

**DIXMIER'S THEOREM FOR SEQUENTIALLY  
ORDER CONTINUOUS BAIRE MEASURES  
ON COMPACT SPACES**

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ABSTRACT. We prove that a Baire measure (or a regular Borel measure) on a compact Hausdorff space is sequentially order continuous as a linear functional on the Banach space of all continuous functions if and only if it vanishes on meager Baire subsets, a result parallel to a much earlier theorem of Dixmier. We also give some results on the relation between sequentially order continuous measures on compact spaces and countably additive measures on Boolean algebras.

INTRODUCTION

In this paper we first prove that a Baire measure (or a regular Borel measure) on a compact Hausdorff space is sequentially order continuous (equivalently,  $\sigma$ -order continuous) as a linear functional on the Banach space of all continuous functions if and only if it vanishes on meager Baire subsets. Thus we obtain a result which is parallel to a much earlier theorem of Dixmier (see [3]) which asserts that a regular Borel measure on a compact Hausdorff space is order continuous if and only if it vanishes on meager Borel sets. Therefore, the relation between sequential order continuity and order continuity of regular Borel measures is just that of meager Baire subsets versus meager Borel subsets. Some related results regarding Baire subsets are also given. In the second half of this paper we study the relation between sequentially order continuous measures on compact spaces and countably additive measures on Boolean algebras.

For unexplained terminology, we refer to [2], [4], [7], [8], and [11]. When  $E$  is a Banach space,  $E^*$  denotes the norm dual of  $E$ , and when  $E$  is a Banach lattice,  $E_c^*$  denotes the subspace of  $E^*$  that consists of all sequentially order continuous linear functionals. It is well-known that  $E_c^*$  is a band of  $E^*$ . Finally, we remark that all spaces considered in this paper are real spaces and that the results can be easily extended to the corresponding complex spaces.

1. AN ANALOGUE OF A THEOREM OF DIXMIER

From now on  $K$  denotes a compact Hausdorff space. Recall that the Borel subsets of  $K$  are the members of the  $\sigma$ -algebra (the Borel field of  $K$ ) generated by all closed subsets of  $K$ , while the Baire subsets of  $K$  are the members of the

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$\sigma$ -algebra (the Baire field of  $K$ ) generated by all compact  $G_\delta$ -subsets of  $K$ . A Borel measure is a countably additive measure on the Borel field, and a Baire measure is a countably additive measure on the Baire field. It is well-known that any Baire measure is Baire regular and can be extended uniquely to a regular Borel measure (see [1] or [6]). As usual  $C(K)$  denotes the real Banach lattice of all real-valued continuous functions on  $K$ . It is well-known that the dual of  $C(K)$  can be identified with  $M(K)$ , the space of all bounded regular Borel (or just Baire) measures on  $K$  (Riesz representation theorem). The following theorem of Dixmier [3] characterizes order continuous functionals on  $C(K)$ . See also [11, Thm. 19.9.2].

**Theorem.** *Let  $0 \leq \mu \in M(K)$ . Then  $\mu$  is order continuous on  $C(K)$  if and only if  $\mu(E) = 0$  for any meager Borel subset  $E$  of  $K$ .*

*Remarks.* (1) In the above theorem the assumption that  $\mu$  is positive can be dropped.

(2) By regularity of the measure  $\mu$ , we can easily see that  $\mu$  vanishes on meager Borel subsets of  $K$  if and only if any one of the following conditions is satisfied:

- (1)  $\mu(E) = 0$  for any Borel subset  $E$  with empty interior.
- (2)  $\mu(E) = \mu(E^\circ)$  for any closed subset  $E$  of  $K$ , where  $E^\circ$  denotes the interior of  $E$ .
- (3)  $\mu(\partial E) = 0$  for any closed subset  $E$  of  $K$ , where  $\partial E$  denotes the boundary of  $E$ .

In the following we focus on sequentially order continuous functionals on  $C(K)$ . It turns out that such functionals (or measures on  $K$ ) can be characterized by their behavior on certain Baire sets in  $K$ , just as order continuous ones are characterized by corresponding behavior on the (generally) wider class of Borel sets. This is of interest, if only because sequential order continuity of functionals occurs much more frequently in analysis than order continuity. For example, if  $\Sigma$  is a  $\sigma$ -algebra of subsets of any nonempty set  $X$  and contains all singletons, then the order continuous measures  $\Sigma \rightarrow \mathbf{R}$  are precisely the atomic ones, while the sequentially order continuous measures  $\Sigma \rightarrow \mathbf{R}$  are exactly the countably additive ones.

Note that any compact Hausdorff space is a Baire space, i.e., any nonempty open subset of such a space is of second category, and that a closed Baire subset of  $K$  is in fact a compact  $G_\delta$ -set. See [6, page 221] for details.

**Theorem 1.1.** *Let  $K$  be a compact space and let  $0 \leq \mu \in M(K)$ . Then the following statements are equivalent:*

- (1)  $\mu$  is sequentially order continuous.
- (2)  $\mu(E) = 0$  whenever  $E$  is a closed, nowhere dense  $G_\delta$ -set in  $K$ .
- (3)  $\mu(A) = 0$  whenever  $A$  is a meager Baire subset of  $K$ .
- (4)  $\mu(A) = 0$  for any Baire subset  $A$  with empty interior.

*Proof.* First it is easy to see that (4) implies (3) and that (3) implies (2). That (2) implies (4) follows from the fact that every Baire measure on  $K$  is regular (see [1, chapter 8]). We now prove that (1) and (2) are equivalent. Suppose that (1) holds. Let  $E$  be any closed, nowhere dense  $G_\delta$ -subset of  $K$ . By [1, Thm. 2, p. 175], there exists a sequence  $\{f_n\} \subset C(K)$  such that  $f_n \downarrow \chi_E$ , i.e.,  $\{f_n\}$  is a decreasing sequence and  $f_n(x) \rightarrow \chi_E(x)$  for all  $x \in K$ . Let  $0 \leq g \in C(K)$  be such that  $0 \leq g \leq f_n$  for all  $n$ . Then  $0 \leq g(x) \leq \chi_E(x)$  for all  $x \in K$ . Since  $E$  is nowhere dense,  $g = 0$ . So  $\{f_n\}$  order converges to zero in  $C(K)$ , and we have  $\mu(f_n) \rightarrow 0$ , from which we conclude that  $\mu(E) = 0$ .

Now suppose that (2) holds. Let  $\{f_n\}$  be a decreasing sequence in  $C(K)$  that order converges to zero. For any  $\varepsilon > 0$ , define

$$E = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \left\{ x \in K : f_n(x) > \varepsilon - \frac{1}{k} \right\} = \bigcap_{n=1}^{\infty} \{x \in K : f_n(x) \geq \varepsilon\}.$$

Then  $E$  is a closed  $G_\delta$ -set. We claim that its interior is empty. If not, then there is a nonempty open set  $W \subseteq E$ . Let  $0 \leq g \in C(K)$  be a nonzero element such that  $0 \leq g \leq \varepsilon \mathbf{1}$  and  $g(x) = 0$  for all  $x \in K - W$ . Then  $0 \leq g \leq f_n$  for all  $n$ . This is a contradiction to the fact that  $\{f_n\}$  order converges to zero. Therefore, we have  $\mu(E) = 0$ . Let  $E_n = \{x \in K : f_n(x) \geq \varepsilon\}$ . Then  $E_n \downarrow E$  and so  $\mu(E_n) \rightarrow \mu(E) = 0$ . Now

$$0 \leq \mu(f_n) = \int_{\{f_n \geq \varepsilon\}} f_n d\mu + \int_{\{f_n < \varepsilon\}} f_n d\mu \leq \|f_n\| \mu(E_n) + \varepsilon \mu(K).$$

From the above, we see that  $\mu(f_n) \rightarrow 0$ . So  $\mu$  is sequentially order continuous.

It follows from the above theorem of Dixmier and Theorem 1.1 that if  $K$  is a compact space such that the Baire subsets coincide with the Borel subsets, then sequentially order continuous functionals on  $C(K)$  coincide with order continuous functionals. For example, this is the case when  $K$  is metrizable.

*Remarks.* (1) It is not difficult to see that the positivity of the measure  $\mu$  in Theorem 1.1 can be dropped.

(2) It should be pointed out that a very special case of Theorem 1.1 was much earlier discussed in [11, section 18.7.3].

**Theorem 1.2.** *Let  $K$  be a compact space and let  $0 \leq \mu \in M(K)$ . Consider the following statements:*

- (1)  $\mu(A) = \mu(A^\circ)$  for any compact  $G_\delta$ -set  $A$  of  $K$ .
- (2)  $\mu$  is sequentially order continuous.

*Then (1) implies (2), and if  $K$  is quasi-Stonian or metrizable, then (1) and (2) are equivalent.*

*Proof.* By Theorem 1.1, (1) implies (2). Now suppose that  $K$  is quasi-Stonian and that  $\mu$  is sequentially order continuous. Let  $A$  be a compact  $G_\delta$ -set. Then there exists a sequence of open subsets  $\{U_n\}$  such that  $A = \bigcap_{n=1}^{\infty} U_n$ . Then  $A^c$ , the complement, is an open  $F_\sigma$ -set. Since  $K$  is quasi-Stonian,  $\overline{A^c}$  is open and closed. Since  $A^\circ = (\overline{A^c})^c$  (this is true in any topological space),  $A^\circ$  is open and closed and thus a Baire subset. Now  $A - A^\circ$  is a nowhere dense Baire subset. By Theorem 1.1, we have  $\mu(A - A^\circ) = 0$ . So (1) follows. Finally, suppose that  $K$  is metrizable. Then any closed subset as well as its interior is a Baire subset, and so (1) also follows from (2).

**Example.** By Theorem 1.1, conditions (1) and (2) of the preceding theorem would be clearly equivalent if the interior of a closed Baire subset were necessarily Baire. The following example shows this to be false, and hence accentuates the care to be exercised when dealing with Baire subsets of a compact space. Recall that a compact Hausdorff space  $K$  is said to be an  $F$ -space if disjoint open  $F_\sigma$ -subsets of  $X$  have disjoint closures. We refer to [5] and [10] for more details. Now let  $K$  be a compact Hausdorff space such that  $K$  is a connected  $F$ -space (such a space exists by [10, page 272]). Let  $A$  be a compact  $G_\delta$ -subset of  $K$  such that  $A \neq K$  and  $A^\circ$  is nonempty (such a subset exists by the fact that continuous functions separate the

points of  $K$ ). We claim that  $A^\circ$  is not a Baire subset. Since  $A^\circ = (\overline{A^c})^c$ , we have  $K = A^\circ \cup \overline{A^c}$ , a disjoint union. If  $A^\circ$  were Baire, then it would be an open  $F_\sigma$ -set by [6, Thm. D, page 221]. Now  $A^\circ$  and  $A^c$  are two disjoint open  $F_\sigma$ -sets. Since  $K$  is an  $F$ -space, we have  $\overline{A^\circ} \cap \overline{A^c} = \emptyset$ , and so  $A^\circ = \overline{A^\circ}$ . This shows that  $A^\circ$  is open and closed, which is a contradiction to the assumption that  $K$  is connected.

More generally, we have the following. We note again that in a compact Hausdorff space a closed subset is Baire if and only if it is a  $G_\delta$ -set, hence an open subset is Baire if and only if it is an  $F_\sigma$ -set ([6, page 221]).

**Theorem 1.3.** *Let  $K$  be a compact Hausdorff space. Then the following statements are equivalent:*

- (1)  *$K$  is an  $F$ -space, and for any compact  $G_\delta$ -subset  $G$ , the interior  $G^\circ$  is a Baire subset.*
- (2)  *$K$  is an  $F$ -space, and for any open  $F_\sigma$ -subset  $O$ , the closure  $\overline{O}$  is a Baire subset.*
- (3)  *$K$  is quasi-Stonian.*

*Proof.* We first prove that (1) implies (2). Let  $O$  be an open  $F_\sigma$ -subset of  $K$ . Then  $O^c$  is a compact  $G_\delta$ -subset. By (1),  $(O^c)^\circ$  is Baire. But  $(O^c)^\circ = (\overline{O})^c$ . So  $\overline{O}$  is Baire. Similarly, we can prove that (2) implies (1), since  $G^\circ = (\overline{G^c})^c$ .

We now prove that (2) implies (3). Let  $O$  be an open  $F_\sigma$ -subset. Then  $\overline{O}$  is Baire by assumption and thus a compact  $G_\delta$ -set. Now  $O$  and  $(\overline{O})^c$  are two disjoint open  $F_\sigma$ -subsets. Since  $K$  is an  $F$ -space, they have disjoint closures. This implies that  $\overline{O} = (\overline{O})^\circ$ . So  $\overline{O}$  is open-and-closed, and thus  $K$  is quasi-Stonian. Finally, that (3) implies (2) is trivial.

## 2. COUNTABLY ADDITIVE MEASURES ON BOOLEAN ALGEBRAS

In this section we consider the special case when  $K$  is 0-dimensional (see [11]). The open-and-closed subsets of such a space form a base of its topology and an algebra of sets, and conversely it follows from the well-known Stone Representation Theorem (see [4]) that any algebra of sets (more generally, any Boolean algebra) is Boolean isomorphic to the algebra of all open-and-closed subsets of a unique (within homeomorphisms) 0-dimensional compact Hausdorff space.

Let  $X$  be a nonempty set, and let  $\mathcal{F}$  be an algebra of subsets of  $X$ . We use  $S(\mathcal{F})$  to denote the vector lattice of all real-valued  $\mathcal{F}$ -simple functions on  $X$  equipped with the sup-norm and use  $B(\mathcal{F})$  to denote the norm closure of  $S(\mathcal{F})$  in the Banach lattice of all bounded real functions on  $X$ . It is well-known [4] that the dual of  $B(\mathcal{F})$  can be identified with  $ba(\mathcal{F})$ , the Banach lattice of all bounded, finitely additive real-valued measures on  $\mathcal{F}$  equipped with the total variation norm and the usual lattice structure. For any  $\mu \in ba(\mathcal{F})$  and any  $f \in B(\mathcal{F})$ , we have  $\mu(f) = \int f d\mu$ . Consider  $\mathcal{F}$  as an abstract Boolean algebra. A measure  $\mu \in ba(\mathcal{F})$  is said to be countably additive in the Boolean sense if for any disjoint sequence  $\{E_i\}$  in  $\mathcal{F}$  such that the supremum  $\bigvee_{i=1}^{\infty} E_i$  exists in  $\mathcal{F}$ ,  $\mu(\bigvee_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ . We use  $cab(\mathcal{F})$  to denote the subspace of all countably additive measures in the Boolean sense, while we use  $cas(\mathcal{F})$  to denote the subspace of all countably additive measures in the usual set-theoretical sense. It is well-known and easy to verify that both  $cab(\mathcal{F})$  and  $cas(\mathcal{F})$  are bands in  $ba(\mathcal{F})$ . Also, it is well-known that  $cab(\mathcal{F}) \subseteq cas(\mathcal{F})$  and that when  $\mathcal{F}$  is a  $\sigma$ -algebra,  $cab(\mathcal{F}) = cas(\mathcal{F})$ . For proofs of the above results, we refer to [4] and [11].

**Theorem 2.1.** *Let  $\mu \in ba(\mathcal{F})$ . Then the following statements are equivalent:*

- (1)  $\mu$  is countably additive in the Boolean sense.
- (2)  $\mu : S(\mathcal{F}) \rightarrow \mathbf{R}$  is sequentially order-continuous.
- (3)  $\mu : B(\mathcal{F}) \rightarrow \mathbf{R}$  is sequentially order-continuous.

To prove Theorem 2.1 we need the following lemma.

**Lemma.** *For any  $0 \leq f \in B(\mathcal{F})$  and any  $\varepsilon > 0$ , there exists  $\phi \in S(\mathcal{F})$  such that  $0 \leq \phi \leq f$  and  $\|f - \phi\| < \varepsilon$ .*

*Proof of the Lemma.* First choose  $g \in S(\mathcal{F})$  such that  $\|f - g\| < \varepsilon$ . Suppose that  $g = \sum_{i=1}^n a_i \chi_{E_i}$  with  $E_i \in \mathcal{F}$  disjoint. It follows that for each  $i$ ,  $|f(x) - a_i| < \varepsilon$  for all  $x \in E_i$ . Define  $b_i = \inf\{f(x) : x \in E_i\}$  and form  $\phi = \sum_{i=1}^n b_i \chi_{E_i}$ . Then it is easy to verify that  $\|f - \phi\| \leq 2\varepsilon$  and  $0 \leq \phi \leq f$ .

*Proof of Theorem 2.1.* We may assume that  $\mu \geq 0$ . Suppose that (1) holds. Let  $\phi_n \downarrow 0$  (order converging to zero) in  $S(\mathcal{F})$ . For any  $\varepsilon > 0$ , let  $E_n = \{x \in X : \phi_n(x) \geq \varepsilon\}$ . Suppose that  $A \in \mathcal{F}$  satisfies  $A \subseteq E_n$  for all  $n$ . Then  $\varepsilon \chi_A \leq \phi_n$  for all  $n$  and so  $A = \emptyset$ . This shows that the infimum of  $\{E_n\}$  is  $\emptyset$ . By the countable additivity of  $\mu$  in the Boolean sense, we have  $\mu(E_n) \rightarrow 0$ . From

$$0 \leq \mu(\phi_n) = \int_{E_n} \phi_n d\mu + \int_{X-E_n} \phi_n d\mu \leq \|\phi_n\| \mu(E_n) + \varepsilon \|\mu\|,$$

we see that  $\mu$  is sequentially order continuous on  $S(\mathcal{F})$ .

Now suppose that (2) holds. Let  $f_n \downarrow 0$  (order converging to zero) in  $B(\mathcal{F})$ , and let  $\varepsilon > 0$ . By the above lemma, there exists  $0 \leq \phi_n \in S(\mathcal{F})$  such that  $0 \leq \phi_n \leq f_n$  and  $\|f_n - \phi_n\| \leq \varepsilon/2^n$ . Observe that  $|f_n - \phi_1 \wedge \cdots \wedge \phi_n| \leq \sum_{i=1}^n |f_i - \phi_i|$ . So  $\|f_n - \phi_1 \wedge \cdots \wedge \phi_n\| \leq \varepsilon$ . It is clear that  $\phi_1 \wedge \cdots \wedge \phi_n \downarrow 0$  in  $S(\mathcal{F})$ . So  $\mu(\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow 0$ . From

$$0 \leq \mu(f_n) - \mu(\phi_1 \wedge \cdots \wedge \phi_n) \leq \varepsilon \|\mu\|,$$

we see that  $\mu(f_n) \rightarrow 0$ .

Finally, we prove that (3) implies (1). Let  $\{E_i\}$  be a disjoint sequence in  $\mathcal{F}$  such that the supremum exists and equals  $E$ . Let  $A_n = E - \bigcup_{i=1}^n E_i$ . Then  $A_n \downarrow \emptyset$  in  $\mathcal{F}$ . It is easy to verify that  $\chi_{A_n} \downarrow 0$  in  $B(\mathcal{F})$  by the above lemma. So  $\mu(A_n) \rightarrow 0$ . This implies that  $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$ . The proof is complete.

*Remark.* Using Theorem 1.1, we can also prove that (1) implies (3). Let  $K$  be the Stone space of  $\mathcal{F}$ . It is well-known that  $B(\mathcal{F})$  can be identified with  $C(K)$ . So it is sufficient to prove that  $\mu$  is sequentially order continuous on  $C(K)$ . Let  $G = \bigcap_{n=1}^{\infty} U_n$  be a compact  $G_\delta$ -subset of  $K$  such that its interior is empty, where each  $U_n$  can be chosen to be open-and-closed since  $K$  is 0-dimensional. We may assume that  $U_{n+1} \subseteq U_n$  for all  $n$ . If  $A$ , open-and-closed, is a lower bound for  $\{U_n\}$  in the Boolean sense, then  $A$  is empty since the interior of  $G$  is empty. So  $U_n \downarrow \emptyset$  in the Boolean sense, and so we have  $\mu(U_n) \rightarrow 0$  by (1). This shows that  $\mu(G) = 0$ . By Theorem 1.1,  $\mu$  is sequentially order continuous on  $C(K)$ .

The next theorem now follows immediately from Theorem 2.1.

**Theorem 2.2.** *Let  $\mathcal{F}$  be an algebra, and let  $K_{\mathcal{F}}$  be the Stone space of  $\mathcal{F}$ . Then  $cab(\mathcal{F}) \cong C(K_{\mathcal{F}})_c^*$ . Conversely, if  $K$  is a 0-dimensional compact Hausdorff space, then  $C(K)_c^* \cong cab(\mathcal{F}_K)$ , where  $\mathcal{F}_K$  is the algebra of all open-and-closed subsets of  $K$ .*

We next consider the measures in  $cas(\mathcal{F})$ . The following theorem is well-known and follows from the extension theorem of Carathéodory and the Lebesgue Dominated Convergence Theorem. One can also give a proof which is similar to that of Theorem 2.1 without using the extension theorem of Carathéodory. Note that in the following theorem,  $\mathcal{F}$  is not understood to be a  $\sigma$ -algebra.

**Theorem 2.3.** *Let  $\mu \in ba(\mathcal{F})$ . Then the following statements are equivalent:*

- (1)  $\mu$  is countably additive in the set-theoretical sense.
- (2a)  $\mu : S(\mathcal{F}) \rightarrow \mathbf{R}$  is sequentially continuous in the sense that if  $\{\phi_n\} \subset S(\mathcal{F})$  and  $\phi_n(x) \downarrow 0$  for all  $x \in X$ , then  $\mu(\phi_n) \rightarrow 0$ .
- (2b)  $\mu : S(\mathcal{F}) \rightarrow \mathbf{R}$  is sequentially continuous in the sense that if  $\{\phi_n\} \subset S(\mathcal{F})$  is bounded and  $\phi_n(x) \rightarrow 0$  for all  $x \in X$ , then  $\mu(\phi_n) \rightarrow 0$ .
- (3a)  $\mu : B(\mathcal{F}) \rightarrow \mathbf{R}$  is sequentially continuous in the sense that if  $\{f_n\} \subset B(\mathcal{F})$  and  $f_n(x) \downarrow 0$  for all  $x \in X$ , then  $\mu(f_n) \rightarrow 0$ .
- (3b)  $\mu : B(\mathcal{F}) \rightarrow \mathbf{R}$  is sequentially continuous in the sense that if  $\{f_n\} \subset B(\mathcal{F})$  is bounded and  $f_n(x) \rightarrow 0$  for all  $x \in X$ , then  $\mu(f_n) \rightarrow 0$ .

Again, let  $\mathcal{F}$  be an algebra of subsets of  $X$ , and let  $\Sigma$  be the  $\sigma$ -algebra generated by  $\mathcal{F}$ . If  $K_{\mathcal{F}}$  and  $K_{\Sigma}$  denote the Stone spaces of  $\mathcal{F}$  and  $\Sigma$  respectively, then  $B(\mathcal{F})$  and  $B(\Sigma)$  can be identified with  $C(K_{\mathcal{F}})$  and  $C(K_{\Sigma})$ , respectively. Here we use  $C(K_{\mathcal{F}})_s^*$  to denote the subspace of  $C(K_{\mathcal{F}})^*$  that consists of all functionals induced by measures in  $cas(\mathcal{F})$ . Let  $T : B(\mathcal{F}) \rightarrow B(\Sigma)$  be the natural imbedding. Then  $T$  is an isometry and a lattice homomorphism. Now regard  $T$  as a mapping from  $C(K_{\mathcal{F}})$  into  $C(K_{\Sigma})$ . It follows from the Banach-Stone Theorem (see [7, page 171]) that there exists a continuous mapping  $\varphi : K_{\Sigma} \rightarrow K_{\mathcal{F}}$  such that  $Tf(x) = f(\varphi(x))$  for all  $f \in C(K_{\mathcal{F}})$  and all  $x \in K_{\Sigma}$ . Note that  $T^*$  is in fact the restriction of an element  $\mu \in ba(\Sigma)$  to the algebra  $\mathcal{F}$ . Now we have the following:

**Theorem 2.4.** *Let  $\mathcal{F}$  be an algebra, and let  $\Sigma$  be the  $\sigma$ -algebra generated by  $\mathcal{F}$ . Let  $T : B(\mathcal{F}) \rightarrow B(\Sigma)$  be the natural imbedding. Then  $cas(\mathcal{F}) = T^*(cab(\Sigma))$ , and  $T^*$ , restricted to  $cab(\Sigma)$  is an isometric lattice isomorphism from  $cab(\Sigma)$  into  $cas(\mathcal{F})$ . Equivalently, if  $K_{\mathcal{F}}$  and  $K_{\Sigma}$  denote the Stone spaces of  $\mathcal{F}$  and  $\Sigma$  respectively, then  $T^*$  induces an isometric lattice isomorphism from  $C(K_{\Sigma})_c^*$  onto  $C(K_{\mathcal{F}})_s^*$ .*

*Proof.* We first prove that  $cas(\mathcal{F}) = T^*(cab(\Sigma))$ . Let  $\mu \in cas(\mathcal{F})$ . Then it follows from Carathéodory's Extension Theorem that  $\mu$  can be extended (uniquely) to a bounded countably additive measure  $\tilde{\mu}$  on  $\Sigma$ , the  $\sigma$ -algebra generated by  $\mathcal{F}$ . So we have  $\tilde{\mu} \in cab(\Sigma)$  and  $\mu = T^*(\tilde{\mu})$ . Conversely, if  $\mu \in cas(\mathcal{F})$  satisfies  $\mu = T^*(\nu)$  for some  $\nu \in cab(\Sigma)$ , then  $\mu$  is countably additive in the set-theoretical sense.

It is easy to see that  $T^*$ , restricted to  $cab(\Sigma)$  is a bijection from  $cab(\Sigma)$  onto  $cas(\mathcal{F})$ , since a countably additive measure defined on an algebra has a unique countably additive extension to the  $\sigma$ -algebra generated by the algebra. Notice that the inverse  $(T^*)^{-1}$  of this restriction maps  $\mu \in cas(\mathcal{F})$  onto its countably additive extension  $\tilde{\mu} \in cab(\Sigma)$  and that a positive element  $\mu \in cas(\mathcal{F})$  has a positive countably additive extension to  $\Sigma$ . So  $(T^*)^{-1}$  is a positive operator. This implies that  $T^*$  is a lattice isomorphism on  $cab(\Sigma)$ . Now it is easy to see that  $T^*$  is an isometry on  $cab(\Sigma)$ , since it preserves the norm of positive elements. The proof is complete.

In conclusion, let us note the not-so-obvious fact that the Carathéodory extension  $\mu \rightarrow \tilde{\mu}$  and the operation of taking the total variation commute, i.e.,  $|\mu|^\sim = |\tilde{\mu}|$ .

## REFERENCES

1. S. K. Berberian, *Measures and Integration*, Macmillan, 1965; reprint, Chelsea, 1970. MR **32**:1315; MR **49**:10840
2. J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Mathematical Surveys, Number 15, American Mathematical Society, 1977. MR **56**:1121b
3. J. Dixmier, *Sur certains espaces considérés par M. H. Stone*, Summa Brasil. Math. **2** (1951), 151–182. MR **14**:69e
4. N. Dunford and J. T. Schwartz, *Linear Operators*, Part I, Interscience, 1958. MR **22**:8302
5. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, 1976. MR **53**:11352
6. P. R. Halmos, *Measure Theory*, Springer-Verlag, 1974. MR **11**:504d (previous ed.)
7. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991. MR **93f**:46025
8. H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, 1974. MR **54**:11023
9. H. H. Schaefer and Xiao-Dong Zhang, *A variant of Grothendieck's theorem on weak\* convergent sequences*, Arch. Math. (Basel) **65** (1995), 251–254. MR **96f**:46035
10. G. L. Seever, *Measures on  $F$ -spaces*, Trans. Amer. Math. Soc. **133** (1968), 267–280. MR **37**:1976
11. Z. Semadeni, *Banach Spaces of Continuous Functions*, Warsaw: Polish Scientific Publishers 1971. MR **45**:5730

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