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# THE CENTRAL INTERTWINING LIFTING AND STRICT CONTRACTIONS

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ABSTRACT. In this paper we give a necessary and sufficient condition for the central intertwining lifting of a strict contraction to be strictly contractive. As an application, we obtain a factorization of $D_{A_c}^{-2}$ when the central intertwining lifting $A_c$ of $A$ is a strict contraction.

Let $\mathcal{H}$, $\mathcal{H}'$ be (complex, separable) Hilbert spaces, $T$ and $T'$ contractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$, respectively, having minimal isometric dilations $U$ and $U'$ in $\mathcal{L}(\mathcal{K})$ and $\mathcal{L}(\mathcal{K}')$. If $A$ is a contraction operator in $\mathcal{L}(\mathcal{H},\mathcal{H}')$ such that $AT = T'A$, the celebrated Sz.-Nagy–Foias commutant lifting theorem ([3, 7, 8]) ensures the existence of an intertwining lifting of $A$, i.e. the existence of a contraction $B$ in $\mathcal{L}(\mathcal{K},\mathcal{K}')$ such that $BU = U'B$ and $B^*{}_{|_{\mathcal{H}'}} = A^*$. Recently, a particular intertwining lifting of $A$, called the *central intertwining lifting* of $A$ (see [4, 5, 6]) has played an important role in solving certain $\mathbf{H}^2$-$\mathbf{H}^\infty$ optimization problems appearing in control theory. Moreover, an example of a strictly contractive Hankel operator whose central intertwining lifting has norm one was given in [2]. The purpose of this note is to prove a necessary and sufficient condition (Theorem 1 below) for the central intertwining lifting $A_c$ of a strict contraction $A$ to be strictly contractive. Necessary and sufficient conditions for the central intertwining lifting to be strictly contractive were also obtained in [5] when $T$ is a unilateral shift. In this case, the central intertwining lifting is strictly contractive if and only if the outer function $(I - A^*A)^{-1}_{|_{\ker T^*}}$ is invertible (see [5]).

For the convenience of the reader we begin by reviewing some notation and terminology (see [3, 4, 5, 6, 8]). Throughout this note all Hilbert spaces are understood to be complex and separable. If $\mathcal{D}$, $\mathcal{D}'$ are Hilbert spaces and $C \in \mathcal{L}(\mathcal{D},\mathcal{D}')$ is a contraction (i.e. $\|C\| \leq 1$), we denote as usual the defect operator of $C$ by $D_C$ $(:= (I - C^*C)^{1/2})$, the defect space of $C$ by $\mathcal{D}_C$ $(:= \overline{D_C\mathcal{D}})$, and $C$ is said to be a strict contraction if $\|C\| < 1$. Let $\mathbf{D}$ denote the open unit disc in $\mathbf{C}$, and $\mathbf{T}=\partial\mathbf{D}$. The spaces $\mathbf{L}^2(\mathcal{D})$ and $\mathbf{H}^2(\mathcal{D})$ are the usual Lebesgue and Hardy spaces of $\mathcal{D}$-valued functions on the unit circle $\mathbf{T}$ relative to normalized Lebesgue measure $\mathbf{m}$ on $\mathbf{T}$, and $S \in \mathcal{L}(\mathbf{H}^2(\mathcal{D}))$ is the multiplication by the variable on $\mathbf{H}^2(\mathcal{D})$. Since the minimal isometric dilation of $T'$ is unique (up to an isomorphism (see [3, 8])), we may and do assume that $U' \in \mathcal{L}(\mathcal{K}')$, the minimal isometric dilation of $T'$ is the Sz.-Nagy–Schäffer minimal isometric dilation of $T'$, i.e. $\mathcal{K}' = \mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ and $U'$

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on $\mathcal{H}'\oplus\mathbf{H}^2(\mathcal{D}_{T'})$ is given by

$$U' = \begin{bmatrix} T' & 0 \\ D_{T'} & S' \end{bmatrix} : \begin{matrix} \mathcal{H}' \\ \oplus \\ \mathbf{H}^2(\mathcal{D}_{T'}) \end{matrix} \to \begin{matrix} \mathcal{H}' \\ \oplus \\ \mathbf{H}^2(\mathcal{D}_{T'}) \end{matrix},$$

where $S' \in \mathcal{L}(\mathbf{H}^2(\mathcal{D}_{T'}))$ is the multiplication by $z$ on $\mathbf{H}^2(\mathcal{D}_{T'})$, (see [3, 8]). If $A$ is a strict contraction, then the central intertwining lifting $A_c$ of $A$ is the contraction in $\mathcal{L}(\mathcal{K}, \mathcal{H}'\oplus\mathbf{H}^2(\mathcal{D}_{T'}))$ given by

$$A_c = \begin{bmatrix} \widetilde{A} \\ D_{T'}\widetilde{A}T^*_{\widetilde{A}}(I - zT^*_{\widetilde{A}})^{-1} \end{bmatrix} : \mathcal{K} \to \begin{matrix} \mathcal{H}' \\ \oplus \\ \mathbf{H}^2(\mathcal{D}_{T'}) \end{matrix}, \tag{1}$$

where $\widetilde{A} := AP$, $T_{\widetilde{A}} := (I - \widetilde{A}^*\widetilde{A})U(I - U^*\widetilde{A}^*\widetilde{A}U)^{-1}$, and $P$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$ (see [4, 5]). Moreover, let $N_* \in \mathcal{L}(\mathcal{D}_{T'})$, $L \in \mathcal{L}(\mathcal{H})$ be the operators defined by

$$\begin{aligned} N_* &= I_{\mathcal{D}_{T'}} + D_{T'}AD_A^{-2}A^*D_{T'}, \\ L &= D_A^{-2} - (I - T^*A^*AT)^{-1}. \end{aligned} \tag{2}$$

Moreover, for any $k \in \mathcal{K}$ the function

$$z \to k_A(z) := D_{T'}AD_A^{-2}P(I - zU^*)^{-1}k \tag{3}$$

on $\mathbf{D}$ will be denoted by $k_A$. Now we may state the main result of this note.

**Theorem 1.** *Let $T$ and $T'$ be contractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$, respectively, having minimal isometric dilations $U \in \mathcal{L}(\mathcal{K})$ and $U' \in \mathcal{L}(\mathcal{H}'\oplus\mathbf{H}^2(\mathcal{D}_{T'}))$. If $A$ is a strict contraction in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $AT = T'A$, then the central intertwining lifting $A_c$ of $A$ is a strict contraction if and only if for any $k$ in $\mathcal{K}$ the function $k_A$ is in $\mathbf{H}^2(\mathcal{D}_{T'})$. Moreover, if $T$ is the multiplication by the variable on $\mathbf{H}^2(\mathcal{D})$, then the central intertwining lifting $A_c$ of $A$ is a strict contraction if and only if the function $\Theta_A\colon \mathbf{D}\to \mathcal{L}(\mathcal{D}_{T'}, \mathcal{D})$ given by*

$$\Theta_A(z)d_{T'} = (D_A^{-2}A^*D_{T'}d_{T'})(z), \quad z \in \mathbf{D},\ d_{T'} \in \mathcal{D}_{T'}$$

*is bounded.*

*Proof.* Without loss of generality we may assume (see [3, 8]) that $T$ is an isometry. In this case $U = T$, $P = I_{\mathcal{H}}$, and for any nonnegative integer $n$ let $i_n : \mathcal{D}_{T'} \to \mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ be defined by $i_n d_{T'} = 0 \oplus e^{int}d_{T'}$, $d_{T'} \in \mathcal{D}_{T'}$, and let $\mathcal{H}'_n := \mathcal{H}' \oplus \overset{n}{\underset{k=0}{\oplus}} e^{ikt}\mathcal{D}_{T'}$ $(\subset \mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'}))$. Furthermore, let $P'_n$ denote the orthogonal projection from $\mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ onto $\mathcal{H}'_n$, $A_n := P'_nA_c$ and $T'_n := P'_nU'_{|\mathcal{H}'_n}$. Then (see [3, 8]), for any nonnegative integer $n$

$$\begin{aligned} P'_n &= U'^*P'_{n-1}U' + i_{n-1}i^*_{n-1}, \\ A_nT &= T'_nA_n, \end{aligned} \tag{4}$$

and according to (1) and Lemma V.1.1 of [3] ($A_{n+1}$ is a one-step intertwining lifting of $A_n$), for any nonnegative integer $n$

$$i^*_nA_n = i^*_{n-1}A_{n-1}T^*_{A_{n-1}}. \tag{5}$$

We will show that for any nonnegative integer $n$

$$D_{A_n}^{-2} = D_A^{-2} + \sum_{k=1}^{n+1} T^k L T^{*k}. \tag{6}$$

Once we show this the proof can be completed as follows. It is easy to see that

$$L = D_A^{-2} A^* D_{T'} N_*^{-1} D_{T'} A D_A^{-2}, \tag{7}$$

so by (6) and (7), $\|A_c\| < 1$ if and only if there exists a positive constant $M$ such that for any $h$ in $\mathcal{H}$ $(= \mathcal{K})$

$$\sum_{n=0}^{\infty} \|L^{1/2} T^{*n} h\|^2 = \sum_{n=0}^{\infty} \|N_*^{-1/2} D_{T'} A D_A^{-2} T^{*n} h\|^2 \leq M \,\|h\|^2. \tag{8}$$

Taking into account (3), it follows that for any vector $h$ in $\mathcal{H}$

$$\|h_A\|^2_{\mathbf{H}^2(\mathcal{D}_{T'})} = \sum_{n=0}^{\infty} \|D_{T'} A D_A^{-2} T^{*n} h\|^2, \tag{9}$$

so by (8) and (9) it follows that $\|A_c\| < 1$ if and only if for any vector $h$ in $\mathcal{H}$, $h_A$ is in $\mathbf{H}^2(\mathcal{D}_{T'})$ and the proof of the first part of the theorem is complete. Assume now that $T$ is the multiplication by the variable on $\mathbf{H}^2(\mathcal{D})$ and $\|A_c\| < 1$. Then the operator $M_A : \mathbf{H}^2(\mathcal{D}) \to \mathbf{H}^2(\mathcal{D}_{T'})$ defined by $M_A := h_A$, $h \in \mathbf{H}^2(\mathcal{D})$ is bounded. It is easy to check that $M_A^*$ is an analytic Toeplitz operator whose symbol is $\Theta_A$ (i.e. $M_A^* h' = \Theta_A h'$, $h' \in \mathbf{H}^2(\mathcal{D}_{T'})$), so $\Theta_A$ is bounded. Conversely, if $\Theta_A$ is bounded, then $M_A$ is bounded, hence for any function $h$ in $\mathbf{H}^2(\mathcal{D})$, $h_A \in \mathbf{H}^2(\mathcal{D}_{T'})$ and $\|A_c\| < 1$.

Now we show that (6) holds for any nonegative integer $n$. From (4) and (5) we obtain that for any nonegative integer $n$

$$\begin{aligned} D_{A_n}^2 &= D_{A_{n-1}}^2 - T_{A_{n-1}} A_{n-1}^* D_{T'}^2 A_{n-1} T_{A_{n-1}}^*, \\ D_{A_{n-1}}^2 &= I - T^* A_n^* A_n T, \end{aligned} \tag{10}$$

and we will show that

$$D_{A_n}^{-2} = D_{A_{n-1}}^{-2} + T[D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}]T^*, \tag{11}$$

from which (6) follows at once. In order to prove (11) we need to check that

$$\begin{aligned} D_{A_{n-1}}^2 T[D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}]T^* - T_{A_{n-1}} A_{n-1}^* D_{T'}^2 A_{n-1} T_{A_{n-1}}^* D_{A_{n-1}}^{-2} \\ - T_{A_{n-1}} A_{n-1}^* D_{T'}^2 A_{n-1} [D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}]T^* = 0, \end{aligned}$$

or equivalently, using (1), (4) and (10),

$$\begin{aligned} D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2} &- D_{A_{n-2}}^{-2} A_{n-1}^* D_{T'}^2 D_{A_{n-1}}^{-2} \\ &- D_{A_{n-1}}^{-2} A_{n-1}^* D_{T'}^2 A_{n-1} [D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2}] \\ &= D_{A_{n-1}}^{-2} - D_{A_{n-2}}^{-2} - D_{A_{n-1}}^{-2} A_{n-1}^* D_{T'_{n-1}}^2 A_{n-1} D_{A_{n-1}}^{-2} \\ &= D_{A_{n-1}}^{-2} [D_{A_{n-1}}^2 - D_{A_{n-2}}^2 - A_{n-1}^* D_{T'_{n-1}}^2 A_{n-1}] D_{A_{n-1}}^{-2} = 0, \end{aligned}$$

which follows easily since by (4) and (10)

$$\begin{aligned} D^2_{A_{n-1}} &- D^2_{A_{n-2}} - A^*_{n-1} D^2_{T'_{n-1}} A_{n-1} \\ &= (I - T^* A^*_{n-1} A_{n-1} T) - D^2_{A_{n-2}} - A^*_{n-1} D^2_{T'_{n-1}} A_{n-1} \\ &= (I - A^*_{n-1} T^{*\prime}_{n-1} T'_{n-1} A_{n-1}) - D^2_{A_{n-2}} - A^*_{n-1} D^2_{T'_{n-1}} A_{n-1} = 0. \end{aligned}$$

So we have proved (6) and the proof of the theorem is complete. ☐

A direct application of (6) yields the following.

**Corollary 2.** *Let $T$ and $T'$ be contractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}')$, respectively, having minimal isometric dilations $U$ in $\mathcal{L}(\mathcal{K})$ and $U'$ in $\mathcal{L}(\mathcal{H}' \oplus \mathbf{H}^2(\mathcal{D}_{T'}))$. Let $A$ in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ be a strict contraction such that $AT = T'A$, and set $\widetilde{A} = AP$ ($P$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$). If the central intertwining lifting $A_c$ of $A$ is a strict contraction, then*

$$D^{-2}_{A_c} = \Psi^* \Psi,$$

*where $\Psi : \mathcal{K} \to \mathcal{K} \oplus \mathbf{H}^2(\mathcal{D}_{T'})$ is given by*

$$\Psi := \begin{bmatrix} D^{-2}_{\widetilde{A}} \\ N_*^{-1/2} D_{T'} \widetilde{A} D^{-2}_{\widetilde{A}} (I - zU^*)^{-1} U^* \end{bmatrix}.$$

**Corollary 3.** *Let $h \in \mathbf{H}^\infty$ be an outer function such that dist$(\overline{h}/h\, , \mathbf{H}^\infty) < 1$ and $h^{-1} \notin \mathbf{H}^\infty$. If $H \in \mathcal{L}(\mathbf{H}^2, \mathbf{H}^2_-)$ is the Hankel operator defined by*

$$Hf = P_- h/\overline{h} f, \qquad f \in \mathbf{H}^2,$$

*then $\|H\| < 1$ and $\|H_c\| = 1$.*

*Proof.* Without loss of generality we may assume that $h(0) = 1$. According to Chapter VIII of [9], $\|H\| < 1$ and $h^{-1} \in \mathbf{H}^2$, so the Toeplitz operator $T_{h/\overline{h}}$ is invertible. Upon setting $g := P_+ |h|^2$ we will show that

$$\Theta_H = D_H^{-2} H^* e^{-it} = -h^{-1} S^* g \notin \mathbf{H}^\infty \tag{12}$$

which by Theorem 1 will finish the proof. To show (12) let us note that since $D_H^2 = T^*_{h/\overline{h}} T_{h/\overline{h}}$ and $T^*_{h/\overline{h}}(-S^* h) = H^* e^{-it}$, it follows that

$$T_{h/\overline{h}} \Theta_H = -S^* h.$$

Thus there exists a function $u \in \mathbf{H}^2$ such that

$$(h/\overline{h}) \Theta_H = -S^* h + e^{-it} \overline{u},$$

hence

$$h \Theta_H = -\overline{h} e^{-it} (h - 1) + e^{-it} \overline{h} \overline{u} = -e^{-it} |h|^2 + e^{-it} \overline{h} (1 + \overline{u}).$$

Therefore,

$$h \Theta_H = -P_+ e^{-it} |h|^2 = -S^* P_+ |h|^2 = -S^* g.$$

Since $|h|^2 \geq 0$ a.e. on $\mathbf{T}$, it follows that

$$|h|^2 = SS^* g + g(0) + \overline{SS^* g}.$$

So

$$\overline{h} = h^{-1} SS^* g + g(0) h^{-1} + h^{-1} \overline{SS^* g},$$

from which it follows that

$$h^{-1} = g(0)^{-1}(\overline{h} - h^{-1}SS^*g - h^{-1}\overline{SS^*g}).$$

Since $h^{-1} \notin \mathbf{H}^\infty$, $h^{-1}SS^*g$ cannot be bounded. So $\Theta_H \notin \mathbf{H}^\infty$, and the proof of the corollary is complete. □

*Remark.* The example in [2] was obtained by considering the function $h(z) = (1-z)^\beta$, where $0 < \beta < 1/2$. Then $h \in \mathbf{H}^\infty$, $\operatorname{dist}(\overline{h}/h, \mathbf{H}^\infty) < 1$ (see [1]), and $h^{-1} \notin \mathbf{H}^\infty$. So by Corollary 3 the central intertwining lifting of $H_{h/\overline{h}}$ has norm one and one regains the main result in [2].

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