ON RATIONAL INVARIANTS OF THE GROUP $E_6$

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Abstract. We prove rationality of the field of invariants in several variables of a minimal irreducible representation of a simple algebraic group of type $E_6$ over an algebraically closed field of characteristic zero.

This note completes the main result of [3], where the reader can find an extensive introduction and basic notations.

Let $A = H(O_3)$ be an Albert algebra [4, p.21] over an algebraically closed field $F$ of characteristic 0. Any element $x \in A$ satisfies a cubic equation over $F$

$$x^3 - x^2 t_1(x) + x t_2(x) - e t_3(x) = 0,$$

where $e$ is the unit of $A$, $t_1(x)$ is the trace $tr(x)$ of $x$ and $t_3(x)$ is the cubic norm $N(x)$ of $x$.

The group of norm-preserving linear transformations $M = \{ g \in End(A) | g \text{ is invertible}, \forall x \in A \ N(x \cdot g) = N(x) \}$

is a simple algebraic group of type $E_6$ [1] and $A$ is an irreducible $M$-module of dimension 27 [5, th.14]. The form $tr(xy)$ is nondegenerate. For $g \in End(A)$ denote by $g^*$ the transposed transformation with respect to this form. We may consider $A$ as an $M$-module $A^*$ under the action $x \rightarrow x \cdot g^*$, where $g \rightarrow g^* = (g^*)^{-1}$ is an (outer) automorphism of $M$ [4, p.370, ex.2] (in other words, $A^*$ is the module contragredient to $A$). Let $V_{k,m} = kA \oplus mA^*$, where $kA$ ($mA^*$) is the direct sum of $k$ ($m$) copies of $A$ ($A^*$) and $M$ acts diagonally.

**Theorem 1.** For any $k, m \geq 0$ the field of rational invariants $F(V_{k,m})^M$ is purely transcendental over $F$.

Proof. Since $F(V_{k,m})^M \simeq F(V_{m,k})^M$, we can assume that $k \geq 1$. Using the slice method [2] (see also, for example, [8]), we shall show that this field is isomorphic to a simple transcendental extension of $F(nA)^G$, where $n = m + k - 1$ and $G$ is the group of automorphisms $Aut(A)$. Since rationality of the latter field is obtained in [3], this will prove the statement. In particular, the transcendence degree of this
field is computed in the following way (cf. [3]):
\[
tr.\ deg_F(V_{k,m})^M = \begin{cases} 
30 + 27(m + k - 4), & m + k \geq 4, \\
11, & m + k = 3, \\
3, & m + k = 2, \\
1, & m + k = 1.
\end{cases}
\]

**Lemma 1** ([5, th.7], [4, p.244]). 1. If \(a \in A\) and \(N(a) = 1\), then there is \(\eta \in M\) such that \(a \cdot \eta = e\).

2. If \(\phi \in M\) and \(e \cdot \phi = e\), then \(\phi\) is an automorphism of \(A\).

From the lemma it follows that the subspace
\[
E = \{\bar{a} = (a_1, \ldots, a_k, b_1, \ldots, b_m) \in V_{k,m} | a_k \in eF\}
\]
is a slice of the \(M\)-module \(V_{k,m}\) [2, sect.3]. Hence, \(F(V_{k,m})^M \simeq F(E)^{N(E)}\), where \(N(E) = \{\phi \in M | E \cdot \phi \subseteq E\}\). Also, \(N(E) = G \times C_3\), where \(C_3\) is the group of linear transformations \(R_x : y \rightarrow yx\), where \(x = \xi^2, \xi \in F\) and \(\xi^3 = 1\).

Notice that \(E = eF \oplus V_{k-1,m}\). Denote by \(t\) the coordinate function of the first summand and by \(y_i\) the projector from \(kA\) \((mA^*)\) onto the \(i\)-th summand (generic elements of \(A\) [3]). Consider a monomorphism \(\psi : F(V_{k-1,m}) \rightarrow F(E)\): if \(f = f(y_2, \ldots, y_k, z_1, \ldots, z_m)\), then \(\psi(f) = f(y_2/t, \ldots, y_k/t, z_1t, \ldots, z_m t)\). Since \(R^*_x = R^*_{1-x} \equiv R_x \in C_3\), the image \(K\) of \(F(V_{k-1,m})^G\) lies in \(F(E)^{N(E)}\). Observe that \(F(E)\) can be considered as the field of rational functions of the \(G\)-module \(V_{k-1,m} \otimes_F F(t)\) over \(F(t)\), where the generic elements are \(y_2/t, \ldots, y_k/t, z_1t, \ldots, z_m t\); hence, \(F(E)^G = K(t)\). Since for the extension degree we have \(|F(E)^{N(E)} : F(E)^G| = 3\), we get \(K(t^3) = F(E)^{N(E)} \simeq F(V_{k,m})^M\).

It is known that the field of invariants \(F(kA)^G\) is generated by trace polynomials [3]. Working out this proof, we shall describe generators of the field \(F(V_{k,m})^M\) in a similar way. Let \(N(x, y, z)\) be the linearization of \(N(x)\) \((N(x, x, x) = 3N(x))\) and let \(x \rightarrow x^\#\) be a quadratic operation such that \(tr(x^\#y) = N(x, x, y)\) [4, 6]. Also, \(x \times y = (x + y)^\# - g^\#\). Consider a pair of subspaces \(\Delta, \Delta^* \subseteq A \otimes_F F[V_{k,m}]\) which are minimal with the following properties:

1. \(y_1, \ldots, y_k \in \Delta, z_1, \ldots, z_m \in \Delta^*\);
2. if \(u, v \in \Delta (u, v \in \Delta^*)\), then \(u \times v \in \Delta^* (\Delta)\).

**Theorem 2.** The field \(F(V_{k,m})^M\) is generated by elements \(N(u), u \in \Delta\).

By the definition of \(x \rightarrow x^\#\), for any \(g \in M\) we have \((x \cdot g)^\# = x^\# \cdot g^\#\). The next statement can be easily checked by induction.

**Lemma 2.** Let \(u \in \Delta (\Delta^*)\). Then for any \(g \in M\) the polynomial
\[
u(y_1 \cdot g, \ldots, y_k \cdot g, z_1 \cdot g^\#, \ldots, z_m \cdot g^\#)
\]
is equal to \(u \cdot g\) \((\text{respectively}, u \cdot g^\#)\).

In particular, this yields \(N(u) \in F(V_{k,m})^M\) for any \(u \in \Delta\). Since \(M\) is a simple linear algebraic group, the field \(F(V_{k,m})^M\) is generated by homogeneous polynomial invariants [7, sect.3.3]. Take such an invariant \(f = f(y_1, \ldots, y_k, z_1, \ldots, z_m)\) and put \(p = \sum_{i=1}^k deg_y(f), q = \sum_{i=1}^m deg_z(f)\). Recall that \(E\) is a slice and, therefore, the mapping \(\pi : f \rightarrow |f|_E\) is a monomorphism from the algebra of polynomial invariants \(F[V_{k,m}]^M\) to \(F(E)^{N(E)}\) and its image generates this field. Denote \(\Omega = \ldots\)

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alg_{F}\{N(u)\mid u \in \Delta\} \subseteq F[V_{k,m}]^M. To prove the theorem it suffices to show that \(\pi(f)\) belongs to the field \(K\) generated by \(\pi(\Omega)\).

Observe that \(t^3 = \pi(N(y_1)) \in K\). Also, \(\pi(f) = ht^{p-q}\), where
\[
h = f(e, y_2/t, \ldots, y_k/t, z_1t, \ldots, z_m t) = \psi(\tilde{f})
\]
and \(\tilde{f} = f|_{y_1=e} \in F[V_{k-1,m}]^G\). Recall that \(\psi(\tilde{f}) \in F(E)^{N(E)}\); hence, \(t^{p-q}\) also lies in this field. Therefore, \(r = (p - q)/3\) is an integer and \(t^{p-q} \in K\). In particular, \(\psi(\tilde{g})\) is defined and belongs to \(K\) for any \(g \in \Omega\) and, therefore, for any \(g \in F(\Omega)\). If we show that \(\tilde{f} = \tilde{g}\) for some \(g \in F(\Omega)\), then we shall get \(h = \psi(\tilde{g}) \in K\) and prove the theorem.

The field \(F(V_{k-1,m})^G\) is generated by polynomials \(tr(u)\), where \(u \in B_{k+m-1} = alg_{F}\{y_2, \ldots, y_k, z_1, \ldots, z_m\}\) \([3]\). It suffices to show that \(tr(u) = \tilde{g}\) for an appropriate \(g \in F(\Omega)\). Let \(\Delta\Omega\) be the \(\Omega\)-module generated by \(\Delta\).

**Lemma 3.** For any \(u \in B_{k+m-1}\) there is \(h_u \in \Delta\Omega\) such that \(h_u|_{y_1=e} = u\).

**Proof.** We shall need the following identities, which follow from \([6, eq.(11),(21)]:\)
\[(2)\]
\[a = -a \times e + tr(a) \cdot e,\]
\[(3)\]
\[a^2 = (a \times e)^\# - N(a, a, e) e.\]

Denote \(\omega(a, b) = -a \times b^\# + N(a, b^\#, b^\#) b\). By \(2\), if \(a \in \Delta^*\) and \(b \in \Delta\), then \(\omega(a, b) \in \Delta\Omega\) and \(\omega(a, e) = a\). We assume that \(u\) is a monomial and carry out an induction on its degree. First, put \(h_{y_i} = y_i, 2 \leq i \leq k\), and \(h_{z_i} = \omega(z_i, y_1)\), \(1 \leq l \leq m\). Now, suppose that \(deg(u) = 2\), i.e. \(u\) has the form \(u_1 = y_1y_j, u_2 = zsy_j\) or \(u_3 = zsz_t\). Using the linearized form of \(3\), we may put
\[h_{u_1} = 1/2(y_i \times y_1) \times (y_j \times y_1) - N(y_1, y_j, y_1)y_1.\]

Next,
\[h_{u_2} = h_{u_1}|_{y_i = \omega(z_i, y_1)}, \quad h_{u_3} = h_{u_2}|_{y_j = \omega(z_i, y_1)}.\]

We can assume that the lemma holds for an arbitrary monomial \(u \in B_{k+m-1}\) of degree \(< n\) and for arbitrary \(k, m\). Every monomial \(u\), \(deg(u) = n \geq 3\), can be represented in the form \(u = v|_{y_k+1 = w}\), where \(deg(v), deg(w) < n\). Hence,
\[h_u = h_v|_{y_k+1 = w}.\]

To conclude the proof of Theorem 2 we may put \(g = N(h_u, y_1, y_1)\).

Recall that \(\Omega\) lies in the algebra of polynomial invariants \(F[V_{k,m}]^M\), and it looks probable that this algebra is generated by elements \(N(u), N(v), tr(uv)\), where \(u \in \Delta, v \in \Delta^*\).

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REFERENCES


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