FACTORISATION IN NEST ALGEBRAS

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Abstract. We give a necessary and sufficient condition on an operator $A$ for the existence of an operator $B$ in the nest algebra $\text{Alg}N$ of a continuous nest $N$ satisfying $AA^* = BB^*$ (resp. $A^*A = B^*B$). We also characterise the operators $A$ in $B(H)$ which have the following property: For every continuous nest $N$ there exists an operator $B_N$ in $\text{Alg}N$ satisfying $AA^* = B_NB_N^*$ (resp. $A^*A = B_N^*B_N$).

1. Introduction–Preliminaries

The problem of factorisation of operators with respect to a nest algebra has been studied by many authors [8], [1], [13], [9], [11], [12], [10]. In this work we give a necessary and sufficient condition on an operator $A$ for the existence of an operator $B$ in the nest algebra $\text{Alg}N$ of a continuous nest $N$ satisfying $AA^* = BB^*$ (resp. $A^*A = B^*B$). This result improves Theorem 4.9 in [9] for continuous nests. We also characterise the operators $A$ in $B(H)$ which have the following property: For every continuous nest $N$ there exists an operator $B_N$ in $\text{Alg}N$ satisfying $AA^* = B_NB_N^*$ (resp. $A^*A = B_N^*B_N$).

Throughout this work $H$ denotes a separable Hilbert space and $B(H)$ the space of all bounded operators from $H$ into itself. If $V$ is a subset of $H$ we denote by $[V]$ the linear span of $V$. By subspace of $H$ we mean a subset of $H$ which is closed under addition of vectors and scalar multiplication. If $(V_n)_{n=1}^\infty$ is a sequence of closed mutually orthogonal subspaces of $H$ we denote by $\bigoplus_{n=1}^\infty V_n$ the closure of their linear span. If $A$ is in $B(H)$ we denote by $r(A)$ the range of $A$ and by $\text{coker}A$ the orthogonal complement of the kernel of $A$. An operator range is the range of a bounded operator in $H$. A nest in $H$ is a totally ordered set of closed subspaces of $H$ containing $\{0\}$ and $H$ which is closed under intersection and closed span. If $N$ is a nest in $H$ and $P$ is in $N$ we will denote by the same symbol the orthogonal projection on the subspace $P$. If $N$ is a nest we denote by $N^\perp$ the nest $\{P^\perp : P \in N\}$. A nest $N$ is continuous if $P = \bigcup_{Q < P} Q$ for every $P$ in $N$. Given a nest $N$ the associated nest algebra $\text{Alg}N$ is the set of operators $A$ in $B(H)$ satisfying $PAP = AP$ for every $P$ in $N$. For a general discussion of nest algebras the reader is referred to [3].
2. Proper subspaces

We introduce in this section the notion of $N$-proper subspace for a nest $N$. We show that a closed subspace of $H$ of co-finite dimension is $N$-proper for every continuous nest $N$.

**Definition 1.** Let $N$ be a nest on $H$. A vector $x$ in $H$ is called $N$-proper if $x = Px$ for some $P$ in $N$, $P \neq I$.

**Definition 2.** Let $N$ be a nest on $H$. A subspace $V$ of $H$ is called $N$-proper if $[V \cap P : P \in N, P \neq I]$ is dense in $V$.

**Lemma 3.** Let $N$ be a continuous nest on $H$. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of elements of $N$ such that: $P_n \neq I$, $P_{n+1} \geq P_n$, and $P_n$ converges strongly to $I$. Let $x_1, x_2, \ldots, x_m$ be orthonormal vectors in $H$. Set $V = [x_1, x_2, \ldots, x_m]^\perp$. Then:

(a) There exists $n_0$ such that $P_n x_1, P_n x_2, \ldots, P_n x_m$ are linearly independent for $n \geq n_0$.
(b) We set $V_1 = P_1 H \oplus P_1 V^\perp$ and we define inductively

$$V_n = P_n H \oplus \left( \sum_{i=1}^{n-1} V_i \oplus P_n V^\perp \right).$$

Then $V = \sum_{i=1}^{\infty} V_i$.

**Proof.** (a) The Grammian of the vectors $P_n x_1, P_n x_2, \ldots, P_n x_m$ converges to the Grammian of the vectors $x_1, x_2, \ldots, x_m$ which equals 1.

(b) It is easy to see that the $V_n$’s are mutually orthogonal and that $V_n$ is contained in $V$ for every $n$. We show that $\left( \sum_{i=1}^{\infty} V_i \right) \oplus V^\perp = H$. Let $x$ be a vector in $H$ which is orthogonal to $\left( \sum_{i=1}^{\infty} V_i \right) \oplus V^\perp$. For each $n$ the vector $P_n x$ is orthogonal to $\sum_{i=1}^{n} V_i$ so $P_n x$ is in $P_n V^\perp$. For $n \geq n_0$ we have $P_n x = P_n \left( \sum_{i=1}^{m} a_i x_i \right)$, where the $a_i$’s are complex numbers not depending on $n$. So $x = \lim_{n \to \infty} P_n x = \sum_{i=1}^{m} a_i x_i$. But $x$ is orthogonal to $V^\perp$, hence it is 0.

**Proposition 4.** Let $N$ be a continuous nest and $V$ a closed subspace of $H$ of co-finite dimension. Then $V$ is $N$-proper.

**Proof.** It follows immediately from Lemma 3.

Let $N$ be a continuous nest on $H$ and $A$ an operator in $B(H)$. Consider the set $\bigcup_{P \in N, P \neq I} A^{-1}(P)$. This set is equal to $\bigcup_{P \in N, P \neq I} \ker(P^2 - A)$. If $A$ is an $\text{Alg}N$ the set $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ contains $\bigcup_{P \in N, P \neq I} P$; hence it is dense in $H$. There exist operators $A$ in $B(H)$ for which $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is not dense in $H$. We construct such an operator in Example 9. We will prove in the next section that $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is dense in $H$ if and only if there exists an operator $B$ in $\text{Alg}N$ such that $AA^* = BB^*$. We first prove some preliminary results.

**Lemma 5.** Let $N$ be a nest on $H$ and $A$ an operator in $B(H)$. The following are equivalent:

(a) The set $\bigcup_{P \in N, P \neq 0} (A^*)^{-1}(P^\perp)$ is dense in $H$.
(b) $\bigcap_{P \in N, P \neq 0} r(AP) = \{0\}$.

**Proof.** We have that $(A^*)^{-1}(P^\perp) = \{x \in H : A^* x \in P_0^\perp\} = \{x \in H : P^\perp A^* x = A^* x\} = \ker(PA^*) = r(AP)^\perp$ and $\bigcup_{P \in N, P \neq 0} r(AP) = \{0\}$ is dense in $(\bigcap_{P \in N, P \neq 0} r(AP))^\perp$. 

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Proposition 6. Let $N$ be a nest on $H$ and $A$ an operator in $B(H)$. 
(a) Suppose that the set $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is dense in $H$. Then $r(A)$ is $N$-proper.
(b) Suppose that $r(A)$ is $N$-proper and closed. Then the set $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is dense in $H$.

Proof. (a) The set $A(\bigcup_{P \in N, P \neq I} A^{-1}(P))$ is contained in $[r(A) \cap P : P \in N, P \neq I]$ and is dense in $r(A)$.
(b) The restriction of $A$ to coker $A$ is an isomorphism from coker $A$ onto $r(A)$. Hence $(\bigcup_{P \in N, P \neq I} A^{-1}(P)) \cap \text{coker} A = A^{-1}(\bigcup_{P \in N, P \neq I} P) \cap \text{coker} A$ is dense in coker $A$. Therefore $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is dense in $H$. \hfill \Box

Proposition 7. Let $N$ be a nest on $H$ and $A$ an operator in $B(H)$. 
(a) Suppose that $\bigcap_{P \in N, P \neq 0} r(AP) = \{0\}$. Then coker $A$ is $N^\perp$-proper.
(b) Suppose that coker $A$ is $N^\perp$-proper and $r(A)$ is closed. Then $\bigcap_{P \in N, P \neq 0} r(AP) = \{0\}$.

Proof. (a) It follows from Lemma 5 that $\bigcup_{P \in N, P \neq 0} (A^*)^{-1}(P^\perp)$ is dense in $H$. It follows from Proposition 6 that $r(A^*)$ is $N^\perp$-proper. Since the closure of an $N^\perp$-proper subspace is an $N^\perp$-proper subspace we conclude that coker $A$ is $N^\perp$-proper.
(b) It follows from [2, Ch. VI, Th. 1.10] that $r(A^*)$ is closed. Hence $r(A^*) = \text{coker} A$. It follows from Proposition 6 that $\bigcup_{P \in N, P \neq 0} (A^*)^{-1}(P^\perp)$ is dense in $H$. Therefore from Lemma 5 we conclude that $\bigcap_{P \in N, P \neq 0} r(AP) = \{0\}$. \hfill \Box

3. Factorisation

In this section we prove our main results and give some applications.

Theorem 8. Let $N$ be a continuous nest and $A$ an operator in $B(H)$. The following are equivalent:
(a) There exists an operator $B$ in $\text{Alg} N$ such that $AA^* = BB^*$.
(b) The set $\bigcup_{P \in N, P \neq I} A^{-1}(P)$ is dense in $H$.

Proof. Assume (a) holds. In order to prove (b) it is enough to prove that the set $\bigcup_{P \in N, P \neq I} (A^{-1}(P) \cap \text{coker} A)$ is dense in coker $A$. Using polar decomposition one can see that there exists a partial isometry $U$ with domain coker $A$ and range coker $B$ such that $A = BU$. We put $R = \bigcup_{P \in N, P \neq I} (A^{-1}(P) \cap \text{coker} A)$ and $M = \text{coker} A \ominus R$. We will show that $M = \{0\}$. Take $m$ in $M$ and $P$ in $N, P \neq I$. Since $r(A) = r(B)$ ([5, Th. 1]), we have $BPUm = AxP$ for some $xP$ in coker $A$. Since $BPUm$ is in $P, xP$ is in $A^{-1}(P) \cap \text{coker} A$ and hence in $R$. We have $BPUm = AxP = BUxP$ and so $PUm - UxP$ is in ker $B$. We have $PUm = PUm - UxP + UxP$ which belongs to ker $B \oplus UR$. Note that the decomposition $H = \ker B \oplus UR \oplus UM$ is orthogonal. Therefore $Um = \lim_{P \in N, P \neq I, \rho \to 1} PPUm$ is in $(\ker B \oplus UR) \cap UM = \{0\}$. We conclude that $m = 0$.

Assume (b) holds. It is then clear that the set $\bigcup_{P \in N, P \neq I} (A^{-1}(P) \cap \text{coker} A)$ is dense in coker $A$. Take a sequence $\{P_n\}_{n=0}^\infty$ of elements of $N$ such that: $P_0 = 0, P_{n+1} > P_n, P_n \neq I$ for every $n$ and $P_n$ converges strongly to $I$. We set: $R_1 = A^{-1}(P_1) \cap \text{coker} A, R_n = (A^{-1}(P_n) \cap \text{coker} A) \ominus R_{n-1}$ for $n > 1$.

It is clear that $R_n$ is orthogonal to $R_m$ for $n \neq m$ and that $R_n$ is contained in coker $A$ for every $n$. We show that coker $A = \sum_{n=1}^\infty \oplus R_n$. Take $y$ in coker $A$. If $y$ is orthogonal to $\sum_{n=1}^\infty \oplus R_n$, then $y$ is orthogonal to $A^{-1}(P_n) \cap \text{coker} A$ for every...
n; hence y is orthogonal to \((\bigcup_{n=1}^{\infty} (A^{-1}(P_n) \cap \text{coker}A))\). Since \((\bigcup_{n=1}^{\infty} (A^{-1}(P_n) \cap \text{coker}A))\) is dense in \text{coker}A, y = 0, and so \text{coker}A = \bigoplus_{n=1}^{\infty} R_n.

Consider for \(n \geq 1\) a partial isometry \(V_n\) with domain contained in \((P_{n+1} - P_n)H\) and range \(R_n\). Put \(V = \bigoplus_{n=1}^{\infty} V_n\). Then \(V\) is a partial isometry with range \text{coker}A. Note that \(A = AVV^*.\) We show that \(AV\) belongs to \(\text{Alg}N\). Let \(P\) be in \(N\) and \(x\) be a vector in \(P\). We show that \(AVx\) is in \(P\). If \(P \leq P_1\) we have \(AVx = 0\). If \(P > P_1\) there exists \(m \geq 1\) such that \(P_m = P \leq P_{m+1}\). Then

\[ AVx = A(\sum_{n=1}^{m} V_n)x = (\sum_{n=1}^{m} V_n)x \text{ is contained in } (\bigoplus_{n=1}^{m} R_n). \]

Therefore \(AVx\) is in \(A(\sum_{n=1}^{m} R_n)\) which is contained in \(P_m\). Since \(P_m < P\) we conclude that \(AVx\) is in \(P\).

Put \(B = AV\). Then \(BB^* = AVV^*A^* = AA^*\) and \(B\) is in \(\text{Alg}N\).

\[ \square \]

**Remark.** Theorem 8 remains true under the weaker assumption that \(N\) is a nest which satisfies \(H = \bigcup_{Q \subset H} Q\).

Let \(N\) be a continuous nest. We give an example of an operator with \(N\)-proper range which does not satisfy condition (b) of Theorem 8.

**Example 9.** Let \(N\) be a continuous nest. Take a sequence \(\{P_n\}_{n=0}^{\infty}\) of elements of \(N\) such that:

\[ P_0 = 0, \quad P_{n+1} > P_n, \quad P_n \neq I \quad \text{for every } n \text{ and } P_n \text{ converges strongly to } I. \]

For each \(n\) consider a vector \(e_n\) of norm 1 and such that \((P_{n+1} - P_n)e_n = e_n\). Put \(y = \sum_{n=1}^{\infty} n^{-1}e_n\). Let \(A\) be the operator defined by: \(Ae_n = n^{-1}e_n\) for \(n \geq 1\), \(Ae_0 = y\) and \(A\) is 0 on \([e_n : n = 0, 1, 2, \ldots]♀\). Then \(r(A)\) is \(N\)-proper and it is easy to see that \(A\) does not satisfy condition (b) of Theorem 8. In fact, \(e_0\) is orthogonal to \(\bigcup_{P \in N, P \neq I} A^{-1}(P)\). So \(A\) does not satisfy condition (a) of Theorem 8.

**Corollary 10.** Let \(N\) be a continuous nest and \(A\) an operator in \(B(H)\). The following are equivalent:

(a) There exists an operator \(B\) in \(\text{Alg}N\) such that \(A^*A = B^*B\).

(b) \(\bigcap_{P \in N, P \neq I} r(AP) = \{0\}\).

**Proof.** There exists an operator \(B\) in \(\text{Alg}N\) such that \(A^*A = B^*B\) if and only if there exists an operator \(C\) in \(\text{Alg}N^{\perp}\) such that \(A^*A = CC^*\). The corollary follows now from Theorem 8 and Lemma 5.

\[ \square \]

**Corollary 11.** Let \(N\) be a continuous nest and \(A\) an operator in \(B(H)\). Suppose \(A\) is onto (resp. one-to-one and \(r(A)\) is closed). Then there exists an operator \(B\) in \(\text{Alg}N\) such that \(AA^* = BB^*\) (resp. \(A^*A = B^*B\)).

**Proof.** It follows from Proposition 6 and Theorem 8 (resp. from Proposition 7 and Corollary 10).

\[ \square \]

**Corollary 12.** Let \(N\) be a continuous nest and \(Q\) a projection in \(B(H)\). Then there exists an operator \(B\) in \(\text{Alg}N\) such that \(Q = BB^*\) (resp. \(Q = B^*B\)) if and only if \(QH\) is \(N\)-proper (resp. \(N^\perp\)-proper).

**Proof.** It follows from Proposition 6 and Theorem 8 (resp. from Proposition 7 and Corollary 10).

\[ \square \]

The following corollary answers a question posed by Shields in [13].
Corollary 13. Let $N$ be a continuous nest and $A$ a positive operator in $B(H)$. Assume there exists an operator $B$ in $\text{Alg}N$ such that $A^2 = B^*B$. Then there exists an operator $C$ in $\text{Alg}N$ such that $A = C^*C$.

Proof. We have to show that if $\bigcap_{P \in N, P \neq 0} \overline{r(AP)} = \{0\}$, then $\bigcap_{P \in N, P \neq 0} \overline{r(A^{1/2}P)} = \{0\}$. Let $y$ be in $\bigcap_{P \in N, P \neq 0} \overline{r(A^{1/2}P)}$. Then $A^{1/2}y$ is in $\bigcap_{P \in N, P \neq 0} \overline{r(AP)} = \{0\}$; hence $A^{1/2}y = 0$. So $y$ is in $\text{Ker} A^{1/2}$. Since $y$ is also in $r(A^{1/2})$ we see that $y = 0$. □

We will characterise now the operators that satisfy condition (a) of Theorem 8 (resp. condition (a) of Corollary 10) for every continuous nest.

Proposition 14. Let $V$ be an operator range. Assume $V$ is not of co-finite dimension in $H$. Then there exists a continuous nest $N$ in $H$ such that $V \cap P = \{0\}$ for every $P$ in $N$, $P \neq I$.

Proof. (i) We first show that there exists a non-closed operator range $W$ which contains $V$. We will use the following fact: If $V_1, V_2$ are operator ranges, then $V_1 + V_2$ is an operator range [7, Ch. I, 1]. If $V$ is closed we consider an operator range $U$ which is non-closed and is contained in $V^\perp$. We set $W = V + U$. Then $W$ is an operator range which is non-closed and contains $V$.

(ii) It follows from (i) above that we may assume that $V$ is non-closed. An operator range $R$ is called of type $J_S$ (Dixmier’s notation) if it is dense and there exists a sequence $\{H_n\}_{n=0}^\infty$ of closed mutually orthogonal infinite dimensional subspaces of $H$ such that $R = \{\sum_{n=0}^\infty x_n : x_n \in H_n \text{ and } \sum_{n=0}^\infty (2^n \|x_n\|)^2 < \infty\}$. It is shown in the proof of Theorem 3.6 in [6] that any non-closed operator range is contained in an operator range of type $J_S$. It follows that there exists an operator range $S$ of type $J_S$ such that $V \subset S$. It follows from Theorem 3.6 in [6] that there exists a unitary operator $U$ on $H$ such that $US \cap S = \{0\}$. We conclude that there exists an operator range $T$ of type $J_S$ such that $V \cap T = \{0\}$. Now it is easy to see that there exists a continuous nest $N$ in $H$ such that $P \subset T$ for every $P$ in $N$, $P \neq I$. It follows that $P \cap V = \{0\}$ for every $P$ in $N$, $P \neq I$. □

Theorem 15. Let $A$ be an operator in $B(H)$.

(a) There exists for every continuous nest $N$ an operator $B_N$ in $\text{Alg}N$ satisfying $AA^* = B_NB_N^*$ if and only if $A$ is a right Fredholm operator.

(b) There exists for every continuous nest $N$ an operator $B_N$ in $\text{Alg}N$ satisfying $A^*A = B_N^*B_N$ if and only if $A$ is a left Fredholm operator.

Proof. (a) Assume that for every continuous nest $N$ there exists an operator $B_N$ in $\text{Alg}N$ satisfying $AA^* = B_NB_N^*$. It follows from Theorem 8 and Proposition 6 that $r(A)$ is $N$-proper for very continuous nest $N$. Proposition 14 implies that $r(A)$ is of co-finite dimension in $H$. If the range of an operator is of co-finite dimension, then it is closed [4, Prop. 3.7]. Therefore $A$ is a right Fredholm operator. Assume now that $A$ is a right Fredholm operator. Then $r(A)$ is closed and of co-finite dimension in $H$. By Proposition 4, $r(A)$ is $N$-proper for every continuous nest $N$. It follows then from Proposition 6 and Theorem 8 that for every continuous nest $N$ there exists an operator $B_N$ in $\text{Alg}N$ satisfying $AA^* = B_NB_N^*$.

(b) Consider the following properties of an operator $A$:

(i) There exists for every continuous nest $N$ an operator $B_N$ in $\text{Alg}N$ satisfying $AA^* = B_NB_N^*$.

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(ii) There exists for every continuous nest $N$ an operator $B_N$ in $\text{Alg} N$ satisfying $A^*A = B_N^*B_N$.

Since a nest $N$ is continuous if and only if the nest $N^\perp$ is continuous we see that an operator $A$ has property (i) if and only if the operator $A^*$ has property (ii). The assertion follows now from (a).

**Added in proof**

After this work was submitted a paper of G. T. Adams, J. Froehlich, P. J. McGuire, and V. I. Paulsen entitled *Analytic reproducing kernels and factorisation*, Indiana Univ. Math. J. 43 (1994), came to our attention. Condition (b) of our Theorem 8 is essentially the same with the density condition given in Theorem 3.1 of this paper in a different but related context.

**References**


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