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ON RATIONAL INVARIANTS OF THE GROUP E_6

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Abstract. We prove rationality of the field of invariants in several variables of a minimal irreducible representation of a simple algebraic group of type E_6 over an algebraically closed field of characteristic zero.

This note completes the main result of [3], where the reader can find an extensive introduction and basic notations.

Let $\mathbf{A} = H(\mathbf{O}_3)$ be an Albert algebra [4, p.21] over an algebraically closed field F of characteristic 0. Any element $x \in \mathbf{A}$ satisfies a cubic equation over F

$$x^3 - x^2 t_1(x) + x\, t_2(x) - e\, t_3(x) = 0\ , \tag{1}$$

where e is the unit of $\mathbf{A}$, $t_1(x)$ is the trace $tr(x)$ of x and $t_3(x)$ is the cubic norm $N(x)$ of x.

The group of norm-preserving linear transformations

$$M = \{g \in End(\mathbf{A}) |\ g \text{ is invertible},\ \forall x \in \mathbf{A}\ N(x \cdot g) = N(x)\}$$

is a simple algebraic group of type E_6 [1] and $\mathbf{A}$ is an irreducible M-module of dimension 27 [5, th.14]. The form $tr(xy)$ is nondegenerate. For $g \in End(\mathbf{A})$ denote by g^* the transposed transformation with respect to this form. We may consider $\mathbf{A}$ as an M-module $\mathbf{A}^*$ under the action $x \to x \cdot g^\tau$, where $g \to g^\tau = (g^*)^{-1}$ is an (outer) automorphism of M [4, p.370, ex.2] (in other words, $\mathbf{A}^*$ is the module contragredient to $\mathbf{A}$). Let $\mathbf{V}_{k,m} = k\mathbf{A} \oplus m\mathbf{A}^*$, where $k\mathbf{A}$ ($m\mathbf{A}^*$) is the direct sum of k (m) copies of $\mathbf{A}$ ($\mathbf{A}^*$) and M acts diagonally.

Theorem 1. *For any $k, m \geq 0$ the field of rational invariants $F(\mathbf{V}_{k,m})^M$ is purely transcendental over F.*

Proof. Since $F(\mathbf{V}_{k,m})^M \simeq F(\mathbf{V}_{m,k})^M$, we can assume that $k \geq 1$. Using the slice method [2] (see also, for example, [8]), we shall show that this field is isomorphic to a simple transcendental extension of $F(n\mathbf{A})^G$, where $n = m + k - 1$ and G is the group of automorphisms $Aut(\mathbf{A})$. Since rationality of the latter field is obtained in [3], this will prove the statement. In particular, the transcendence degree of this

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field is computed in the following way (cf. [3]):

$$tr.\,deg_F F(\mathbf{V}_{k,m})^M = \begin{cases} 30+27(m+k-4), & m+k\geq 4,\\ 11, & m+k=3,\\ 3, & m+k=2,\\ 1, & m+k=1.\end{cases}$$

Lemma 1 ([5, th.7], [4, p.244])**.** 1. *If $a\in\mathbf{A}$ and $N(a)=1$, then there is $\eta\in M$ such that $a\cdot\eta=e$.*

2. *If $\phi\in M$ and $e\cdot\phi=e$, then ϕ is an automorphism of $\mathbf{A}$.*

From the lemma it follows that the subspace

$$E=\{\bar a=(a_1,\dots,a_k,b_1,\dots,b_m)\in\mathbf{V}_{k,m}|\ a_k\in eF\}$$

is a slice of the M-module $\mathbf{V}_{k,m}$ [2, sect.3]. Hence, $F(\mathbf{V}_{k,m})^M\simeq F(E)^{N(E)}$, where $N(E)=\{\phi\in M|\ E\cdot\phi\subseteq E\}$. Also, $N(E)=G\times C_3$, where C_3 is the group of linear transformations $R_x: y\to yx$, where $x=e\xi$, $\xi\in F$ and $\xi^3=1$.

Notice that $E=eF\oplus\mathbf{V}_{k-1,m}$. Denote by t the coordinate function of the first summand and by y_i (z_i) the projector from $k\mathbf{A}$ ($m\mathbf{A}^*$) onto the i-th summand (generic elements of $\mathbf{A}$ [3]). Consider a monomorphism $\psi: F(\mathbf{V}_{k-1,m})\to F(E)$: if $f=f(y_2,\dots,y_k,z_1,\dots,z_m)$, then $\psi(f)=f(y_2/t,\dots,y_k/t,z_1t,\dots,z_mt)$. Since $R_x^\tau=R_{x^{-1}}$ for $R_x\in C_3$, the image K of $F(\mathbf{V}_{k-1,m})^G$ lies in $F(E)^{N(E)}=(F(E)^G)^{C_3}$. Observe that $F(E)$ can be considered as the field of rational functions of the G-module $\mathbf{V}_{k-1,m}\otimes_F F(t)$ over $F(t)$, where the generic elements are $y_2/t,\dots,y_k/t,z_1t,\dots,z_mt$; hence, $F(E)^G=K(t)$. Since for the extension degree we have $|F(E)^{N(E)}:F(E)^G|=3$, we get $K(t^3)=F(E)^{N(E)}\simeq F(\mathbf{V}_{k,m})^M$. □

It is known that the field of invariants $F(k\mathbf{A})^G$ is generated by trace polynomials [3]. Working out this proof, we shall describe generators of the field $F(\mathbf{V}_{k,m})^M$ in a similar way. Let $N(x,y,z)$ be the linearization of $N(x)$ ($N(x,x,x)=3N(x)$) and let $x\to x^\#$ be a quadratic operation such that $tr(x^\#y)=N(x,x,y)$ [4, 6]. Also, $x\times y=(x+y)^\#-x^\#-y^\#$. Consider a pair of subspaces Δ, $\Delta^*\subseteq\mathbf{A}\otimes_F F[\mathbf{V}_{k,m}]$ which are minimal with the following properties:

1. $y_1,\dots,y_k\in\Delta$, $z_1,\dots,z_m\in\Delta^*$;
2. if $u,v\in\Delta$ ($u,v\in\Delta^*$), then $u\times v\in\Delta^*$ (Δ);

Theorem 2. *The field $F(\mathbf{V}_{k,m})^M$ is generated by elements $N(u)$, $u\in\Delta$.*

By the definition of $x\to x^\#$, for any $g\in M$ we have $(x\cdot g)^\#=x^\#\cdot g^\tau$. The next statement can be easily checked by induction.

Lemma 2. *Let $u\in\Delta$ (Δ^*). Then for any $g\in M$ the polynomial*

$$u(y_1\cdot g,\dots,y_k\cdot g,z_1\cdot g^\tau,\dots,z_m\cdot g^\tau)$$

is equal to $u\cdot g$ (respectively, $u\cdot g^\tau$).

In particular, this yields $N(u)\in F(\mathbf{V}_{k,m})^M$ for any $u\in\Delta$. Since M is a simple linear algebraic group, the field $F(\mathbf{V}_{k,m})^M$ is generated by homogeneous polynomial invariants [7, sect.3.3]. Take such an invariant $f=f(y_1,\dots,y_k,z_1,\dots,z_m)$ and put $p=\sum_{i=1}^k deg_{y_i}(f)$, $q=\sum_{i=1}^m deg_{z_i}(f)$. Recall that E is a slice and, therefore, the mapping $\pi: f\to f|_E$ is a monomorphism from the algebra of polynomial invariants $F[\mathbf{V}_{k,m}]^M$ to $F(E)^{N(E)}$ and its image generates this field. Denote $\Omega=$

$alg_F\{N(u)|\ u \in \Delta\} \subseteq F[\mathbf{V}_{k,m}]^M$. To prove the theorem it suffices to show that $\pi(f)$ belongs to the field K generated by $\pi(\Omega)$.

Observe that $t^3 = \pi(N(y_1)) \in K$. Also, $\pi(f) = ht^{p-q}$, where

$$h = f(e, y_2/t, \dots, y_k/t, z_1 t, \dots, z_m t) = \psi(\tilde{f})$$

and $\tilde{f} = f|_{y_1=e} \in F(\mathbf{V}_{k-1,m})^G$. Recall that $\psi(\tilde{f}) \in F(E)^{N(E)}$; hence, t^{p-q} also lies in this field. Therefore, $r = (p-q)/3$ is an integer and $t^{p-q} \in K$. In particular, $\psi(\tilde{g})$ is defined and belongs to K for any $g \in \Omega$ and, therefore, for any $g \in F(\Omega)$. If we show that $\tilde{f} = \tilde{g}$ for some $g \in F(\Omega)$, then we shall get $h = \psi(\tilde{g}) \in K$ and prove the theorem.

The field $F(\mathbf{V}_{k-1,m})^G$ is generated by polynomials $tr(u)$, where $u \in B_{k+m-1} = alg_F\{y_2, \dots, y_k, z_1, \dots, z_m\}$ [3]. It suffices to show that $tr(u) = \tilde{g}$ for an appropriate $g \in F(\Omega)$. Let $\Delta\Omega$ be the Ω-module generated by Δ.

Lemma 3. *For any $u \in B_{k+m-1}$ there is $h_u \in \Delta\Omega$ such that $h_u|_{y_1=e} = u$.*

Proof. We shall need the following identities, which follow from [6, eq.(11),(21)]:

$$a = -a \times e + tr(a) \cdot e\ , \tag{2}$$

$$a^2 = (a \times e)^{\#} - N(a, a, e)e\ . \tag{3}$$

Denote $\omega(a, b) = -a \times b^{\#} + N(a, b^{\#}, b^{\#})b$. By (2), if $a \in \Delta^*$ and $b \in \Delta$, then $\omega(a, b) \in \Delta\Omega$ and $\omega(a, e) = a$. We assume that u is a monomial and carry out an induction on its degree. First, put $h_{y_i} = y_i$, $2 \le i \le k$, and $h_{z_l} = \omega(z_l, y_1)$, $1 \le l \le m$. Now, suppose that $deg(u) = 2$, i.e. u has the form $u_1 = y_i y_j$, $u_2 = z_s y_j$ or $u_3 = z_s z_t$. Using the linearized form of (3), we may put

$$h_{u_1} = 1/2(y_i \times y_1) \times (y_j \times y_1) - N(y_i, y_j, y_1)y_1\ .$$

Next,

$$h_{u_2} = h_{u_1}|_{y_i=\omega(z_s,y_1)}, \quad h_{u_3} = h_{u_2}|_{y_j=\omega(z_t,y_1)}.$$

We can assume that the lemma holds for an arbitrary monomial $u \in B_{k+m-1}$ of degree $< n$ and for arbitrary k, m. Every monomial u, $deg(u) = n \ge 3$, can be represented in the form $u = v|_{y_{k+1}=w}$, where $deg(v), deg(w) < n$. Hence,

$$h_u = h_v|_{y_{k+1}=h_w}.$$

To conclude the proof of Theorem 2 we may put $g = N(h_u, y_1, y_1)$. □

Recall that Ω lies in the algebra of polynomial invariants $F[\mathbf{V}_{k,m}]^M$, and it looks probable that this algebra is generated by elements $N(u)$, $N(v)$, $tr(uv)$, where $u \in \Delta$, $v \in \Delta^*$.

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