

SOME REMARKS ON BEILINSON ADELES

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ABSTRACT. Let X be a scheme of finite type over a field k . Denote by \mathcal{A}_X the sheaf of Beilinson adeles with values in the algebraic De Rham complex $\Omega_{X/k}$. Then $\Omega_{X/k} \rightarrow \mathcal{A}_X$ is a flasque resolution. So if X is smooth, \mathcal{A}_X calculates De Rham cohomology. In this note we rewrite the proof of Deligne-Illusie for the degeneration of the Hodge spectral sequence in terms of adeles. We also give a counterexample to show that the filtration $\mathcal{A}_X^{\geq q}$ does not induce Hodge decomposition.

0. INTRODUCTION

In this note we consider two aspects of Beilinson adeles on schemes.

Let X be a scheme of finite type over a field k . Given a quasi-coherent sheaf \mathcal{M} let $\underline{\mathbb{A}}_{\text{red}}^q(\mathcal{M})$ be the sheaf of reduced Beilinson adeles of degree q (see [Be], [Hr], [HY1]). It is known that $\underline{\mathbb{A}}_{\text{red}}^q(\mathcal{M}) \cong \underline{\mathbb{A}}_{\text{red}}^q(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M}$. For any open set $U \subset X$

$$(0.1) \quad \Gamma(U, \underline{\mathbb{A}}_{\text{red}}^q(\mathcal{M})) \subset \prod_{\xi \in S(U)_q^{\text{red}}} \mathcal{M}_\xi$$

where $S(U)_q^{\text{red}}$ is the set of reduced chains of points in U of length q , and \mathcal{M}_ξ is the Beilinson completion of \mathcal{M} along the chain ξ (cf. [Ye1]). For $q = 0$ and \mathcal{M} coherent one has $\mathcal{M}_{(x)} = \widehat{\mathcal{M}}_x$, the \mathfrak{m}_x -adic completion, and (0.1) is an equality.

Let $\Omega_{X/k}$ be the De Rham complex on X , relative to k . As shown in [HY1], setting $\mathcal{A}_X^{p,q} := \underline{\mathbb{A}}_{\text{red}}^q(\Omega_{X/k}^p)$ and $\mathcal{A}_X^i := \bigoplus_{p+q=i} \mathcal{A}_X^{p,q}$ we get a differential graded algebra (DGA) which is quasi-isomorphic to $\Omega_{X/k}$ and is flasque. Thus $H^*(X, \Omega_{X/k}) = H^*(X, \mathcal{A}_X)$. In particular if X is smooth, we get the De Rham cohomology $H_{\text{DR}}^*(X/k)$.

More generally, let \mathfrak{X} be a formal scheme, of formally finite type (f.f.t.) over k (see [Ye2]). Then applying the adelic construction to the complete De Rham complex $\widehat{\Omega}_{\mathfrak{X}/k}$ we get a DGA $\mathcal{A}_{\mathfrak{X}}$. If $X \subset \mathfrak{X}$ is a smooth formal embedding (op. cit.) and $\text{char } k = 0$, then $H^*(X, \mathcal{A}_{\mathfrak{X}}) = H_{\text{DR}}^*(X/k)$.

There is an analogy between the sheaf $\mathcal{A}_X^{p,q}$ on a smooth n -dimensional variety X and the sheaf of smooth (p, q) -forms on a complex manifold. The coboundary operator D of \mathcal{A}_X is defined as a sum $D := D' + D''$, and $D'' : \mathcal{A}_X^{p,q} \rightarrow \mathcal{A}_X^{p,q+1}$ plays

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the role of the anti-holomorphic derivative. The map $\int_X = \sum_{\xi} \text{Res}_{\xi} : \Gamma(X, \mathcal{A}_X^{2n}) \rightarrow k$ is the counterpart of the integral (Res_{ξ} is the Parshin-Lomadze residue along the maximal chain ξ in X , see [Ye1]). This analogy to the complex manifold picture is quite solid; for example, in [HY2] there is an algebraic proof of the Bott residue formula, which in some parts is just a translation of the original proof of Bott to the setting of adèles.

The main purpose of this note is to examine the potential applicability of adèles for the study of algebraic De Rham cohomology. In §1 the construction of Deligne and Illusie [DI] is rewritten in terms of adèles. In §2 we consider a possibility to relate adèles to Hodge theory, and show by example its failure.

1. LIFTING MODULO p^2

We interpret, in terms of adèles, the result of Deligne and Illusie on the decomposition of the De Rham complex in characteristic p . In this section we shall follow closely the ideas and notation of [DI].

Let k be a perfect field of characteristic p . Write $\tilde{k} := W_2(k)$. Let $F_k : \text{Spec } k \rightarrow \text{Spec } k$ be the Frobenius morphism, i.e. $F_k^*(a) = a^p$ for $a \in k$. By pullback along F_k we get a scheme $X' := X \times_k k$ and a finite, bijective k -morphism $F = F_{X/k} : X \rightarrow X'$.

Assume we are given some lifting \tilde{X} of X to \tilde{k} . By this we mean a smooth scheme \tilde{X} over \tilde{k} s.t. $X \cong \tilde{X} \times_{\tilde{k}} k$. Using the Frobenius $F_{\tilde{k}}$ we also define a scheme \tilde{X}' , and a \tilde{k} -morphism $F_{\tilde{X}} : \tilde{X} \rightarrow \tilde{X}'$. For any point $x \in X$ the relative Frobenius homomorphism $F_x^* : \mathcal{O}_{X',F(x)} \rightarrow \mathcal{O}_{X,x}$ can be lifted to a \tilde{k} -algebra homomorphism $\tilde{F}_x^* : \mathcal{O}_{\tilde{X}',F(x)} \rightarrow \mathcal{O}_{\tilde{X},x}$ (cf. [DI]). In view of (0.1), the collection $\{\tilde{F}_x^*\}_{x \in X}$ induces a homomorphism of sheaves of DG \tilde{k} -algebras

$$\tilde{F}^* : \underline{\mathbb{A}}_{\text{red}}^0(\Omega_{\tilde{X}'/\tilde{k}}) \rightarrow F_* \underline{\mathbb{A}}_{\text{red}}^0(\Omega_{\tilde{X}/\tilde{k}}).$$

Lemma 1.1. *The liftings $\{\tilde{F}_x^*\}_{x \in X}$ determine $\mathcal{O}_{X'}$ -linear homomorphisms*

$$f : \Omega_{X'/k}^1 \rightarrow F_* \mathcal{A}_X^{1,0},$$

$$h : \Omega_{X'/k}^1 \rightarrow F_* \mathcal{A}_X^{0,1}$$

such that

$$D(f + h) = 0.$$

Proof. Let $\mathbf{p} : \Omega_{X/k}^1 \xrightarrow{\cong} p \Omega_{\tilde{X}/\tilde{k}}^1$ be multiplication by p . This extends to an $\underline{\mathbb{A}}_{\text{red}}^0(\mathcal{O}_X)$ -linear isomorphism

$$\mathbf{p} : \mathcal{A}_X^{1,0} = \underline{\mathbb{A}}_{\text{red}}^0(\Omega_{X/k}^1) \xrightarrow{\cong} p \underline{\mathbb{A}}_{\text{red}}^0(\Omega_{\tilde{X}/\tilde{k}}^1).$$

Just as in [DI] we get a homomorphism f making the diagram

$$\begin{array}{ccc} \Omega_{\tilde{X}'/\tilde{k}}^1 & \xrightarrow{\tilde{F}^*} & p F_* \underline{\mathbb{A}}_{\text{red}}^0(\Omega_{\tilde{X}/\tilde{k}}^1) \\ \downarrow & & \uparrow \mathbf{p} \\ \Omega_{X'/k}^1 & \xrightarrow{f} & F_* \underline{\mathbb{A}}_{\text{red}}^0(\Omega_{X/k}^1) \end{array}$$

commutative.

Next, for any chain of points (x_0, x_1) in X and a local section $a \in \mathcal{O}_{\tilde{X}}$, we have

$$D''\tilde{F}^*(a) = \tilde{F}_{x_0}^*(a) - \tilde{F}_{x_1}^*(a) \in p\mathcal{O}_{\tilde{X},(x_0,x_1)}.$$

Therefore

$$D''\tilde{F}^* : \mathcal{O}_{\tilde{X}'} \rightarrow pF_*\mathbb{A}_{\text{red}}^1(\mathcal{O}_{\tilde{X}})$$

is a derivation which kills $p\mathcal{O}_{\tilde{X}'}$, and we get an $\mathcal{O}_{X'}$ -linear homomorphism h s.t. the diagram

$$\begin{array}{ccc} \mathcal{O}_{\tilde{X}'} & \xrightarrow{D''\tilde{F}^*} & F_*p\mathbb{A}_{\text{red}}^1(\mathcal{O}_{\tilde{X}}) \\ \downarrow d & & \uparrow \mathbf{p} \\ \Omega_{X'/k}^1 & \xrightarrow{h} & F_*\mathbb{A}_{\text{red}}^1(\mathcal{O}_X) \end{array}$$

commutes.

Reinterpreting the calculations of [DI] in terms of adèles we see that the following hold: for each point $x_0 \in X$, $D'f = 0$ in $\Omega_{X/k,(x_0)}^1$; for each chain (x_0, x_1, x_2) in X , $D''h = 0$ in $\mathcal{O}_{X,(x_0,x_1,x_2)}$; lastly, for each chain (x_0, x_1) , $D''f = -D'h$ in $\Omega_{X/k,(x_0,x_1)}^1$. This implies that on the level of sheaves $D(f+h) = 0$. \square

Proposition 1.2. *The liftings $\{\tilde{F}_x^*\}_{x \in X}$ determine an $\mathcal{O}_{X'}$ -linear homomorphism of complexes*

$$\psi_{\tilde{X}} : \bigoplus_{i=0}^n \mathbb{A}_{\text{red}}^i(\Omega_{X'/k}^i)[-i] \rightarrow F_*\mathcal{A}_X$$

making the diagram

$$(1.1) \quad \begin{array}{ccc} \bigoplus_i \Omega_{X'/k}^i & \xrightarrow{C^{-1}} & F_*\mathbf{H}^i\Omega_{X'/k} \\ \downarrow & & \downarrow \\ \bigoplus_i \mathbf{H}^i\mathbb{A}_{\text{red}}^i(\Omega_{X'/k}^i)[-i] & \xrightarrow{\mathbf{H}(\psi_{\tilde{X}})} & F_*\mathbf{H}^i\mathcal{A}_X \end{array}$$

commute. Here C^{-1} is the Cartier operation, and the vertical arrows are the canonical isomorphisms. Therefore $\psi_{\tilde{X}}$ is a quasi-isomorphism.

Proof. Since

$$\mathbb{A}_{\text{red}}^j(\Omega_{X'/k}^i) \cong \mathbb{A}_{\text{red}}^j(\mathcal{O}_{X'}) \otimes_{\mathcal{O}_{X'}} \Omega_{X'/k}^i,$$

and since $F^* : \mathbb{A}_{\text{red}}^i(\mathcal{O}_{X'}) \rightarrow F_*\mathbb{A}_{\text{red}}^i(\mathcal{O}_X)$ commutes with D'' and is killed by D' (i.e. $D'F^* = 0$), it suffices to define $\mathcal{O}_{X'}$ -linear homomorphisms $\psi_{\tilde{X}}^i : \Omega_{X'/k}^i \rightarrow F_*\mathcal{A}_X^i$ s.t. $D\psi_{\tilde{X}}^i = 0$. Define $\psi_{\tilde{X}}^0 := F^*$, and $\psi_{\tilde{X}}^1 := f+h$ as in Lemma 1.1. For $1 \leq i \leq n$ let $\mathbf{a} : \Omega_{X'/k}^i \rightarrow (\Omega_{X'/k}^1)^{\otimes i}$ be the anti-symmetrizing operator (this makes sense

since $n < p$; cf. [DI]), and define $\psi_{\tilde{X}}^i$ by

$$\begin{array}{ccc}
 (\Omega_{X'/k}^1)^{\otimes i} & \xrightarrow{(\psi_{\tilde{X}}^1)^{\otimes i}} & (F_*\mathcal{A}_X^1)^{\otimes i} \\
 \uparrow \mathbf{a} & & \downarrow \text{product} \\
 \Omega_{X'/k}^i & \xrightarrow{\psi_{\tilde{X}}^i} & F_*\mathcal{A}_X^i
 \end{array}$$

Let $a \in \mathcal{O}_{\tilde{X}}$ be a local section, with corresponding pullback $a \otimes 1 \in \mathcal{O}_{\tilde{X}'}$, and with image $a_0 \in \mathcal{O}_X$. Then according to the calculations in [DI], we have $\tilde{F}^*(a \otimes 1) = a^p + \mathbf{p}u$ for some local section $u \in \mathbb{A}_{\text{red}}^0(\mathcal{O}_X)$. Therefore $f(da_0 \otimes 1) = a_0^{p-1}da_0 + D'u$ and $h(da_0 \otimes 1) = D''u$, so $\psi_{\tilde{X}}(da_0 \otimes 1) = a_0^{p-1}da_0 + Du$. This means that

$$H^1(\psi_{\tilde{X}}) = C^{-1} : \Omega_{X'/k}^1 \xrightarrow{\cong} F_*H^1\Omega_{X'/k} \cong F_*H^1\mathcal{A}_X.$$

Clearly in degree 0, $H^0(\psi_{\tilde{X}}) = F^* = C^{-1}$. Since the vertical arrows in diagram (1.1) are isomorphisms of (sheaves of) graded algebras, it follows that $H^*\mathcal{A}_X$ is a graded-commutative algebra, and therefore

$$H^*(\psi_{\tilde{X}}) : \bigoplus_i \Omega_{X'/k}^i \rightarrow F_*H^*\mathcal{A}_X$$

is a homomorphism of graded algebras. But then $H^*(\psi_{\tilde{X}}) = C^{-1}$ in all degrees, and it's an isomorphism. \square

Of course in the derived category the map $\psi_{\tilde{X}}$ is independent of the choices of Frobenius liftings.

2. A HODGE-TYPE DECOMPOSITION?

The second aspect is a naive attempt to use adèles for a Hodge-type decomposition of De Rham cohomology. Suppose $\text{char } k = 0$ and X is smooth over k , of dimension n . For any $0 \leq p, q \leq n$ define a canonical subspace

$$(2.1) \quad H^{p,q} := \frac{\Gamma(X, \mathcal{A}_X^{p,q}) \cap \text{Ker } D}{\Gamma(X, \mathcal{A}_X^{p,q}) \cap \text{Im } D} \subset H_{\text{DR}}^{p+q}(X/k)$$

(cf. [GH], p. 116). Since the sheaves $\mathcal{A}_X^{p,q}$ imitate the Dolbeault sheaves on a complex manifold so nicely, one can imagine that

$$H_{\text{DR}}^i(X/k) = \bigoplus_{p+q=i} H^{p,q}$$

if X is proper. Yet this is *false*, as can be seen from the example below.

What we get is a serious breakdown in the analogy to smooth forms on a complex manifold. I should mention that even in [HY2] there was a breakdown in this analogy; there it was not possible to define a connection on the adelic sections of a vector bundle, and hence an auxiliary algebraic device, the sheaf $\tilde{\mathcal{A}}_X$ of Thom-Sullivan adèles, had to be introduced.

Problem 2.1. Is it true that for X smooth, the filtration on \mathcal{A}_X by the subcomplexes $\mathcal{A}_X^{\geq q}$ induces the coniveau filtration on $H_{\text{DR}}^1(X/k)$?

Example 2.2. Suppose k is algebraically closed and X is an elliptic curve. Then $\dim H_{\text{DR}}^1(X/k) = 2$. Consider the nondegenerate pairing on $H_{\text{DR}}^1(X/k)$ given by

$$\langle [\alpha], [\beta] \rangle = \int_X [\alpha] \smile [\beta] = \sum_{\xi} \text{Res}_{\xi}(\alpha \cdot \beta)$$

for adèles $\alpha, \beta \in \mathcal{A}_X^1$. We see that $\langle H^{1,0}, H^{1,0} \rangle = \langle H^{0,1}, H^{0,1} \rangle = 0$. Therefore if $H_{\text{DR}}^1(X) = H^{1,0} + H^{0,1}$, then $\dim H^{1,0} = \dim H^{0,1} = 1$. It is easy to find $0 \neq [\alpha] \in H^{1,0}$; take any $0 \neq [\alpha] \in \Gamma(X, \Omega_{X/k}^1)$. On the other hand an adèle

$$\beta = (b_{(\text{gen},x)}) \in \Gamma(X, \mathcal{A}_X^{0,1}) = \mathbb{A}_{\text{red}}^1(X, \mathcal{O}_X)$$

(where x runs over the set X_0 of closed points, and gen is the generic point) satisfies $D\beta = 0$ iff $db_{(\text{gen},x)} = 0$ for every x . This forces $b_{(\text{gen},x)} \in k$. But taking $b = (b_{(\text{gen})}, b_{(x)}) \in \mathbb{A}_{\text{red}}^0(X, \mathcal{O}_X)$, with $b_{(\text{gen})} = 0$, $b_{(x)} = b_{(\text{gen},x)}$ we get $\beta = Db$. Hence $H^{0,1} = 0$.

Problem 2.3. For α as above find explicitly a cocycle $\beta \in \Gamma(X, \mathcal{A}_X^1)$ s.t. $\langle \alpha, \beta \rangle = 1$.

The best I can do is:

Proposition 2.4. *Suppose X is a smooth proper curve and k is algebraically closed. Let $\alpha_{(\text{gen})} \in \Omega_{k(X)/k}^1$ be a differential of the 2-nd kind, namely $\text{Res}_{(\text{gen},x)} \alpha_{(\text{gen})} = 0$ for every $x \in X_0$. Then it defines a cocycle $\alpha \in \Gamma(X, \mathcal{A}_X^1)$ whose component at (gen) is $\alpha_{(\text{gen})}$. Every cohomology class in $H_{\text{DR}}^1(X/k)$ is gotten in this way. The Hodge filtration is induced by the differentials of the 1-st kind.*

Proof. The adèle α will be given by its bihomogeneous components, $\alpha = \alpha^{1,0} + \alpha^{0,1}$. We set $\alpha^{1,0} := (\alpha_{(\text{gen})}, \alpha_{(x)})$ where for $x \in X_0$, $\alpha_{(x)} = 0$. Since $\text{Res}_{(\text{gen},x)} \alpha_{(\text{gen})} = 0$ there is some $a_{(\text{gen},x)} \in k(X)_{(\text{gen},x)}$ (unique up to adding a constant) s.t. $da_{(\text{gen},x)} = \alpha_{(\text{gen})}$. Set $\alpha^{0,1} := (a_{(\text{gen},x)})$. Then α is evidently a cocycle.

If $\alpha_{(\text{gen})}$ is of the 1-st kind then actually we get $a_{(\text{gen},x)} \in \mathcal{O}_{X,(x)}$; call this element also $a_{(x)}$. So we can define an adèle $\tilde{\alpha} = \tilde{\alpha}^{1,0} + \tilde{\alpha}^{0,1}$ with $\tilde{\alpha}^{1,0} := (\alpha_{(\text{gen})}, da_{(x)})$ and $\tilde{\alpha}^{0,1} := 0$. We get a cocycle (cohomologous to α), and conversely any cocycle in $\Gamma(X, \mathcal{A}_X^{1,0})$ looks like this.

Consider the niveau spectral sequence of De Rham homology (cf. [Ye3]). A comparison of dimensions shows that this degenerates at the E_2 term. Also the niveau filtration on $H_1^{\text{DR}}(X/k)$ is trivial. Hence we get

$$\begin{aligned} H_1^{\text{DR}}(X/k) &= \text{Ker} \left(H^1 \Omega_{k(X)/k} \rightarrow \bigoplus_{x \in X_0} k \right) \\ &\cong (\text{forms of the 2-nd kind}) / (\text{exact forms}). \end{aligned}$$

Now the map $H_{\text{DR}}^1(X/k) \rightarrow H_1^{\text{DR}}(X/k)$, $[\alpha] \mapsto [\alpha] \smile [X] = \pm [C_X \cdot \alpha]$ is bijective. A direct inspection reveals that the adèle $\alpha = \alpha^{1,0} + \alpha^{0,1}$ is sent to the differential of the second kind $\alpha_{(\text{gen})} \in \Omega_{k(X)/k}^1$. \square

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