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ALMOST EVERYWHERE CONVERGENCE OF LACUNARY PARTIAL SUMS OF VILENKIN-FOURIER SERIES

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ABSTRACT. We prove that if $f \in L^p$, $p > 1$, and $\{n_k\}$ is any lacunary sequence of positive integers, then the sequence of n_kth partial sums of Vilenkin-Fourier series of f converges almost everywhere to f.

1. INTRODUCTION

Let $G = \prod_{i=0}^{\infty} Z_{p_i}$ be the direct product of cyclic groups of order p_i, where $\{p_i\}_{i\geq 0}$ is a sequence of integers with $p_i \geq 2$. Let μ be the Haar measure on G normalized by $\mu(G) = 1$. For $x = \{x_k\} \in G$, let $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $k = 0, 1, 2, \ldots$. The Vilenkin system $\{\chi_n\}$ is the set of all finite products of $\{\phi_k\}$, and is enumerated in the following manner. Let $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. We express each nonnegative integer n as a finite sum in the form $n = \sum_{k=0}^{\infty} \alpha_k m_k$, where $0 \leq \alpha_k < p_k$, and define $\chi_n = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of G, and they form a complete orthonormal system on G. If $p_i = 2$, $i = 0, 1, 2, \ldots$, then $\{\phi_k\}$ are the Rademacher functions and $\{\chi_n\}$ are the Walsh functions. In this paper there is no restriction on the orders $\{p_i\}$.

We consider Fourier series with respect to $\{\chi_n\}$. For $f \in L^1$, let $\widehat{f}(j) = \int_G f(t)\overline{\chi_j(t)}\, d\mu(t)$, $j = 0, 1, 2, \ldots$, and $S_n f = \sum_{j=0}^{n-1} \widehat{f}(j)\chi_j$, $n = 1, 2, \ldots$. Not much is known about the almost everywhere convergence of partial sums when there is no restriction on the orders $\{p_i\}$. We have the following result.

Theorem. *Let $1 < p < \infty$ and $\{n_k\}_{k\geq 1}$ be a lacunary sequence of positive integers, i.e., there is $q > 1$ such that $n_{k+1}/n_k \geq q$, $k = 1, 2, \ldots$. Then there is a constant C_p such that*

$$\|\sup_k |S_{n_k} f|\,\|_p \leq C_p \|f\|_p, \quad f \in L^p. \tag{1.1}$$

It follows that $\lim_{k\to\infty} S_{n_k} f(x) = f(x)$ a.e. for all $f \in L^p$.

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In the proof that follows, C and C_p will denote absolute constants which may vary from line to line.

2. Proof of the Theorem

We first prove (1.1). For $k = 0, 1, \ldots,$ let L_k be the positive integer such that $2^{L_k} \leq p_k < 2^{L_k+1}$. Since every lacunary sequence can be decomposed into a finite number of lacunary subsequences with ratio $q = 2$, we may assume, by adding more terms to the sequence if necessary, that $\{n_j\}$ can be relabelled $\{n_{k,\ell}\}_{k=0,1,\ldots,\, \ell=0,1,\ldots,L_k}$, such that for $k = 0, 1, \ldots,$ $2^\ell m_k \leq n_{k,\ell} < 2^{\ell+1} m_k$ if $\ell = 0, 1, \ldots, L_k - 1$, and $2^{L_k} m_k \leq n_{k,L_k} < m_{k+1}$. (There is no n_{k,L_k} term if $2^{L_k} m_k = m_{k+1}$.) Also, it is sufficient to show that there is a constant C_p such that

$$\Big\| \sup_{\substack{\ell=0,\ldots,L_k \\ k=0,\ldots,N-1}} |S_{n_{k,\ell}} f| \Big\|_p \leq C_p \|f\|_p \tag{2.1}$$

for all $f \in L^p,\ N = 1, 2, \ldots$.

Let $f_k = S_{m_{k+1}} f - S_{m_k} f,\ k = 0, 1, \ldots,$ and $f_{-1} = S_1 f$. We observe that

$$\sup_{\substack{\ell=0,\ldots,L_k \\ k=0,\ldots,N-1}} |S_{n_{k,\ell}} f| \leq \sup_{\substack{\ell=0,\ldots,L_k \\ k=0,\ldots,N-1}} |S_{n_{k,\ell}} f_k| + \sup_{k=0,\ldots,N-1} |S_{m_k} f|.$$

Since $\{S_{m_k} f\}$ is a martingale (see, e.g., [5]), it follows from Doob's inequality ($\|\sup_{k\geq 0} |S_{m_k} f|\,\|_p \leq C_p \|f\|_p,\ f \in L^p$) that (2.1) will be proved if we have

$$\Big\| \sup_{\substack{\ell=0,\ldots,L_k \\ k=0,\ldots,N-1}} |S_{n_{k,\ell}} f_k| \Big\|_p \leq C_p \|f\|_p, \tag{2.2}$$

for all $f \in L^p,\ N = 1, 2, \ldots$.

Now,

$$\begin{aligned} \sup_{\substack{\ell=0,\ldots,L_k \\ k=0,\ldots,N-1}} |S_{n_{k,\ell}} f_k| \leq & \left(\sum_{k=0}^{N-1} |S_{n_{k,0}} f_k|^2 \right)^{1/2} + \sup_{\substack{\ell=1,\ldots,L_k-2 \\ k=0,\ldots,N-1}} |S_{n_{k,\ell}} f_k| \\ & + \left(\sum_{k=0}^{N-1} |S_{n_{k,L_k-1}} f_k|^2 \right)^{1/2} + \left(\sum_{k=0}^{N-1} |S_{n_{k,L_k}} f_k|^2 \right)^{1/2}. \end{aligned}$$

For each of $\ell = 0,\ L_k - 1$ and L_k, we apply [5, Theorem 2] and Burkholder's result for martingales [1, Theorem 3.2] to get

$$\begin{aligned} \Big\| \Big(\sum_{k=0}^{N-1} |S_{n_{k,\ell}} f_k|^2 \Big)^{1/2} \Big\|_p & \leq C_p \Big\| \Big(\sum_{k=0}^{N-1} |f_k|^2 \Big)^{1/2} \Big\|_p \\ & \leq C_p \|f\|_p. \end{aligned}$$

To prove (2.2) it remains to show

$$\Big\| \sup_{\substack{\ell=1,\ldots,L_k-2 \\ k=0,\ldots,N-1}} |S_{n_{k,\ell}} f_k| \Big\|_p \leq C_p \|f\|_p \tag{2.3}$$

for all $f \in L^p,\ N = 1, 2, \ldots$.

We shall use the following operators. Let $k = 0, 1, \ldots$. If $L_k > 2$, define, for $\ell = 1, \ldots, L_k - 2$, the sequence $\{a_{k,\ell}(n)\}_{n\geq 0}$ by

$$a_{k,\ell}(n) = \begin{cases} 1 & \text{if } 2^\ell m_k \leq n < 2^{\ell+1} m_k, \\ \dfrac{j}{2^{\ell-1}} & \text{if } (2^{\ell-1}+j)m_k \leq n < (2^{\ell-1}+j+1)m_k, \\ & \quad j = 0, 1, \ldots, 2^{\ell-1}-1, \\ 1 - \dfrac{j+1}{2^{\ell-1}} & \text{if } (2^{\ell+1}+j)m_k \leq n < (2^{\ell+1}+j+1)m_k, \\ & \quad j = 0, 1, \ldots, 2^{\ell-1}-1, \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$A_{k,\ell} f = \sum_{n=0}^{\infty} a_{k,\ell}(n) \widehat{f}(n) \chi_n.$$

It is proved in [6, pp. 665-666] that

$$\left\| \Big(\sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k - 2} |A_{k,\ell} f|^2 \Big)^{1/2} \right\|_p \leq C_p \|f\|_p, \quad f \in L^p, \quad N = 1, 2, \ldots . \tag{2.4}$$

(We interpret a sum $\sum_{j=k}^{\ell}$ with $\ell < k$ as zero.)

The second operator is defined as follows. Let $k = 0, 1, \ldots$. If $L_k \geq 2$, define, for $\ell = 1, \ldots, L_k - 1$, the sequence $\{b_{k,\ell}(n)\}_{n\geq 0}$ by

$$b_{k,\ell}(n) = \begin{cases} 1 & \text{if } m_k \leq n < 2^{\ell-1} m_k \text{ or } m_{k+1} - (2^{\ell-1}-1)m_k \leq n < m_{k+1}, \\ 1 - \dfrac{j}{2^{\ell-1}} & \text{if } (2^{\ell-1}+j)m_k \leq n < (2^{\ell-1}+j+1)m_k \text{ or} \\ & \quad m_{k+1} - (2^{\ell-1}+j)m_k \leq n < m_{k+1} - (2^{\ell-1}+j-1)m_k, \\ & \quad j = 0, 1, \ldots, 2^{\ell-1}-1, \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in L^1$, set

$$B_{k,\ell} f = \sum_{n=0}^{\infty} b_{k,\ell}(n) \widehat{f}(n) \chi_n.$$

To estimate $B_{k,\ell} f$, we need the following definitions. Let $\{G_k\}$ be the sequence of subgroups of G defined by

$$G_0 = G, \quad G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \quad k = 1, 2, \ldots .$$

We shall identify G with the unit interval $(0,1)$ by associating with each $\{x_i\} \in G$, $0 \leq x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0,1)$. If we disregard the countable set of p_i-rationals, this mapping is one-to-one, onto and measure preserving. On the interval $(0,1)$, cosets of G_k are intervals of the form $\left(jm_k^{-1},(j+1)m_k^{-1}\right)$, $j = 0,1,\ldots,m_k-1$. A set I is called a generalized interval if $I \subset x+G_k$ for some $x \in G$, $k=0,1,\ldots$, I is a union of cosets of G_{k+1}, and I is an interval if we consider $x+G_k$ as a circle.

Let

$$Mf(x) = \sup_{\substack{x\in I\\ I \text{ generalized}\\ \text{interval}}} \frac{1}{\mu(I)}\int_I |f|\,d\mu$$

be the Hardy-Littlewood maximal function for the Vilenkin system. The following pointwise estimate will be obtained in §3.

Lemma. *There is a constant C such that*

$$\sup_{\substack{\ell=1,\ldots,L_k-1\\ k=0,1,\ldots}} |B_{k,\ell}f(x)| \le CMf(x), \qquad f\in L^1,\ x\in G.$$

To estimate $S_{n_{k,\ell}}f_k$, $\ell=1,\ldots,L_k-2$, $k=0,1,\ldots$, we shall use the part of $b_{k,\ell}(n)$ with $n\in[m_k,2^\ell m_k)$. In order to get rid of the remaining part of $b_{k,\ell}(n)$, we define

$$H^N f = \sum_{k=0}^{N-1}\left(S_{2^{L_k-1}m_k}f - S_{m_k}f\right).$$

Since $H^N f = S_{m_N}f - \sum\limits_{k=0}^{N-1}\left(S_{m_{k+1}}f - S_{2^{L_k-1}m_k}f\right)$, and it is proved in [4, Theorem 1*] that

$$\left\|\sum_{k=0}^{N-1}\left(S_{m_{k+1}}f - S_{2^{L_k-1}m_k}f\right)\right\|_p \le C_p\|f\|_p$$

for all $f\in L^p$, $N=1,2,\ldots$, we have

(2.5) $$\|H^N f\|_p \le C_p\|f\|_p, \quad f\in L^p, \quad N=1,2,\ldots .$$

(See also [2].)

We are now ready to prove (2.3). We observe that for $k=0,\ldots,N-1$, $\ell=1,\ldots,L_k-2$,

$$S_{n_{k,\ell}}f_k = B_{k,\ell}(H^N f) + S_{n_{k,\ell}}(A_{k,\ell}f).$$

Hence

(2.6)

$$\sup_{\substack{\ell=1,\ldots,L_k-2\\ k=0,\ldots,N-1}} |S_{n_{k,\ell}}f_k| \le \sup_{\substack{\ell=1,\ldots,L_k-2\\ k=0,\ldots,N-1}} B_{k,\ell}(H^N f) + \left(\sum_{k=0}^{N-1}\sum_{\ell=1}^{L_k-2}|S_{n_{k,\ell}}(A_{k,\ell}f)|^2\right)^{1/2}.$$

To estimate the first term on the right, we apply the lemma, the fact that the Hardy-Littlewood maximal operator M is bounded in L^p (see [3]) and (2.5). We have

$$\begin{aligned}\left\|\sup_{\substack{\ell=1,\ldots,L_k-2\\ k=0,\ldots,N-1}} |B_{k,\ell}(H^N f)|\right\|_p &\le C\|M(H^N f)\|_p\\ &\le C_p\|H^N f\|_p \le C_p\|f\|_p\end{aligned}$$

for all $f \in L^p,\ N = 1, 2, \dots$. For the last term in (2.6) we use [5, Theorem 2] and (2.4). We obtain

$$\Big\|\Big(\sum_{k=0}^{N-1}\sum_{\ell=1}^{L_k-2}|S_{n_{k,\ell}}(A_{k,\ell}f)|^2\Big)^{1/2}\Big\|_p \le C_p\Big\|\Big(\sum_{k=0}^{N-1}\sum_{\ell=1}^{L_k-2}|A_{k,\ell}f|^2\Big)^{1/2}\Big\|_p$$
$$\le C_p\|f\|_p$$

for all $f \in L^p,\ N = 1, 2, \dots$. This proves (2.3). The proof of (1.1) will be complete once we prove the lemma.

Finally, if $f \in L^p$, $\lim_{k\to\infty}\|S_{m_k}f - f\|_p = 0$ since $S_{m_k}f$ is the average of f over the cosets of G_k. As a consequence of this and (1.1), we have $\lim_{k\to\infty} S_{n_k}f(x) = f(x)$ a.e.

3. Proof of the Lemma

We shall use the following notation. For each generalized interval I, we define the generalized interval $3I$ as follows. If $I = G$, let $3I = G$. Suppose I is a proper subset of $x + G_k$, $x \in G$, $k = 0, 1, 2, \dots$, and is the union of cosets of G_{k+1}. If $\mu(I) \ge \mu(G_k)/3$, let $3I = x + G_k$. If $\mu(I) < \mu(G_k)/3$, consider $x + G_k$ as a circle and define $3I$ to be the interval in this circle which has the same center as I and has measure $\mu(3I) = 3\mu(I)$. For all cases, we have $\mu(3I) \le 3\mu(I)$.

For $k = 0, 1, \dots,\ \ell = 1, \dots,\ L_k - 1$ and $f \in L^1$, we have

$$B_{k,\ell}f(x) = \int_G f(t)\Big[\sum_{n=0}^{\infty} b_{k,\ell}(n)\chi_n(x-t)\Big]d\mu(t).$$

Since $b_{k,\ell}(n)$ vanishes for $n \notin [m_k, m_{k+1})$ and is constant for $n \in [\alpha m_k, (\alpha+1)m_k)$, $\alpha = 0, 1, \dots$, we get

$$\sum_{n=0}^{\infty} b_{k,\ell}(n)\chi_n = \sum_{\alpha=1}^{p_k-1} b_{k,\ell}(\alpha m_k)\sum_{n=\alpha m_k}^{(\alpha+1)m_k-1}\chi_n$$
$$= \sum_{\alpha=1}^{p_k-1} b_{k,\ell}(\alpha m_k)\phi_k^\alpha D_{m_k},$$

where $D_n = \sum_{j=0}^{n-1}\chi_j$, $n = 1, 2, \dots$, denotes the n^{th} Dirichlet kernel. Since $D_{m_k} = \mu(G_k)^{-1}\chi_{G_k}$, we have

$$B_{k,\ell}f(x) = \frac{1}{\mu(G_k)}\int_{x+G_k} f(t)M_{k,\ell}(x-t)\,d\mu(t), \tag{3.1}$$

where

$$M_{k,\ell} = \sum_{\alpha=1}^{p_k-1} b_{k,\ell}(\alpha m_k)\phi_k^\alpha$$
$$= \sum_{\alpha=1}^{2^\ell-1} b_{k,\ell}(\alpha m_k)\phi_k^\alpha + \phi_k^0 + \sum_{\alpha=-(2^\ell-1)}^{-1} b_{k,\ell}((p_k+\alpha)m_k)\phi_k^\alpha - \phi_k^0.$$

Let

$$D_{k,j} = \sum_{\alpha=-j}^{j} \phi_k^\alpha, \qquad j = 0, 1, \ldots, 2^{L_k - 1} - 1,$$

and

$$K_{k,n} = \frac{1}{n} \sum_{j=0}^{n-1} D_{k,j}, \qquad n = 1, \ldots, 2^{L_k - 1}.$$

Then

$$\begin{aligned} M_{k,\ell} &= \frac{1}{2^{\ell-1}} \big(D_{k,2^{\ell-1}} + D_{k,2^{\ell-1}+1} + \cdots + D_{k,2^\ell - 1} \big) - \phi_k^0 \\ &= 2K_{k,2^\ell} - K_{k,2^{\ell-1}} - \phi_k^0. \end{aligned} \tag{3.2}$$

For $n = 1, \ldots, 2^{L_k - 1}$, let

$$\sigma_{k,n} f(x) = \frac{1}{\mu(G_k)} \int_{x+G_k} f(t) K_{k,n}(x-t)\, d\mu(t).$$

From (3.1) and (3.2) we obtain

$$|B_{k,\ell} f(x)| \leq 2|\sigma_{k,2^\ell} f(x)| + |\sigma_{k,2^{\ell-1}} f(x)| + Mf(x).$$

The lemma will be proved if we show

$$|\sigma_{k,n} f(x)| \leq C\, Mf(x), \tag{3.3}$$

for all $k = 0, 1, \ldots,\ \ n = 1, \ldots, 2^{L_k - 1},\ \ x \in G$.

By a direct computation we have

$$K_{k,n}(x) = \begin{cases} \dfrac{1}{n} \left(\sin^2 \dfrac{\pi n x_k}{p_k} \Big/ \sin^2 \dfrac{\pi x_k}{p_k} \right) & \text{if } x_k \neq 0, \\ n & \text{if } x_k = 0, \end{cases}$$

and hence

$$0 \leq K_{k,n}(x) \leq \min \left\{ \left(n \sin^2 \frac{\pi x_k}{p_k} \right)^{-1}, n \right\}. \tag{3.4}$$

Let I be a generalized interval containing x such that $I \subset x + G_k$, I is a union of cosets of G_{k+1} and

$$\frac{p_k}{n} - 1 < \frac{\mu(I)}{\mu(G_{k+1})} \leq \frac{p_k}{n}.$$

(Note that $p_k/n \geq 2$ since $n \leq 2^{L_k - 1}$.) For $j = 1, 2, \ldots$, let $3^{j+1} I = 3(3^j I)$, and $J = \min\{ j \geq 1 : 3^j I = x + G_k \}$. Then

$$\begin{aligned} |\sigma_{k,n} f(x)| \leq{} & \frac{1}{\mu(G_k)} \int_{3I} |f(t)| K_{k,n}(x-t)\, d\mu(t) \\ & + \sum_{j=1}^{J-1} \frac{1}{\mu(G_k)} \int_{3^{j+1} I \setminus 3^j I} |f(t)| K_{k,n}(x-t)\, d\mu(t). \end{aligned} \tag{3.5}$$

By (3.4),

$$\frac{1}{\mu(G_k)}\int_{3I}|f(t)|K_{k,n}(x-t)\,d\mu(t)\le\frac{3\mu(I)}{\mu(G_k)}\,n\,\frac{1}{\mu(3I)}\int_{3I}|f(t)|\,d\mu(t)\le CM\,f(x).$$

To estimate the last term in (3.5), we apply the other estimate in (3.4). We observe that when $x\in I$, $t\notin 3^jI$, $j=1,\dots,J-1$,

$$\left|\sin\frac{\pi(x_k-t_k)}{p_k}\right|\ge C\mu(3^{j-1}I)/\mu(G_k).$$

Hence

$$\begin{aligned}
&\frac{1}{\mu(G_k)}\int_{3^{j+1}I\setminus 3^jI}|f(t)|K_{k,n}(x-t)\,d\mu(t)\\
&\quad\le\frac{\mu(3^{j+1}I)}{\mu(G_k)}\,\frac{1}{\mu(3^{j+1}I)}\int_{3^{j+1}I\setminus 3^jI}|f(t)|\left(n\sin^2\frac{\pi(x_k-t_k)}{p_k}\right)^{-1}\,d\mu(t)\\
&\quad\le C\frac{\mu(3^{j+1}I)}{\mu(G_k)}\,\frac{1}{n}\left[\frac{\mu(G_k)}{\mu(3^{j-1}I)}\right]^2\frac{1}{\mu(3^{j+1}I)}\int_{3^{j+1}I}|f(t)|\,d\mu(t)\\
&\quad\le C3^{-j}\frac{\mu(G_k)}{n\mu(I)}\,Mf(x)\\
&\quad\le C3^{-j}M\,f(x).
\end{aligned}$$

Substituting these estimates into (3.5) and summing over j, we obtain (3.3). The lemma follows.

This completes the proof of the theorem.

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