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THE MULTIPLIER OPERATORS ON THE WEIGHTED PRODUCT SPACES

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ABSTRACT. In this paper, we proved the boundedness of multiplier operators on the weighted L^p product spaces.

Let $m(\xi)$ be a function on R^n and let f be a smooth function on R^n. Suppose

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi).$$

Then $T_m f$ is called a multiplier operator. It is well known that a multiplier operator is bounded on the weighted L^p, $1 < p < \infty$, spaces for some suitable weights if the function $m(\xi)$ satisfies Hörmander's condition

$$\int_{s\le|\xi|\le 2s} |\partial_\xi^\alpha m(\xi)|^2 d\xi \le s^{n-2|\alpha|}$$

for $|\alpha| \le [n/2]+1$ (see [3, page 418]). The keys to proving the boundedness of multiplier operator in the weighted L^p spaces are basically

(i) the Hardy-Littlewood maximal operator is bounded by the sharp function, more precisely,

$$\int (Mf(x))^p W(x)dx \le C \int (f^{\#}(x))^p W(x)dx \tag{1}$$

where $W \in A_p$, $1 < p \le \infty$, and

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

f_Q being the average of f over cube Q in R^n;

(ii) an estimate,

$$(T_m f)^{\#}(x) \le C(M|f|^q(x))^{1/q}$$

for some $q > 1$.

The purpose of this paper is to study the boundedness of multiplier operators on the weighted L^p product spaces.

Denote

$$\operatorname{osc}_R f = \inf_{\substack{f_1\\ f_2}} \left(\frac{1}{|R|}\int_R |f(x_1,x_2) - f_1(x_1) - f_2(x_2)|^2 dx_1 dx_2\right)^{1/2}$$

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where R is any rectangle in $R^{n_1} \times R^{n_2}$ and the inf is taken over all functions f_1 and f_2 depending on the variables x_1 and x_2 respectively.

As in the "one dimensional case", if one defines the sharp function by

$$f^{\#}(x) = \sup_{x \in R} \operatorname{osc}_R f,$$

then one might expect to show that the strong maximal operator

$$M_s f(x) = \sup_{\substack{t_1>0 \\ t_2>0}} \frac{1}{t_1^{n_1} t_2^{n_2}} \int_{|y_1|\le t_1} \int_{|y_2|\le t_2} |f(x_1 - y_1, x_2 - y_2)| dy_1 dy_2$$

is bounded by the sharp function. Unfortunately, such an observation is not true due to Carleson's counterexample [1]. To remove the difficulty on the sharp function in order to obtain an inequality similar to (1), R. Fefferman considered a sharp operator (see [2]) defined as follows.

Definition. Let T be an L^2 bounded linear operator. Suppose there exists an operator $T^{\#}$ defined on positive locally square integrable functions which is monotone, i.e.

$$T^{\#} f(x) \le T^{\#} g(x)$$

if $f(x) \le g(x)$ for all $x \in R^{n_1} \times R^{n_2}$ such that $\operatorname{osc}_R(Tf) \le \gamma^{-\sigma} T^{\#} f(x)$ for all $x \in R$, R a rectangle on $R^{n_1} \times R^{n_2}$, and for some $\sigma > 0$, where f is supported outside of the γ-fold dilation of R, $\gamma \ge 2$.

Based on this definition of a sharp operator, R. Fefferman [2] obtained the following inequality:

$$\int\int S^2(Tf)(x)\phi(x)dx \le C \int\int [(I + T^{\#})(|f|)]^2(x) M_s(M_s(M_s(M_s(\phi))))(x)dx$$

where I denotes the identity operator and S is the area function defined on the product spaces. Using this inequality, he obtained the following theorem, which we will apply in this paper.

Theorem A ([2]). *If T is a bounded linear operator on $L^2(R^{n_1} \times R^{n_2})$ whose sharp operator is*

$$T^{\#} f = M_s(f^2)^{1/2},$$

then for $p > 2$

$$\int_{R^{n_1} \times R^{n_2}} |Tf|^p W \le C \int_{R^{n_1} \times R^{n_2}} |f|^p W$$

whenever $W \in A_{p/2}(R^{n_1} \times R^{n_2})$.

Proof. See [2, page 123]. □

In this paper, we will prove the following Theorem.

Theorem. *Let $m(\xi_1,\xi_2)$ be a function and $m \in C^{p_1}(R^{n_1}\setminus\{0\})\times C^{p_2}(R^{n_2}\setminus\{0\})$, where $p_1=[n_1/2]+1$, and $p_2=[n_2/2]+1$. Suppose*

$$\begin{aligned}
&\int_{|\xi_1|\approx s_1}\int_{|\xi_2|\approx s_2}|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}m(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2 \le C s_1^{-2|\alpha_1|+n_1} s_2^{-2|\alpha_2|+n_2};\\
&\sup_{\xi_2}\int_{|\xi_1|\approx s_1}|\partial_{\xi_1}^{\alpha_1}m(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2 \le C s_1^{-2|\alpha_1|+n_1};\\
&\sup_{\xi_1}\int_{|\xi_2|\approx s_2}|\partial_{\xi_2}^{\alpha_2}m(\xi_1,\xi_2)|^2 d\xi_1 d\xi_2 \le C s_2^{-2|\alpha_2|+n_2}
\end{aligned} \tag{2}$$

for every $|\alpha_1|\le p_1$ and $|\alpha_2|\le p_2$, where $|\xi_i|\approx s_i$ signifies that $s_i\le|\xi_i|\le 2s_i$.

Then i) for $2<p<\infty$,

$$\int |T_m f|^p W \le C\int |f|^p W$$

whenever $W\in A_{p/2}(R^{n_1}\times R^{n_2})$; and ii) for $1<p<2$,

$$\int |T_m f|^p W \le C\int |f|^p W$$

whenever $W^{\frac{2}{2-p}}\in A_{\frac{p}{2-p}}(R^{n_1}\times R^{n_2})$.

Remark. The weighted norm inequality for $p=2$ can be obtained by using the interpolation theorem.

Proof. From Theorem A, we need to show that the sharp operator $T^{\#}f = M_s(f^2)^{1/2}$. Then (i) of our Theorem follows from Theorem A.

Let us take a smooth function ϕ on R^1 whose Fourier transform $\widehat{\phi}(t)$ has compact support $\{1/2<|t|<2\}$ such that $\sum\widehat{\phi}(2^{-j}|t|)=1$ for all $t\neq 0$. Let

$$m_{i,j}(\xi_1,\xi_2)=m(\xi_1,\xi_2)\widehat{\phi}(2^{-i}|\xi_1|)\widehat{\phi}(2^{-j}|\xi_2|)$$

and

$$\widehat{T}_{i,j}f(\xi_1,\xi_2)=m_{i,j}(\xi_1,\xi_2)\widehat{f}(\xi_1,\xi_2)\equiv(\widehat{k_{i,j}*f})(\xi_1,\xi_2).$$

It is clear that $Tf=\sum_{i,j}T_{i,j}f$.

To prove $T^{\#}f(x)=(M_s f^2(x))^{1/2}$, one needs to estimate, for every rectangle R,

$$\operatorname{osc}_R(Tf)(x)\le C\gamma^{-\delta}(M_s f^2(x))^{1/2} \tag{3}$$

for every $x\in R$ where f is supported outside of the γ-fold dilation of the rectangle R, $\gamma\ge 2$, i.e. supp $f\subset\tilde{R}_\gamma^c$. By the homogeneity of multiplier operators, it suffices to assume R is the unit square. Since the estimates are translation invariant, we may assume the center of R is at the origin. Let us write a function f, supp $f\subset\tilde{R}_\gamma^c$, as the sum of the functions $g+h+G$ where

$$\text{support of } g\subset {}^c\tilde{R}_\gamma^1\equiv\{|y_1|>\gamma,\ |y_2|\le\gamma\};$$

$$\text{support of } h\subset {}^c\tilde{R}_\gamma^3\equiv\{|y_1|>\gamma,\ |y_2|>\gamma\};$$

$$\text{support of } G\subset {}^c\tilde{R}_\gamma^2\equiv\{|y_1|\le\gamma,\ |y_2|>\gamma\}.$$

Without loss of generality it suffices to show (3) for a function $f = g + h$ where

$$\text{supp } g \subset {}^c\tilde{R}^1_\gamma \subset \{|y_1| > \gamma,\ |y_2| \le 2\};$$

$$\text{supp } h \subset {}^c\tilde{R}^3_\gamma \subset \{|y_1| > \gamma,\ |y_2| > 2\}.$$

We are going to estimate

$$\text{osc}_R(Tg)(x) \le C\gamma^{-\sigma}(M_s g^2(x))^{1/2} \quad \text{and} \quad \text{osc}_R(Th)(x) \le C\gamma^{-\sigma}(M_s h^2(x))^{1/2},$$

since $\text{osc}_R Tf \le \text{osc}_R Tg + \text{osc}_R Th$. Let us write

$$\begin{aligned} \text{osc}_R(Tf) \le & \sum_{i\ge 0} \text{osc}_R(\sum_j T_{i,j} g) + \sum_{i<0} \text{osc}_R(\sum_j T_{i,j} g) \\ & + \sum_{i\ge 0}\sum_{j\ge 0} \text{osc}_R T_{i,j} h + \sum_{i\ge 0}\sum_{j\le 0} \text{osc}_R T_{i,j} h \\ & + \sum_{i<0}\sum_{j\ge 0} \text{osc}_R T_{i,j} h + \sum_{i<0}\sum_{j<0} \text{osc}_R T_{i,j} h \\ \equiv & I + II + III + IV + V + VI. \end{aligned}$$

We will estimate that the first two terms are dominated by $\gamma^{-\sigma}(M_s g^2(x))^{1/2}$ and the last four terms are dominated by $\gamma^{-\sigma}(M_s h^2(x))^{1/2}$ for every $x \in R$. Denote $\sum_j T_{i,j} g \equiv T_i g$ and write

$$\begin{aligned} I &= \sum_{i\ge 0} \text{osc}_R(T_i g) \\ &\le \sum_{i\ge 0} \left(\frac{1}{|R|} \int_R |T_i g(x_1, x_2)|^2 dx_1 dx_2 \right)^{1/2} \\ &= \sum_{i\ge 0} \left(\int_{|x_1|\le 1} \int_{|x_2|\le 1} \Big| \int_{|y_2|<2} \int_{|y_1|\ge\gamma} K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) \right. \\ &\qquad\qquad \left. \cdot\, dy_1 dy_2 \Big|^2 dx_2 dx_1 \right)^{1/2} \qquad (4) \\ &\le \sum_{i\ge 0} \sum_{2^{k_1} \ge \gamma/2} \left(\int_{|x_1|\le 1} \int_{|x_2|\le 1} \Big| \int_{|y_2|<2} \int_{|y_1|\approx 2^{k_1}} \right. \\ &\qquad\qquad \left. \cdot\, K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) dy_1 dy_2 \Big|^2 dx_2 dx_1 \right)^{1/2}. \end{aligned}$$

Since

$$\int_{|y_1|\approx 2^{k_1}} \int_{|y_2|<2} K_i(x_1 - y_1, x_2 - y_2) g(y_1, y_2) dy_2 dy_1$$

is a convolution operator in the variable y_2, applying Plancherel's Theorem for the variable x_2 on (4), one has

$$I \le \sum_{i\ge 0} \sum_{2^{k_1} \ge \gamma/2} \left(\int_{|x_1|\le 1} \int_{\xi_2} \Big| \int_{|y_1|\approx 2^{k_1}} \widehat{K}_i^2(x_1 - y_1, \xi_2) \widehat{g}^2(y_1, \xi_2) dy_1 \Big|^2 d\xi_2 dx_1 \right)^{1/2}$$

where $\wedge^2$ denotes the Fourier transform on the second variable. For $|x_1| \leq 1$, $|y_1| \geq \gamma$, $\gamma \geq 2$, $|y_1| \approx 2^{k_1}$ then $|x_1 - y_1| \approx 2^{k_1}$. Hence

$$I \leq \sum_{i\geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 p_1} \left(\int_{|x_1|\leq 1} \int_{\xi_2} | \int_{|y_1|\approx 2^{k_1}} |x_1 - y_1|^{p_1} \widehat{K}_i^2(x_1 - y_1, \xi_2) \right.$$
$$\left. \cdot\, \widehat{g}^2(y_1, \xi_2) dy_1|^2 d\xi_2 dx_1 \right)^{1/2}.$$

By Hölder's inequality and changing variable y_1,

$$I \leq \sum_{i\geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 p_1} \left(\int_{|x_1|\leq 1} \int_{\xi_2} \int ||x_1 - y_1|^{p_1} \widehat{K}_i^2(x_1 - y_1, \xi_2)|^2 dy_1 \right.$$
$$\left. \cdot \int_{|y_1|\approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1 d\xi_2 dx_1 \right)^{1/2}$$
$$\leq \sum_{i\geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{-k_1 p_1} \left(\sup_{\xi_2} \int ||y_1|^{p_1} \widehat{K}_i^2(y_1, \xi_2)|^2 dy_1 \right)^{1/2}$$
$$\cdot \left(\int \int_{|y_1|\approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1 d\xi_2 \right)^{1/2}$$
$$\leq \sum_{i\geq 0} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1 + n_1/2)} \left(\sup_{\xi_2} \sum_{|\alpha_1| = p_1} \int |\partial_{\xi_1}^{\alpha_1} m_i(\xi_1, \xi_2)|^2 d\xi_1 \right)^{1/2}$$
$$\cdot \left(\frac{1}{2^{k_1 n_1}} \int_{|y_2|\leq 2} \int_{|y_1|\approx 2^{k_1}} |g(y_1, y_2)|^2 dy_1 dy_2 \right)^{1/2}$$

where the last inequality is obtained by applying Plancherel's Theorem to both integrals and the support of g is contained by $\{|y_1| > \gamma,\ |y_2| \leq 2\}$. Hence, by the hypothesis (2) and $-p_1 + n_1/2 < 0$,

$$I \leq C \sum_{i\geq 0} \gamma^{-\sigma} (M_s g^2(0))^{1/2} 2^{i(-p_1 + n_1/2)} \leq C\gamma^{-\sigma} (M_s g^2(0))^{1/2}.$$

For estimating II, we write

$$II = \sum_{i<0} \mathrm{osc}_R(\sum_j T_{i,j} g) = \sum_{i<0} \mathrm{osc}_R(T_i g)$$
$$\leq \sum_{i<0} \left(\frac{1}{|R|} \int_R |T_i g(x_1, x_2) - T_i g(0, x_2)|^2 dx_1 dx_2 \right)^{1/2}$$
$$= C \sum_{i<0} \left(\int_R | \int_{{}^c\tilde{R}_\gamma^1} (K_i(x_1 - y_1, x_2 - y_2) - K_i(0 - y_1, x_2 - y_2)) \right.$$
$$\left. \cdot\, g(y_1, y_2) dy_1 dy_2|^2 dx_2 dx_1 \right)^{1/2}$$
$$\leq C \sum_{i<0} \sum_{2^{k_1} \geq \gamma/2} \left(\int_R | \int_{|y_1|\approx 2^{k_1}} \int_{|y_2|\leq 2} (K_i(x_1 - y_1, x_2 - y_2) \right.$$
$$\left. - K_i(0 - y_1, x_2 - y_2)) g(y_1, y_2) dy_1 dy_2|^2 dx_2 dx_1 \right)^{1/2}.$$

Here, we follow the same procedures as we did in proving I, applying Plancherel's Theorem for the variable x_2,

$$\begin{aligned}
II &\leq C \sum_{i<0} \sum_{2^{k_1} \geq \gamma/2} \Big(\int_{|x_1|\leq 1} \int_{\xi_2} | \int_{|y_1|\approx 2^{k_1}} (\widehat{K}_i^2(x_1 - y_1, \xi_2) - \widehat{K}_i^2(0 - y_1, \xi_2)) \\
&\qquad \cdot \widehat{g}^2(y_1, \xi_2) dy_1|^2 d\xi_2 dx_1 \Big)^{1/2} \\
&= C \sum_{i<0} \sum_{2^{k_1} \geq \gamma/2} \Big(\int_{|x_1|\leq 1} \int_{\xi_2} | \int_{|y_1|\approx 2^{k_1}} \int_0^1 x_1 \partial_{y_1} \widehat{K}_i^2(x_1 s_1 - y_1, \xi_2) \\
&\qquad \cdot \widehat{g}^2(y_1, \xi_2) ds_1 dy_1|^2 d\xi_2 dx_1 \Big)^{1/2} \\
&\leq C \sum_{i<0} \sum_{2^{k_1} \geq \gamma/2} \Big(\int_0^1 \int_{|x_1|\leq 1} \int_{\xi_2} (\int_{|y_1|\approx 2^{k_1}} |\partial_{y_1} \widehat{K}_i^2(x_1 s_1 - y_1, \xi_2)|^2 dy_1) \\
&\qquad \cdot (\int_{|y_1|\approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1) d\xi_2 dx_1 ds_1 \Big)^{1/2} \\
&\leq C \sum_{i<0} \sum_{2^{k_1} \geq \gamma/2} 2^{k_1(-p_1+\epsilon_1+n_1/2)} \Big[\int_0^1 \int_{|x_1|\leq 1} \sup_{\xi_2} (\int_{|y_1|\approx 2^{k_1}} ||x_1 s_1 - y_1|^{p_1-\epsilon_1} \\
&\qquad \cdot \partial_{y_1} \widehat{K}_i^2(x_1 s_1 - y_1, \xi_2)|^2 dy_1) \\
&\qquad \cdot (\frac{1}{2^{k_1 n_1}} \int \int_{|y_1|\approx 2^{k_1}} |\widehat{g}^2(y_1, \xi_2)|^2 dy_1 d\xi_2) dx_1 ds_1 \Big]^{1/2}.
\end{aligned}$$

Taking a very small $\epsilon_1 > 0$ such that $-p_1 + \epsilon_1 + n_1/2 < 0$, changing variable (i.e. $x_1 s_1 - y_1 \to y_1$) in the integral in the first parentheses and applying Plancherel's Theorem for the integral in the second parentheses, one has

$$\begin{aligned}
II &\leq C\gamma^{-\sigma} \sum_{i<0} \sup_{\xi_2} \Big(\int ||y_1|^{-\epsilon_1} |y_1|^{p_1} \partial_{y_1} \widehat{K}_i^2(y_1, \xi_2)|^2 dy_1 \Big)^{1/2} (M_s g^2(0))^{1/2} \\
&\leq C\gamma^{-\sigma} (M_s g^2(0))^{1/2} \sum_{i<0} \sup_{\xi_2} \sum_{|\alpha_1|=p_1} \Big(\int ||\xi_1|^{-n_1+\epsilon_1} *^1 \partial_{\xi_1}^{\alpha_1} (\xi_1 m_i(\xi_1, \xi_2))|^2 d\xi_1 \Big)^{1/2}
\end{aligned}$$

for some $\sigma > 0$, where $*^1$ is the convolution operator on the first variable. By fractional integration,

$$II \leq C\gamma^{-\sigma} (M_s g^2(0))^{1/2} \sum_{i<0} \sum_{|\alpha_1|=p_1} \sup_{\xi_2} \Big(\int |\partial_{\xi_1}^{\alpha_1} (\xi_1 m_i(\xi_1, \xi_2))|^q d\xi_1 \Big)^{1/q}$$

where $1/q = 1/2 + \epsilon_1/n_1$ (clearly $q < 2$). By Hölder's inequality,

$$\begin{aligned}
II &\le C\gamma^{-\sigma}(M_s g^2(0))^{1/2} \sum_{i<0} \sum_{|\alpha_1|=p_1} 2^{in_1(1/q-1/2)} \sup_{\xi_2} \left(\int |\partial_{\xi_1}^{\alpha_1}(\xi_1 m_i(\xi_1,\xi_2))|^2 d\xi_1 \right)^{1/2} \\
&\le C\gamma^{-\sigma}(M_s g^2(0))^{1/2} \sum_{i<0} 2^{i(-p_1+1+n_1/q)} \\
&\le C\gamma^{-\sigma}(M_s g^2(0))^{1/2}
\end{aligned}$$

(since $\epsilon_1 > 0$ then $-p_1 + 1 + n_1/q > 0$).

For estimating III, we write

$$\begin{aligned}
&|T_{i,j}h(x_1,x_2)| \\
&\le \sum_{k_2\ge 1} \sum_{2^{k_1}\ge\gamma/2} \int_{|y_1|\approx 2^{k_1}} \int_{|y_2|\approx 2^{k_2}} |K_{i,j}(x_1-y_1,x_2-y_2)h(y_1,y_2)| dy_1 dy_2 \\
&\le \sum_{k_2\ge 1} \sum_{2^{k_1}\ge\gamma/2} \left(2^{k_1 n_1} 2^{k_2 n_2} \int_{|y_1|\approx 2^{k_1}} \int_{|y_2|\approx 2^{k_2}} |K_{i,j}(x_1-y_1,x_2-y_2)|^2 dy_1 dy_2 \right)^{1/2} \\
&\qquad\qquad \cdot \left(\frac{1}{2^{k_1 n_1} 2^{k_2 n_2}} \int_{|y_1|\approx 2^{k_1}} \int_{|y_2|\approx 2^{k_2}} |h|^2 dy_1 dy_2 \right)^{1/2} \\
&\le C(M_s h^2(0))^{1/2} \sum_{k_2\ge 1} \sum_{2^{k_1}\ge\gamma/2} 2^{k_1(-p_1+n_1/2)} 2^{k_2(-p_2+n_2/2)} \\
&\qquad\qquad \cdot \left(\int\int ||x_1-y_1|^{p_1} |x_2-y_2|^{p_2} K_{i,j}(x_1-y_1,x_2-y_2)|^2 dy_1 dy_2 \right)^{1/2}.
\end{aligned}$$

As before, applying a change of variable and Plancherel's Theorem,

$$\begin{aligned}
&|T_{i,j}h(x_1,x_2)| \\
&\quad \le C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1} \sum_{|\alpha_2|=p_2} \left(\int\int |\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} m_{i,j}(\xi_1,\xi_2)|^2 d\xi_1 \xi_2 \right)^{1/2} \\
&\quad \le C\gamma^{-\sigma}(M_s h^2(0))^{1/2} 2^{i(-p_1+n_1/2)} 2^{j(-p_2+n_2/2)}.
\end{aligned}$$

Hence

$$III = \sum_{i\ge 0} \sum_{j\ge 0} \operatorname{osc}_R T_{i,j}h \le C\gamma^{-\sigma}(M_s h^2(0))^{1/2}.$$

Since the estimates for IV and V are similar, we estimate only term V. First let us write

$$
\begin{aligned}
&|T_{i,j}h(x_1,x_2)-T_{i,j}h(0,x_2)| \\
&\le \sum_{k_2\ge 1}\sum_{2^{k_1}\ge\gamma/2}\int_{|y_1|\approx 2^{k_1}}\int_{|y_2|\approx 2^{k_2}} |K_{i,j}(x_1-y_1,x_2-y_2)-K_{i,j}(0-y_1,x_2-y_2)| \\
&\qquad\qquad \cdot |h(y_1,y_2)|dy_1dy_2 \\
&\le \sum_{k_2\ge 1}\sum_{2^{k_1}\ge\gamma/2} 2^{k_1n_1/2}2^{k_2n_2/2}\Big(\int_{|y_1|\approx 2^{k_1}}\int_{|y_2|\approx 2^{k_2}}|K_{i,j}(x_1-y_1,x_2-y_2) \\
&\qquad\qquad -K_{i,j}(0-y_1,x_2-y_2)|^2dy_1dy_2\Big)^{1/2}(M_sh^2(0))^{1/2} \\
&\le \sum_{k_2\ge 1}\sum_{2^{k_1}\ge\gamma/2} 2^{k_1n_1/2}2^{k_2n_2/2}(M_sh^2(0))^{1/2}\Big(\int_0^1\int\int|\partial_{y_1}K_{i,j}(x_1s_1-y_1,x_2-y_2)|^2 \\
&\qquad\qquad \cdot dy_1dy_2ds\Big)^{1/2} \\
&\approx \sum_{k_2\ge 1}\sum_{2^{k_1}\ge\gamma/2} 2^{k_1(-p_1+\epsilon_1+n_1/2)}2^{k_2(-p_2+n_2/2)}(M_sh^2(0))^{1/2}(\int_0^1\int\int \\
&\qquad \cdot ||x_1s_1-y_1|^{p_1-\epsilon_1}|x_2-y_2|^{p_2}\partial_{y_1}K_{i,j}(x_1s_1-y_1,x_2-y_2)|^2dy_1dy_2ds\Big)^{1/2}.
\end{aligned}
$$

Taking a positive small ϵ_1, changing variables, applying Plancherel's Theorem and fractional integration, we get

$$
\begin{aligned}
&|T_{i,j}h(x_1,x_2)-T_{i,j}h(0,x_2)| \\
&\le C\gamma^{-\sigma}(M_sh^2(0))^{1/2} \\
&\quad \cdot \sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2}\Big(\int\int||\xi_1|^{-n_1+\epsilon_1} *^1 \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\xi_1 m_{i,j}(\xi_1,\xi_2))|^2d\xi_1d\xi_2\Big)^{1/2} \\
&\le C\gamma^{-\sigma}(M_sh^2(0))^{1/2} \\
&\quad \cdot \sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2}\Big(\int\Big(\int|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}|(\xi_1 m_{i,j}(\xi_1,\xi_2))|^qd\xi_1\Big)^{2/q}d\xi_2\Big)^{1/2}
\end{aligned}
$$

where $1/q = 1/2+\epsilon_1/n_1$. By Hölder's inequality and the hypothesis (2),

$$
\begin{aligned}
&|T_{i,j}h(x_1,x_2)-T_{i,j}h(0,x_2)| \\
&\le C\gamma^{-\sigma}(M_sh^2(0))^{1/2}2^{in_1(1/q-1/2)}\Big(\int\int|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\xi_1 m_{i,j}(\xi_1,\xi_2))|^2d\xi_1d\xi_2\Big)^{1/2} \\
&\le C\gamma^{-\sigma}(M_sh^2(0))^{1/2}2^{i(-p_1+1+n_1/q)}2^{j(-p_2+n_2/2)}.
\end{aligned}
$$

Since $-p_1+1+n_1/q>0$ for positive small ϵ_1 and $-p_2+n_2/2<0$, we have

$$V = \sum_{i<0}\sum_{j\ge 0}\operatorname{osc}_R T_{i,j}h \le C\gamma^{-\sigma}(M_sh^2(0))^{1/2}.$$

For the last term VI, we consider

$$
\begin{aligned}
&|T_{i,j}h(x_1,x_2) - T_{i,j}h(0,x_2) - T_{i,j}h(x_1,0) + T_{i,j}h(0,0)| \\
&\leq \sum_{k_2\geq 1}\sum_{2^{k_1}\geq \gamma/2} \int_{|y_1|\approx 2^{k_1}}\int_{|y_2|\approx 2^{k_2}} (K_{i,j}(x_1-y_1,x_2-y_2) - K_{i,j}(-y_1,x_2-y_2) \\
&\qquad\qquad - K_{i,j}(x_2-y_1,-y_2) + K_{i,j}(-y_1,-y_2))h(y_1,y_2)dy_1dy_2 \\
&\leq \sum_{k_2\geq 1}\sum_{2^{k_1}\geq \gamma/2} \bigg(2^{k_1n_1}2^{k_2n_2}\int_{|y_1|\approx 2^{k_1}}\int_{|y_2|\approx 2^{k_2}} |K_{i,j}(x_1-y_1,x_2-y_2) \\
&\quad - K_{ij}(-y_1,x_2-y_2) - K_{ij}(x_1-y_1,-y_2) + K_{ij}(-y_1,-y_2)|^2 dy_1dy_2\bigg)^{1/2} \\
&\qquad \cdot \left(\frac{1}{2^{k_1n_1}2^{k_2n_2}}\int_{|y_1|\approx 2^{k_1}}\int_{|y_2|\approx 2^{k_2}} |h|^2 dy_1dy_2\right)^{1/2}.
\end{aligned}
$$

By the Taylor formula and Hölder's inequality,

$$
\begin{aligned}
&|T_{i,j}h(x_1,x_2) - T_{i,j}h(0,x_2) - T_{i,j}h(x_1,0) + T_{i,j}h(0,0)| \\
&\leq \sum_{k_2\geq 1}\sum_{2^{k_1}\geq \gamma/2} (M_sh^2(0))^{1/2}\bigg(2^{k_1n_1}2^{k_2n_2}\int_0^1\int_0^1\int_{|y_1|\approx 2^{k_1}}\int_{|y_2|\approx 2^{k_2}} \\
&\qquad \cdot |\partial_{y_1}\partial_{y_2}K_{ij}(x_1s_1-y_1,x_2s_2-y_2)|^2 dy_1dy_2ds_1ds_2\bigg)^{1/2} \\
&\approx C(M_sh^2(0))^{1/2}\sum_{k_2\geq 1}\sum_{2^{k_1}\geq \gamma/2} 2^{k_1(-p_1+\epsilon_1+n_1/2)}2^{k_2(-p_2+\epsilon_2+n_2/2)} \\
&\qquad \cdot \bigg(\int_0^1\int_0^1\int_{|y_1|\approx 2^{k_1}}\int_{|y_2|\approx 2^{k_2}} ||x_1s_1-y_1|^{p_1-\epsilon_1}|x_2s_2-y_2|^{p_2-\epsilon_2} \\
&\qquad \cdot \partial_{y_1}\partial_{y_2}K_{ij}(x_1s_1-y_1,x_2s_2-y_2)|^2 dy_1dy_2ds_1ds_2\bigg)^{1/2} \\
&\leq C\gamma^{-\sigma}(M_sh^2(0))^{1/2}\int\int ||y_1|^{-\epsilon_1}|y_2|^{-\epsilon_2}|y_1|^{p_1}|y_2|^{p_2} \\
&\qquad\qquad \cdot \partial_{y_1}\partial_{y_2}K_{ij}(y_1,y_2)|^2 dy_1dy_2\bigg)^{1/2} \\
&\leq C\gamma^{-\sigma}(M_sh^2(0))^{1/2}\sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2}\bigg(\int ||\xi_1|^{-n_1+\epsilon_1}|\xi_2|^{-n_2+\epsilon_2} \\
&\qquad\qquad * \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\xi_1\xi_2m_{i,j}(\xi_1,\xi_2))|^2 d\xi_1d\xi_2\bigg)^{1/2}.
\end{aligned}
$$

Next, we use fractional integration twice on the variables ξ_1 and ξ_2 respectively. Let $1/q_1 = 1/2 + \epsilon_1/n_1$ and $1/q_2 = 1/2 + \epsilon_2/n_2$. Then

$$\begin{aligned}&|T_{i,j}h(x_1,x_2) - T_{i,j}h(0,x_2) - T_{i,j}h(x_1,0) - T_{i,j}h(0,0)|\\ &\le C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2}\left(\int\left(\int ||\xi_2|^{-n_2+\epsilon_2}\right.\right.\\ &\qquad \left.\left. *^2\, \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\xi_1\xi_2 m_{i,j}(\xi_1,\xi_2))|^{q_1} d\xi_1\right)^{2/q_1} d\xi_2\right)^{1/2}\end{aligned}$$

where $*^2$ denotes the convolution operator on the second variable. By Minkowski's inequality, the last inequality is less than

$$\begin{aligned}&C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2}\left(\int\left(\int ||\xi_2|^{-n_2+\epsilon_2}\right.\right.\\ &\qquad\qquad \left.\left. *^2\, \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\xi_1\xi_2 m_{i,j}(\xi_1,\xi_2))|^{2} d\xi_2\right)^{q_1/2} d\xi_1\right)^{1/q_1}\\ &\le C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2}\left(\int\left(\int |\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\xi_1\xi_2 m_{i,j}(\xi_1,\xi_2))|^{q_2}\right.\right.\\ &\qquad\qquad \left.\left. d\xi_2\right)^{q_1/q_2} d\xi_1\right)^{1/q_1}\\ &\le C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2} 2^{jn_2(1/q_2-1/2)}\\ &\qquad \cdot\left(\int\left(\int |\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}(\xi_1\xi_2 m_{i,j}(\xi_1,\xi_2))|^{2} d\xi_2\right)^{q_1/2} d\xi_1\right)^{2/q_1}\\ &\le C\gamma^{-\sigma}(M_s h^2(0))^{1/2} \sum_{|\alpha_1|=p_1}\sum_{|\alpha_2|=p_2} 2^{jn_2(1/q_2-1/2)} 2^{in_1(1/q_1-1/2)}\\ &\qquad \cdot\left(\int\int \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}|(\xi_1\xi_2 m_{i,j}(\xi_1,\xi_2))|^{2} d\xi_1 d\xi_2\right)^{1/2}\\ &\le C\gamma^{-\sigma}(M_s h^2(0))^{1/2} 2^{(-p_1+1+n_1/q_1)} 2^{(-p_2+1+n_2/q_2)}.\end{aligned}$$

Hence

$$VI \le \sum_{i<0}\sum_{j<0} \operatorname{osc}_R T_{i,j}h \le C\gamma^{-\sigma}(M_s h^2(0))^{1/2}.$$

Combining the above estimates, we conclude that

$$T^{\#}f(x_1,x_2) = (M_s f^2(x))^{1/2}.$$

(i) is proved.

For the proof of (ii), we use duality. Let $U = W^{-1/(p-1)}$. Then the dual space of $L^p_W(R^{n_1}\times R^{n_2})$ is $L^{p'}_U(R^{n_1}\times R^{n_2})$. There exists a function $g \in L^{p'}_U(R^{n_1}\times R^{n_2})$ such that

$$\|T_m f\|_{L^p_W} = \int T_m f\bar{g} = \int f\overline{T_m g} \le \|f\|_{L^p_W}\|T_m g\|_{L^{p'}_U}.$$

It is easy to see that

$$W^{2/(2-p)} \in A_{p/(2-p)} \Longleftrightarrow U \in A_{p'/2}.$$

Applying (i),

$$\|T_m g\|_{L_U^{p'}} \le C\|g\|_{L_U^{p'}}$$

where $U \in A_{p'/2}$. The Theorem is proved. □

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