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AN ABSTRACT ERGODIC THEOREM AND SOME INEQUALITIES FOR OPERATORS ON BANACH SPACES

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ABSTRACT. We prove an abstract mean ergodic theorem and use it to show that if $\{A_n\}$ is a sequence of commuting m-dissipative (or normal) operators on a Banach space X, then the intersection of their null spaces is orthogonal to the linear span of their ranges. It is also proved that the inequality $\|x+Ay\| \geq \|x\|-2\sqrt{\|Ax\|\,\|y\|}$ $(x, y \in D(A))$ holds for any m-dissipative operator A. These results either generalize or improve the corresponding results of Shaw, Mattila, and Crabb and Sinclair, respectively.

1. INTRODUCTION

Let X be a real (or complex) Banach space and $B(X)$ the Banach algebra of all bounded linear operators on X. Given a family $\mathbf{A}$ of closed linear operators on X, a net $\{A_\alpha\}$ in $B(X)$ will be called an $\mathbf{A}$-*ergodic net* if the following conditions hold:

(a) There is an $M > 0$ such that $\|A_\alpha\| \leq M$ for all α;
(b) $\|(A_\alpha - I)x\| \to 0$ for all $x \in \bigcap_{A\in\mathbf{A}} N(A)$ and $R(A_\alpha - I) \subset \overline{\sum_{A\in\mathbf{A}} R(A)}$ eventually α;
(c) For every $A \in \mathbf{A}$, $R(A_\alpha) \subset D(A)$ eventually α and $w\text{-}\lim_\alpha AA_\alpha x = 0$ for all $x \in X$, and $\|A_\alpha Ax\| \to 0$ for all $x \in D(A)$.

Note that $\mathbf{A}$-ergodic nets with $\mathbf{A} = \{T - I; T \in \mathbf{S}\}$ for some set $\mathbf{S} \subset B(X)$ were first studied by Eberlein [8]; such an operator net is named a right, weakly left $\mathbf{S}$-ergodic net in [12, p. 75].

An abstract mean ergodic theorem from [17, Theorem 1.1] asserts that, for an $\mathbf{A}$-ergodic net with $\mathbf{A}$ consisting of a single closed operator A, the operator P, defined by

$$\begin{cases} D(P) := \{x \in X; s\text{-}\lim_\alpha A_\alpha x \text{ exists}\}, \\ Px = s\text{-}\lim_\alpha A_\alpha x,\ x \in D(P), \end{cases}$$

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is a bounded linear projection with norm $\|P\| \le M$, range $R(P) = N(A)$, and null space $N(P) = \overline{R(A)}$. Applications of it to studies of ergodic properties of many particular operator families, such as integrated semigroups, cosine operator functions, and tensor product semigroups were also discussed in [18] and [19]. The purposes of this paper are: (1) to extend the above ergodic theorem to the general case that $\mathbf{A}$ is not a singleton; (2) to deduce from the generalized ergodic theorem Dunford's ergodic theorem [7] for multi-parameter semigroups; (3) using the abstract ergodic theorem to investigate orthogonality properties of some operators.

The generalized abstract mean ergodic theorem states as follows.

Theorem 1. *Let $\mathbf{A}$ be a family of closed linear operators on X, and let $\{A_\alpha\}$ be an $\mathbf{A}$-ergodic net. Then the operator P is a linear projection with norm $\|P\| \le M$, range $R(P) = \bigcap_{A\in\mathbf{A}} N(A)$, null space $N(P) = \overline{\sum_{A\in\mathbf{A}} R(A)}$, and domain*

$$\begin{aligned} D(P) &= \bigcap_{A\in\mathbf{A}} N(A) \oplus \overline{\sum_{A\in\mathbf{A}} R(A)} \\ &= \{x \in X; \{A_\alpha x\} \text{ has a weak cluster point}\}. \end{aligned}$$

Here $\sum_{A\in\mathbf{A}} R(A)$ denotes the linear space spanned by the spaces $R(A)$, $A \in \mathbf{A}$. If the space X is reflexive, then $D(P) = X$.

The proof of Theorem 1 and some consequences including an application to multi-parameter semigroups will be given in Section 2. Section 3 will be concerned with applications to orthogonality properties of m-dissipative operators and normal operators on Banach spaces.

We recall some necessary definitions. Let X be a complex Banach space. The (*spatial*) *numerical range* of a linear operator $A : D(A) \subset X \to X$ is the set $W(A) := \{\langle Ax, x^*\rangle;\ x \in D(A),\ x^* \in X^*,\ \|x\| = \|x^*\| = \langle x, x^*\rangle = 1\}$. An operator $H \in B(X)$ is said to be *hermitian* if its numerical range $W(H)$ is contained in the real line R, or equivalently, if $\|\exp(itH)\| = 1$ for all $t \in R$ (see [2], [3]). H is said to be *positive* if $W(H) \subset [0,\infty)$. If an operator T can be expressed as $T = H + iK$ with H and K hermitian, then $i(HK - KH)$ is hermitian. T is called *normal* if $T = H + iK$ for some commuting hermitian operators H and K. These definitions generalize those familiar concepts of operators on Hilbert spaces. An operator $A : D(A) \subset X \to X$ is called *dissipative* if its numerical range $W(A)$ is contained in the half plane $\{z \in C, \operatorname{Re} z \le 0\}$. A dissipative operator A is called *m-dissipative* if $\rho(A) \cap (0,\infty) \ne \varnothing$.

Mattila [16] and Fong [9] proved that the null space $N(T)$ of a normal operator $T \in B(X)$ is *orthogonal* to the range $R(T)$ in the sense that $\|x + Ty\| \ge \|x\|$ for all $x \in N(T)$ and all $y \in X$. In particular, a scalar multiple of a hermitian operator has this orthogonality property. This special case is readily contained in a theorem of Crabb and Sinclair [3, Theorem 20.6], which says that if 0 is not an interior point of the closed convex hull $\overline{\mathrm{co}}\, W(T)$ of the numerical range $W(T)$ of a bounded operator T (in other words, T is a scalar multiple of a bounded m-dissipative operator), then $\|x + Ty\| \ge \|x\| - \sqrt{8\|Tx\|\,\|y\|}$ for all $x, y \in X$.

In Section 3 we establish the following generalizations or improvements of the above known results:

(1) Let $\{A_1, A_2, \dots\}$ be a countable family of m-dissipative operators or normal operators on X. If the resolvents of the A_k's are commutative, then $\bigcap_{k=1}^\infty N(A_k)$ is orthogonal to the linear span $\sum_{k=1}^\infty R(A_k)$ of $\{R(A_k)\}$ (Theorem 7, Corollary 8).

(2) If A is an m-dissipative operator, then $\|x + Ay\| \geq \|x\| - 2\sqrt{\|Ax\|\,\|y\|}$ for $x, y \in D(A)$ (Theorem 9).

(3) If 0 is not an interior point of $\overline{\mathrm{co}}\, W(T)$, $T \in B(X)$, then $\|x + Ty\| \geq \|x\| - 2\sqrt{\|Tx\|\,\|y\|}$ for all $x, y \in X$, and $\|T\|^2 \leq 4\|T^2\|$ (Corollary 10, Corollary 11).

2. Proof of Theorem 1 and applications to products of A-ergodic nets

Proof of Theorem 1. Clearly, (a) implies that P is a bounded linear operator with $\|P\| \leq M$, and both $D(P)$ and $N(P)$ are closed. The first part of (c) and the closedness of A imply that $R(P) \subset N(A)$ for all $A \in \mathbf{A}$. Hence we have $R(P) \subset \bigcap_{A \in \mathbf{A}} N(A)$. Conversely, if $y \in \bigcap_{A \in \mathbf{A}} N(A)$, then by the first part of (b) we have $y \in D(P)$ and $Py = y$. Therefore $R(P) = \bigcap_{A \in \mathbf{A}} N(A)$. Moreover, since $Px \in \bigcap_{A \in \mathbf{A}} N(A)$, one has $PPx = Px$ for all $x \in D(P)$, i.e. P is a projection.

Next, we show that $N(P) = \overline{\sum_{A \in \mathbf{A}} R(A)}$. The second part of (c) implies that $R(A) \subset N(P)$ for all $A \in \mathbf{A}$, so that $\overline{\sum_{A \in \mathbf{A}} R(A)} \subset N(P)$. Conversely, if $x \in N(P)$, the second part of (b) implies that

$$x = s\text{-}\lim_{\alpha} (I - A_\alpha)x \in \overline{\sum_{A \in \mathbf{A}} R(A)}.$$

Therefore $N(P) = \overline{\sum_{A \in \mathbf{A}} R(A)}$.

Finally, if $x \in X$ is such that $\{A_\alpha x\}$ has a weakly convergent subnet $\{A_\beta x\}$, say $y = w\text{-}\lim_\beta A_\beta x$, it follows from the first part of (c) and the closedness of each $A \in \mathbf{A}$ that $y \in D(A)$ and $Ay = w\text{-}\lim_\beta AA_\beta x = 0$. That is, $y \in \bigcap_{A \in \mathbf{A}} N(A) = R(P)$. On the other hand, the second part of (b) implies that

$$y - x = w\text{-}\lim_{\beta} (A_\beta - I)x \in \overline{\sum_{A \in \mathbf{A}} R(A)} = N(P).$$

Hence $x = y - (y - x) \in R(P) \oplus N(P) = D(P)$, and the proof has been completed.

From Theorem 1 we now deduce some ergodic theorems for (countable) products of A-ergodic nets.

Corollary 2. *Let $\{A^{(k)}\}$ be a sequence of closed linear operators on X. For each k, let $\{A_\alpha^{(k)}\}$ be an $A^{(k)}$-ergodic net satisfying*

(a′) $M := \sup\{\|\prod_{k \in F} A_\alpha^{(k)}\|$*; F a finite set of natural numbers and for all* $\alpha\} < \infty$*;*

(b′) $\|(A_\alpha^{(k)} - I)x\| \to$ *for all* $x \in N(A^{(k)})$ *and* $R(A_\alpha^{(k)} - I) \subset \overline{R(A^{(k)})}$ *for all* α*;*

(c′) $R(A_\alpha^{(k)}) \subset D(A^{(k)})$ *and* $A_\alpha^{(k)} A^{(k)} \subset A^{(k)} A_\alpha^{(k)}$ *for all* α*, and* $\|A^{(k)} A_\alpha^{(k)}\| \to 0$.

(d′) $A_\alpha^{(j)} A_\alpha^{(k)} = A_\alpha^{(k)} A_\alpha^{(j)}$ *for all* $\alpha, j,$ *and* k.

For a nondecreasing net $\{r_\alpha\}$ of positive integers, let $P : D(P) \subset X \to X$ be the operator defined by

$$\begin{cases} D(P) := \{x \in X; s\text{-}\lim_\alpha A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(r_\alpha)} x \text{ exists}\}, \\ Px = s\text{-}\lim_\alpha A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(r_\alpha)} x, x \in D(P). \end{cases}$$

Then P is a linear projection with norm $\|P\| \leq M$, range $R(P) = \bigcap_{k=1}^{\infty} N(A^{(k)})$, null space $N(P) = \overline{\sum_{k=1}^{\infty} A^{(k)}}$, and domain

$$D(P) = \{x \in X; \{A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(r_\alpha)} x\} \text{ has a weak cluster point}\}.$$

Proof. Define $A_\alpha := A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(r_\alpha)}$ for all α. It suffices to check that the net $\{A_\alpha\}$ is an $\mathbf{A}$-ergodic net with $\mathbf{A} = \{A^{(k)}\}$, i.e. (a)–(c) of Theorem 1 are satisfied. (a′) implies (a). For every $x \in X$ and α, we have, by the commutativity of $A_\alpha^{(j)}$ and $A_\alpha^{(k)}$, that

$$\begin{aligned}(A_\alpha - I)x &= A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(r_\alpha)} x - x \\ &= \sum_{k=2}^{r_\alpha} A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(k-1)} (A_\alpha^{(k)} - I)x + (A_\alpha^{(1)} - I)x \\ &= \sum_{k=2}^{r_\alpha} (A_\alpha^{(k)} - I) A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(k-1)} x + (A_\alpha^{(1)} - I)x,\end{aligned}$$

from which and (a′) it is seen that parts 1 and 2 of (b) follow respectively from the corresponding parts of (b′).

Also we have for every k

$$A_\alpha = A_\alpha^{(1)} A_\alpha^{(2)} \cdots A A_\alpha^{(r_\alpha)} = A_\alpha^{(k)} \prod_{1 \le j \ne k \le r_\alpha} A_\alpha^{(j)} = \left\{ \prod_{1 \le j \ne k \le r_\alpha} A_\alpha^{(j)} \right\} A_\alpha^{(k)},$$

which and (c′) imply that $R(A_\alpha) \subset D(A^{(k)})$,

$$\|A^{(k)} A_\alpha\| \le \|A^{(k)} A_\alpha^{(k)}\| \left\| \prod_{1 \le j \ne k \le r_\alpha} A_\alpha^{(j)} \right\| \to 0,$$

$$\|A_\alpha A^{(k)} x\| \le \left\| \prod_{1 \le j \ne k \le r_\alpha} A_\alpha^{(j)} \right\| \|A_\alpha^{(k)} A^{(k)}\| \, \|x\| \to 0, \qquad x \in D(A^{(k)}).$$

Therefore $\{A_\alpha\}$ is an $\mathbf{A}$-ergodic net, and the conclusion now follows from Theorem 1.

In the special cases $r_\alpha \equiv m$ and $r_n = n$, Corollary 2 becomes the following two corollaries.

Corollary 3. *Let $\{A^{(k)}\}$, $k = 1, 2, \ldots, m$, be closed linear operators on X. For each k, let $\{A_\alpha^{(k)}\}$ be an $A^{(k)}$-ergodic net satisfying conditions* (b′), (c′), *and* (d′). *Let $P : D(P) \subset X \to X$ be the operator defined by*

$$\begin{cases} D(P) := \{x \in X; s\text{-}\lim_\alpha A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(m)} x \text{ exists}\}, \\ Px = s\text{-}\lim_\alpha A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(m)} x, \ x \in D(P). \end{cases}$$

Suppose that $\|A_\alpha^{(k)}\| \le M_k$ for all $1 \le k \le m$. Then P is a linear projection with norm $\|P\| \le M_1 \cdots M_m$, range $R(P) = \bigcap_{k=1}^m N(A^{(k)})$, null space $N(P) = \overline{\sum_{k=1}^m A^{(k)}}$, and domain

$$D(P) = \{x \in X; \{A_\alpha^{(1)} A_\alpha^{(2)} \cdots A_\alpha^{(m)} x\} \text{ has a weak cluster point}\}.$$

Corollary 4. *Let $\{A^{(k)}\}$, $k = 1, 2, \ldots,$ be a sequence of closed linear operators on X. For each k, let $\{A_n^{(k)}\}$ be an $A^{(k)}$-ergodic sequence satisfying* (b′), (c′), (d′) *(with α replaced by n). Let $P : D(P) \subset X \to X$ be the operator defined by*

$$\begin{cases} D(P) := \{x \in X; s\text{-}\lim_n A_n^{(1)} A_n^{(2)} \cdots A_n^{(n)} x \text{ exists}\}, \\ Px = s\text{-}\lim_n A_n^{(1)} A_n^{(2)} \cdots A_n^{(n)} x, \ x \in D(P). \end{cases}$$

Suppose that $M := \sup\{\|\prod_{k\in F} A_n^{(k)}\|;\ F$ *a finite set of natural numbers and* $n \geq 1\} < \infty$. *Then* P *is a linear projection with norm* $\|P\| \leq M$, *range* $R(P) = \bigcap_{k=1}^{\infty} N(A^{(k)})$, *null space* $N(P) = \overline{\sum_{k=1}^{\infty} A^{(k)}}$, *and domain*

$$D(P) = \{x \in X; \{A_n^{(1)} A_n^{(2)} \cdots A_n^{(n)} x\} \text{ has a weak cluster point}\}.$$

Next, we give two illustrative examples.

Example 1. Let $A^{(1)}, A^{(2)}, \ldots$ be a sequence of closed operators on a Banach space X satisfying $(0,\infty) \subset \rho(A^{(k)})$, $(\lambda - A^{(j)})^{-1}(\lambda - A^{(k)})^{-1} = (\lambda - A^{(k)})^{-1}(\lambda - A^{(j)})^{-1}$, and $\|\lambda(\lambda - A^{(k)})^{-1}\| \leq M_k$ for all $\lambda > 0$ and $j, k = 1, 2, \ldots$. Suppose further that $M = \sup\{\prod_{k\in F} M_k;\ F$ a finite set of natural numbers$\} < \infty$ (particularly, $M_k = 1$ for all k). If we put $A_n^{(k)} := \frac{1}{n}(\frac{1}{n} - A^{(k)})^{-1}$ for $n \geq 1$, then we have $A_n^{(k)} A^{(k)} \subset A^{(k)} A_n^{(k)} = \frac{1}{n}(A_n^{(k)} - I)$, so that conditions in Corollary 4 are satisfied. Hence we can formulate the next theorem.

Theorem 5. *Under the above assumption on closed operators* $A^{(1)}, A^{(2)}, \ldots$, *the limits* $\lim_{n\to\infty} n^{-n}(\frac{1}{n} - A^{(1)})^{-1} \cdots (\frac{1}{n} - A^{(n)})^{-1} x$ *define a linear projection* P *with norm* $\|P\| \leq M$, *range* $R(P) = \bigcap_{k=1}^{\infty} N(A^{(k)})$, *null space* $N(P) = \overline{\sum_{k=1}^{\infty} R(A^{(k)})}$, *and domain*

$$\begin{aligned} D(P) &= \left[\bigcap_{k=1}^{\infty} N(A^{(k)})\right] \oplus \overline{\sum_{k=1}^{\infty} R(A^{(k)})} \\ &= \left\{ x \in X; \exists n_k \to \infty \ni w\text{-}\lim_{k\to\infty} n_k^{-n_k} \left(\frac{1}{n_k} - A^{(1)}\right)^{-1} \right. \\ &\qquad\qquad \left. \cdots \left(\frac{1}{n_k} - A^{(n_k)}\right)^{-1} x \text{ exists} \right\}. \end{aligned}$$

Example 2. For α, β in Euclidean N-space R^N, $\alpha > \beta$ means that $\alpha_1 > \beta_1, \ldots, \alpha_N > \beta_N$; and $\alpha \to \infty$ means that $\alpha_1 \to \infty, \ldots, \alpha_N \to \infty$. Let $\{T(t); t \in R^N, t > 0\} \subset B(X)$ be a strongly continuous semigroup (see [11]) such that $\|T(u)\| \leq M$ for all $0 < u \in R^N$. The averages $A(\alpha)$; $\alpha = (a, \ldots, a)$, are defined in terms of $T(u)$ and the N-dimensional interval $\sigma(\alpha) \equiv (0, a]^N$, by the equation

$$A(\alpha)x = \frac{1}{a^N} \int_{\sigma(\alpha)} T(u)x\, du, \qquad x \in X.$$

Let $\mathbf{A} = \{T(t) - I; t \in R^N, t > 0\}$.

Since $(A(\alpha) - I)x = \frac{1}{a^N} \int_{\sigma(\alpha)} (T(u) - I)x\, du$, $\{A(\alpha)\}$ satisfies condition (b). Since

$$(T(t) - I)A(\alpha)x = \frac{1}{a^N}\left[\int_{\sigma(\alpha)' \cap (t+\sigma(\alpha))} T(u)x\, du - \int_{\sigma(\alpha)\cap(t+\sigma(\alpha))'} T(u)x\, du\right],$$

and since the measures of the sets $\sigma(\alpha)' \cap (t + \sigma(\alpha))$ and $\sigma(\alpha) \cap (t + \sigma(\alpha))'$ are both $O(a^{N-1})$ as $\alpha \to \infty$ (see [7]), we have that $\|(T(t) - I)A(\alpha)x\| \to 0$ as $\alpha \to \infty$. Hence $\{A(\alpha)\}$ is an $\mathbf{A}$-ergodic net.

In case the semigroup $T(\cdot)$ is strongly continuous on $\{\alpha \in R^N; \alpha_k \geq 0, 0 \leq k \leq N\}$ (i.e. an N-parameter C_0-semigroup), the families $T_k(\cdot) := \{T(te_k); 0 \leq t < \infty\}$, $k = 1, \ldots, N$, are commuting one-parameter C_0-semigroups. Let A_k be the infinitesimal generator of $T_k(\cdot)$. It can be verified that $\{A(\alpha)\}$ is an $\mathbf{A}$-ergodic net with $\mathbf{A} = \{A_1, A_2, \ldots, A_N\}$.

Now we can deduce the following theorem from Theorem 1.

Theorem 6. *Let $\{T(t); t \in R^N, t > 0\}$ be a uniformly bounded, strongly continuous semigroup. The operator $P : x \to \lim_{\alpha\to\infty} A(\alpha)x$ is a bounded linear projection with range $R(P) = \bigcap_{0<u\in R^N} N(T(u) - I)$, null space $N(P) = \overline{\bigcup_{0<u\in R^N} R(T(u) - I)}$, and domain*

$$D(P) = \{x \in X; \{A(\alpha)x\} \text{ has a weak cluster point}\}.$$

In case that $T(\cdot)$ is a uniformly bounded N-parameter C_0-semigroup, one also has $R(P) = \bigcap_{k=1}^{N} N(A_k)$ and $N(P) = \overline{\sum_{k=1}^{N} R(A_k)}$.

3. Orthogonality properties of some operators

Let Y and Z be two subspaces of the Banach space X. We say that Y is *orthogonal* to Z, and denote this by $Y \perp Z$, if $\|y + z\| \geq \|y\|$ for all $y \in Y$ and $z \in Z$. This definition of *orthogonality* is consistent with the usual concept of orthogonality in Hilbert spaces, and is equivalent to $Y \cap Z = \{0\}$ and the projection P onto Y along Z has norm $\|P\| = 1$.

Theorem 7. *Let $\{A_n\}$ be a sequence of m-dissipative operators such that their resolvents are commutative. Then*

$$\left[\bigcap_{k=1}^{\infty} N(A_k)\right] \perp \overline{\sum_{k=1}^{\infty} R(A_k)}. \tag{3.1}$$

If the space X is reflexive, then

$$X = \left[\bigcap_{k=1}^{\infty} N(A_k)\right] \overset{\perp}{\oplus} \overline{\sum_{k=1}^{\infty} R(A_k)}. \tag{3.2}$$

Proof. Since an operator A is dissipative if and only if for each $\lambda > 0$, $\lambda - A$ is injective and $\|\lambda(\lambda - A)^{-1}\| \leq 1$ (see [10, p. 26]), the theorem follows immediately from Theorem 5. Noting that A and $\lambda(\lambda - A)^{-1} - I$ $(\lambda > 0)$ have the same null space and range, one can also deduce the theorem directly from Corollary 4 by setting $A^{(k)} = \lambda(\lambda - A_k)^{-1} - I$ and $A_n^{(k)} = \frac{1}{n}\sum_{j=0}^{n-1}(\lambda(\lambda - A_k)^{-1})^j$.

The conclusion of Theorem 7 holds in particular for any sequence of commutative hermitian operators, because i times a hermitian operator is m-dissipative. The next corollary shows that normal operators share the same property. For the proof of it we need the generalized Fuglede theorem (see [5], [6], [9]), which states that if $T = H + iK$ is a normal operator, where H and K are commuting hermitian operators, then $N(T) = N(H) \cap N(K)$. Applying this to the normal derivation $\Delta_{T,T} = \Delta_{H,H} + i\Delta_{K,K}$ (see [1], [13], [17]), one obtains the Fuglede theorem for normal operators on Banach spaces, that is, $TB = BT$ if and only if $HB = BH$ and $KB = BK$.

Corollary 8. *Let $\{T_n\}$ be a sequence of commuting normal operators on X.Then*

$$\left[\bigcap_{k=1}^{\infty} N(T_k)\right] \perp \overline{\sum_{k=1}^{\infty} R(T_k)}.$$

If the space X is reflexive, then

$$X = \left[\bigcap_{k=1}^{\infty} N(T_k)\right] \overset{\perp}{\oplus} \overline{\sum_{k=1}^{\infty} R(T_k)}. \tag{3.3}$$

Proof. Since normal operators $T_k = H_k + iK_k$, $k = 1, 2, \ldots$, are commutative, the above observation shows that the hermitian operators $H_1, H_2, \ldots, K_1, K_2, \ldots$, are commutative. It follows from Theorem 7 that

$$\left\{\bigcap_{k=1}^{\infty} [N(H_k) \cap N(K_k)]\right\} \perp \overline{\sum_{k=1}^{\infty} [R(H_K) + R(K_k)]},$$

and in case X is reflexive,

$$X = \left\{\bigcap_{k=1}^{\infty} [N(H_k) \cap N(K_k)]\right\} \overset{\perp}{\oplus} \overline{\sum_{k=1}^{\infty} [R(H_k) + R(K_k)]}.$$

Since $N(T_k) = N(H_k) \cap N(K_k)$ and $R(T_k) \subset R(H_k) + R(K_k)$, we have proved the first part of the corollary. In particular, $X_0 := [\bigcap_{k=1}^{\infty} N(T_k)] \overset{\perp}{\oplus} \overline{\sum_{k=1}^{\infty} R(T_k)} = {}^{\perp}[\sum_{k=1}^{\infty} R(T_k^*)] \overset{\perp}{\oplus} \overline{\sum_{k=1}^{\infty} R(T_k)}$ is a closed linear subspace of X.

Since the dual operators T_k^* are also normal operators, the first part of this theorem implies that

$$\left[\bigcap_{k=1}^{\infty} N(T_k^*)\right] \perp \overline{\sum_{k=1}^{\infty} R(T_k^*)}.$$

If X is reflexive, then $X_0^{\perp} = \overline{\sum_{k=1}^{\infty} R(T_k^*)} \cap [\bigcap_{k=1}^{\infty} N(T_k^*)] = \{0\}$. Hence $X_0 = X$, i.e. (3.3) holds.

Remarks. (1) In the case of bounded operators, Theorem 7 and Corollary 8 both follow also from [14, Theroem 3.3] and [15, Theorem 2].

(2) If X is smooth, i.e. the norm of X is Gâteaux differentiable, then, because $x \perp y$ and $x \perp z$ imply $x \perp y+z$, the first part of Theorem 7 and that of Corollary 8 follow immediately from the case of a single operator, even when the resolvents of the concerned family of operators are not commutative.

(3) Mattila [15] showed that if $T = H+iK$ is a hyponormal operator on a strictly c-convex Banach space, then $N(T) = N(H) \cap N(K)$. Thus when X is a smooth and strictly c-convex Banach space (e.g. L_p, $1 < p < \infty$, or Hilbert spaces), the first part of Corollary 8 also holds for noncommutative hypernormal operators.

Theorem 7 implies that if A is an m-dissipative operator, then $\|x + Ay\| \geq \|x\|$ for all $x \in N(A)$ and $y \in D(A)$. The next theorem gives a generalized inequality.

Theorem 9. *If A is an m-dissipative operator in a Banach space X, then*

$$\|x + Ay\| \geq \|x\| - 2\sqrt{\|Ax\|\,\|y\|} \quad \textit{for all } x, y \in D(A). \tag{3.4}$$

Proof. For each $n \geq 1$ we define an operator function $T_n(\cdot)$ by

$$T_n(t)x := t^{-1} \int_0^t (I - \frac{s}{n}A)^{-n} x\, ds, \qquad x \in X, t > 0.$$

The m-dissipativity of A implies $\|T_n(t)\| \leq 1$ for all n and t. Then we have for $x, y \in D(A)$, $n \geq 2$ and $t > 0$

$$\begin{aligned}
\|x + Ay\| &\geq \|T_n(t)(x + Ay)\| \\
&\geq \|x\| - \|(T_n(t) - I)x + T_n(t)Ay\| \\
&\geq \|x\| - \left\| \frac{1}{t} \int_0^t \left[\left(I - \frac{s}{n} A \right)^{-n} - I \right] x \, ds \right\| - \left\| \frac{1}{t} \int_0^t \left(I - \frac{s}{n} A \right)^{-n} Ay \, ds \right\| \\
&= \|x\| - \left\| \frac{1}{t} \int_0^t \frac{s}{n} \sum_{k=0}^{n-1} \left(I - \frac{s}{n} A \right)^{-k-1} Ax \, ds \right\| \\
&\quad - \left\| \frac{1}{t} \int_0^t \frac{n}{n-1} \frac{d}{ds} \left[\left(I - \frac{s}{n} A \right)^{1-n} y \right] ds \right\| \\
&\geq \|x\| - \frac{1}{t} \int_0^t s \, ds \|Ax\| - \left\| \frac{1}{t} \frac{n}{n-1} \left[\left(I - \frac{t}{n} A \right)^{1-n} y - y \right] \right\| \\
&\geq \|x\| - \left(\frac{t}{2} \|Ax\| + \frac{n}{n-1} \frac{2}{t} \|y\| \right).
\end{aligned}$$

Hence $\|x + Ay\| \geq \|x\| - (\frac{t}{2}\|Ax\| + \frac{2}{t}\|y\|)$ for all $t > 0$. Minimizing the function $\frac{t}{2}\|Ax\| + \frac{2}{t}\|y\|$, we obtain its minimum $2\sqrt{\|Ax\| \, \|y\|}$, and hence the estimate (3.4).

Theorem 9 contains as a corollary the following slight improvement of a theorem of Crabb and Sinclair (see [4] or [3, Theorem 20.6]).

Corollary 10. *If an operator $T \in B(X)$ is such that 0 is not an interior point of $\overline{\text{co}}\, W(T)$, then $\|x + Ty\| \geq \|x\| - 2\sqrt{\|Tx\| \, \|y\|}$ for all $x, y \in X$.*

Using Corollary 10 one can deduce some consequences. For instance, by modifying the proof of Corollary 20.12 in [3] we obtain the following improvement.

Corollary 11. *Under the hypothesis of Corollary 10, the inequality $\|T\|^2 \leq 4\|T^2\|$ holds.*

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