{bmatrix}. \]

**Corollary 3.** Let \( h \in \mathcal{H}^\infty \) be an outer function such that \( \text{dist}(\overline{h}/h, \mathcal{H}^\infty) < 1 \) and \( h^{-1} \notin \mathcal{H}^\infty \). If \( H \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^2) \) is the Hankel operator defined by
\[ Hf = P_-h/\overline{h}f, \quad f \in \mathcal{H}^2, \]
then \( \|H\| < 1 \) and \( \|H_c\| = 1 \).

**Proof.** Without loss of generality we may assume that \( h(0) = 1 \). According to Chapter VIII of \([9]\), \( \|H\| < 1 \) and \( h^{-1} \in \mathcal{H}^2 \), so the Toeplitz operator \( T_{h/\overline{h}} \) is invertible. Upon setting \( g := P_+|h|^2 \) we will show that
\[ \Theta_H = D_H^{-2}H^*e^{-it} = -h^{-1}S^*g \notin \mathcal{H}^\infty \]
which by Theorem 1 will finish the proof. To show (12) let us note that since
\[ D_H^2 = T_{h/\overline{h}}^*T_{h/\overline{h}} \text{ and } T_{h/\overline{h}}^*(-S^*h) = H^*e^{-it}, \]
it follows that
\[ T_{h/\overline{h}}\Theta_H = -S^*h. \]
Thus there exists a function \( u \in \mathcal{H}^2 \) such that
\[ (h/\overline{h})\Theta_H = -S^*h + e^{-it}\overline{\pi}, \]
hence
\[ h\Theta_H = -\overline{h}e^{-it}(h - 1) + e^{-it}\overline{h}\pi = -e^{-it}|h|^2 + e^{-it}\overline{h}(1 + \pi). \]
Therefore,
\[ h\Theta_H = -P_+e^{-it}|h|^2 = -S^*P_+|h|^2 = -S^*g. \]
Since \( |h|^2 \geq 0 \) a.e. on \( \mathcal{T} \), it follows that
\[ |h|^2 = SS^*g + g(0) + \overline{SS^*g}. \]
So
\[ \overline{h} = h^{-1}SS^*g + g(0)h^{-1} + h^{-1}\overline{SS^*g}, \]

from which it follows that
\[ h^{-1} = g(0)^{-1}(\sqrt{h} - h^{-1}SS^*g - h^{-1}SS^*g). \]
Since \( h^{-1} \notin H^\infty \), \( h^{-1}SS^*g \) cannot be bounded. So \( \Theta \notin H^\infty \), and the proof of the corollary is complete.

Remark. The example in [2] was obtained by considering the function \( h(z) = (1 - z)^\beta \), where \( 0 < \beta < 1/2 \). Then \( h \in H^\infty \), \( \text{dist}(h,h,H^\infty) < 1 \) (see [1]), and \( h^{-1} \notin H^\infty \). So by Corollary 3 the central intertwining lifting of \( H^\infty \) has norm one and one regains the main result in [2].

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