

EXTENSIONS OF MULTIPLIERS AND DILATIONS OF PROJECTIVE ISOMETRIC REPRESENTATIONS

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ABSTRACT. An elementary proof of a multiplier extension theorem of M. Laca and I. Raeburn is presented. A very general dilation theorem is also derived.

1. INTRODUCTION

The problem of extending a multiplier on a subsemigroup to a multiplier on a group has arisen in a number of contexts in recent years. It arises, for instance, in the context of W. B. Arveson's theory of semigroups of endomorphisms. In [1] Arveson showed that every multiplier on the semigroup \mathbf{R}^+ extends to \mathbf{R} . This result was extended to the semigroup $[t, \infty)$ ($t > 0$) by P. R. Chernoff in [2] and H. Dinh has shown in [3] that a multiplier on an arbitrary discrete subsemigroup of \mathbf{R} admits an extension to the group it generates. More recently, M. Laca and I. Raeburn [5] have obtained far-reaching generalisations of these results. The method they use involves dilation theory. Their multiplier extension theorem is deduced as a consequence. The dilation theory they derived has been used by the author in [6], where aspects of the theory of crossed products of C^* -algebras by semigroups of endomorphisms are developed. It seems likely that the Laca-Raeburn dilation theorem will admit other applications.

In this paper we present extension and dilation theorems related to those of [5]. In particular, we give a new proof of the multiplier extension theorem. Although our method is similar to that of [5], our proof is completely elementary and does not require the dilation theorem. In addition, we present a number of other extension theorems.

We also prove a dilation theorem more general than that proved by Laca and Raeburn. Our approach is radically different from that of these authors. Their proofs, although very elegant, are developed along lines rather different from those of classical dilation theory as presented, for example, by D. E. Evans and J. T. Lewis in [4]. The theory in [4] involves positive definite kernels and their Kolmogorov decompositions. The many results that fall within its scope include the unitary dilation results on groups due to M. Naimark and B. Sz.-Nagy and the dilation theorem of W. F. Stinespring on completely positive maps. In this paper we apply this theory to obtain the Laca-Raeburn dilation theorem. As well as having the advantage of placing the theorem in the classical setting and thereby presenting a

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“natural” proof, our approach also has the advantage that we are able to prove a more general result (Theorem 3.1).

The paper is organised as follows: In §2 we obtain a number of extension results, including the one on multiplier extensions discussed above. Section 3 is concerned with dilation theory and the principal result is Theorem 3.1. The Laca-Raeburn dilation theorem (Theorem 3.3) is shown to be a consequence of it.

2. EXTENSION OF MULTIPLIERS

Throughout the paper the term *semigroup* will signify a semigroup with unit. A *subsemigroup* of a semigroup signifies a subset closed under the operation and containing the unit. We shall usually write the operation multiplicatively and denote the unity by e .

An *involution* on a semigroup S is a function, $s \mapsto s^*$, from S to itself having the properties that for all $s, t \in S$, we have $(st)^* = t^*s^*$ and $(s^*)^* = s$. Automatically (and trivially), $e^* = e$.

We call a pair consisting of a semigroup together with an involution a **-semigroup*.

2.1. Example. An involution on a group is defined by setting $x^* = x^{-1}$.

2.2. Example. Let A be a unital C^* -algebra. Define an equivalence relation \sim on A by setting $a \sim b$ if there exists a scalar λ of unit modulus for which $b = \lambda a$. Let S be the set of equivalence classes and π the canonical surjection from A onto S . We make S into a semigroup by setting $\pi(a)\pi(b) = \pi(ab)$ and we define an involution on S by setting $(\pi(a))^* = \pi(a^*)$. We shall call S the *projective semigroup* associated to A .

If $U(A)$ denotes the group of unitaries of A , then $\pi U(A)$ is, of course, a group, the *projective unitary* group of A .

If S is a semigroup and S' a *-semigroup, an *isometric representation* of S in S' is a unital homomorphism φ from S to S' such that $\varphi(s)^*\varphi(s) = e$ for all $s \in S$.

A subsemigroup S of a group G is *normal* if $xSx^{-1} \subseteq S$ for all $x \in G$.

2.3. Theorem. *Let S be a normal, generating subsemigroup of a group G and let φ be an isometric representation of S in a *-semigroup S' . Then there exists a unique extension φ' of φ to G having the following properties:*

- (1) $\varphi'(xs) = \varphi'(x)\varphi(s)$ for all $x \in G$ and $s \in S$;
- (2) $\varphi'(x^{-1}) = \varphi'(x)^*$ for all $x \in G$.

Proof. Since S is normal and generates G , every element x of G can be written in the form $x = s^{-1}t$, for some elements s and t of S . From this, uniqueness of φ' is trivial. To get existence, set $\varphi'(x) = \varphi(s)^*\varphi(t)$, if $x = s^{-1}t$, where $s, t \in S$. Of course, we must show that this is well defined. To see this, suppose that we can also write $x = u^{-1}v$, where $u, v \in S$. Then $ut = (usu^{-1})v$ and since usu^{-1} belongs to S and φ is a homomorphism, we have

$$(1) \quad \varphi(u)\varphi(t) = \varphi(usu^{-1})\varphi(v).$$

However, $\varphi(usu^{-1})\varphi(u) = \varphi(u)\varphi(s)$, so $\varphi(u)^*\varphi(usu^{-1})\varphi(u) = \varphi(s)$, and therefore, $\varphi(u)^*\varphi(usu^{-1})^*\varphi(u) = \varphi(s)^*$. Hence,

$$\begin{aligned}\varphi(s)^*\varphi(t) &= \varphi(u)^*\varphi(usu^{-1})^*\varphi(u)\varphi(t) \\ &= \varphi(u)^*\varphi(usu^{-1})^*\varphi(usu^{-1})\varphi(v) \quad (\text{by (1)}) \\ &= \varphi(u)^*\varphi(v).\end{aligned}$$

Thus, φ' is well defined, as claimed. It follows immediately from the definition of φ' that conditions (1) and (2) are satisfied. \square

By taking adjoints in condition (1) of the preceding theorem, one sees that $\varphi'(s^{-1}x) = \varphi(s)^*\varphi'(x) = \varphi'(s^{-1})\varphi'(x)$ for all $s \in S$ and $x \in G$.

The following simple result is in [5]. We shall use it below.

2.4. Corollary. *Let S be a normal, generating subsemigroup of a group G and let φ be a homomorphism from S to a group G' . Then φ has a unique extension to a homomorphism from G to G' .*

Proof. We apply the theorem with $S' = G'$, where G' has the involution given by inversion. We claim that the extension φ' of φ that results is a homomorphism and a moment's reflection shows that to prove this we need only show that if $s, t \in S$, then $\varphi'(st^{-1}) = \varphi(s)\varphi(t)^{-1}$. However, this follows immediately from the observation that $\varphi'(st^{-1})\varphi(t) = \varphi'(st^{-1}t) = \varphi(s)$. \square

A *multiplier* on a semigroup S is a function σ , from the Cartesian product $S \times S$ to the unit circle \mathbf{T} in the plane, such that for all elements s, t and u of S :

- (1) $\sigma(e, s) = \sigma(s, e) = 1$;
- (2) $\sigma(s, t)\sigma(st, u) = \sigma(s, tu)\sigma(t, u)$.

Let S be a semigroup and H a Hilbert space. A *projective isometric representation* of S on H is a map, $W : x \mapsto W_x$, from S to $B(H)$ having the following properties for all elements $s, t \in S$:

- (1) W_s is an isometry and $W_e = 1$;
- (2) $W_{st} = \sigma(s, t)W_sW_t$, where $\sigma(s, t)$ is some scalar of unit modulus.

It follows from the equations

$$W_{(st)u} = \sigma(st, u)W_{st}W_u = \sigma(st, u)\sigma(s, t)W_sW_tW_u$$

and

$$W_{s(tu)} = \sigma(s, tu)W_sW_{tu} = \sigma(s, tu)\sigma(t, u)W_sW_tW_u$$

that $\sigma(s, t)\sigma(st, u) = \sigma(s, tu)\sigma(t, u)$. Also, since

$$W_s = W_{se} = \sigma(s, e)W_sW_e = W_{es} = \sigma(e, s)W_eW_s,$$

we have $\sigma(e, s) = \sigma(s, e) = 1$. Hence, the function, $\sigma : (s, t) \mapsto \sigma(s, t)$, is a multiplier on S , said to be the multiplier *associated* to W .

If all the W_x are unitaries, we say that W is a *projective unitary representation*.

It is clear that if W is a projective isometric representation on a group, it is automatically a projective unitary representation. Also, $W_x^* = \sigma(x^{-1}, x)W_{x^{-1}}$ for all x .

A projective isometric representation or projective unitary representation with σ an associated multiplier will sometimes be referred to as an *isometric σ -representation* or, respectively, a *unitary σ -representation*.

2.5. Example. Given a multiplier σ on a left-cancellative semigroup S , a projective isometric representation having σ associated to it exists. To see this, set $H = l^2(S)$ and denote by $(\delta_s)_{s \in S}$ the canonical orthonormal basis. For each element $s \in S$, let W_s be the isometry on H that maps δ_t onto $\overline{\sigma(s, t)}\delta_{st}$. A simple computation shows that $W_{st} = \sigma(s, t)W_sW_t$.

Now suppose that S is a normal subsemigroup of a group G . Obviously, δ_{st} belongs to the range of W_s and, since $\delta_{st} = \delta_{t(t^{-1}st)}$, this vector δ_{st} also belongs to the range of W_t . Therefore, the ranges of W_s and W_t are not orthogonal.

2.6. Theorem (Laca-Raeburn [5]). *Let S be a normal, generating subsemigroup of a group G and let σ be a multiplier on S . Then σ extends to a multiplier on G .*

Proof. Let W be the projective isometric representation on $H = l^2(S)$, having σ as associated multiplier, of the preceding example. Let S' be the projective semigroup associated to the C^* -algebra $B(H)$. The map φ obtained by composing W with the canonical map from $B(H)$ to S' is clearly an isometric representation of S in S' . It has an extension to G given by Theorem 2.3 and we denote this extension by φ also. If $x \in G$, choose W_x in the equivalence class $\varphi(x)$; if $s \in S$, we may, and we do, keep W_s as before. Since $\varphi(x^{-1}) = \varphi(x)^*$, there is a scalar $\mu(x)$ of unit modulus such that $W_{x^{-1}} = \mu(x)W_x^*$. If $x = s^{-1}t$, where $s, t \in S$, then $\varphi(x) = \varphi(s)^*\varphi(t)$, so $W_x = \lambda W_s^*W_t$, for some scalar λ of unit modulus. Since the ranges of W_s and W_t are not orthogonal, $W_s^*W_t$ is non-zero. Hence, $W_x \neq 0$. If $t \in S$ and $x \in G$, then $\varphi(t^{-1})\varphi(x) = \varphi(t^{-1}x)$, so there exists a scalar $\sigma'(t^{-1}, x)$ of unit modulus such that

$$(2) \quad W_{t^{-1}x} = \sigma'(t^{-1}, x)W_{t^{-1}}W_x.$$

We relate σ' to σ as follows: If $s, t \in S$, then $W_{st} = \sigma(s, t)W_sW_t$ and taking adjoints gives $\overline{\mu(st)W_{t^{-1}s^{-1}}} = \overline{\sigma(s, t)\mu(t)\mu(s)W_{t^{-1}}W_{s^{-1}}}$. Comparing with equation (2) (with $x = s^{-1}$) and using the fact that $W_{t^{-1}s^{-1}} \neq 0$, we see that

$$(3) \quad \sigma'(t^{-1}, s^{-1}) = \mu(st)\overline{\mu(s)\mu(t)\sigma(s, t)}.$$

Now we define a projective unitary representation U of the semigroup S^{-1} on $K = l^2(G)$: If $s \in S$, the unitary $U_{s^{-1}}$ is defined on the canonical orthonormal basis $(e_x)_x$ of K by setting $U_{s^{-1}}(e_x) = \overline{\sigma'(s^{-1}, x)}e_{s^{-1}x}$. We claim that

$$(4) \quad U_{t^{-1}s^{-1}} = \sigma'(t^{-1}, s^{-1})U_{t^{-1}}U_{s^{-1}}$$

and to see this we need only show that for all $x \in G$ we have

$$\sigma'(t^{-1}, s^{-1}x)\sigma'(s^{-1}, x) = \sigma'(t^{-1}, s^{-1})\sigma'(t^{-1}s^{-1}, x).$$

However, this follows by comparing the two ways of writing $W_{t^{-1}(s^{-1}x)} = W_{(t^{-1}s^{-1})x}$ in terms of $W_{t^{-1}}W_{s^{-1}}W_x$ (using equation (2)). The subsemigroup S^{-1} of G is normal and generates G , and the map ψ from S^{-1} to the projective unitary group G' of $B(K)$ obtained by composing U with the canonical map from $U(B(K))$ to G' is a homomorphism. Hence, by Corollary 2.4, ψ extends to a homomorphism from G to G' , that we shall also denote by ψ . For each element $x \in G$, choose a unitary U_x on K belonging to the equivalence class $\psi(x)$. In particular, if $s \in S$, choose $U_{s^{-1}}$ as before and choose $U_s = \mu(s)U_{s^{-1}}^*$. Of course, if $s \in S \cap S^{-1}$, we need to check that these choices are compatible. That they are follows from the easily verified fact that if $s \in S \cap S^{-1}$, then $\mu(s) = \overline{\sigma'(s^{-1}, s)}$.

If $x, y \in G$, then $\psi(xy) = \psi(x)\psi(y)$ and therefore there exists a scalar $\sigma''(x, y)$ of unit modulus such that $U_{xy} = \sigma''(x, y)U_xU_y$. Hence, the function, $\sigma'' : (x, y) \mapsto$

$\sigma''(x, y)$, is a multiplier on G . If $s, t \in S$, then $U_{st} = \sigma''(s, t)U_s U_t$. We get $\overline{\mu(st)U_{t^{-1}s^{-1}}} = \overline{\sigma''(s, t)\mu(t)\mu(s)U_{t^{-1}}U_{s^{-1}}}$ if we take adjoints; hence, by equation (4), $\sigma'(t^{-1}, s^{-1}) = \mu(st)\mu(s)\mu(t)\sigma''(s, t)$ and consequently, by equation (3), $\sigma''(s, t) = \sigma(s, t)$. Thus, the multiplier σ'' on G extends σ . \square

3. DILATION THEORY

Let X be a non-empty set and let H be a Hilbert space. Recall that a map K from $X \times X$ to $B(H)$ is a *positive definite kernel* if for every positive integer n and elements x_1, \dots, x_n of X , the operator matrix $(K(x_i, x_j))$ in the C^* -algebra $M_n(B(H))$ is positive.

If K can be written in the form $K(x, y) = V(x)^*V(y)$, where V is a map from X to $B(H, H_V)$, for some Hilbert space H_V , then K is automatically positive definite. Such a map V is said to be a *Kolmogorov decomposition* of K . Moreover, if, in addition, H_V is the closed linear span of the set $\bigcup_x V(x)H$, then V is said to be *minimal*.

The fundamental result in this connection asserts that if K is a positive definite kernel, then it admits a minimal Kolmogorov decomposition; moreover, if V and V' are two such decompositions for K , then there exists a unique unitary operator U from H_V onto $H_{V'}$ such that $V'(x) = UV(x)$, for all $x \in X$.

For a nice presentation of this theory, see [4].

For the rest of this section, G will denote a group, σ a multiplier of G and S a normal, generating subsemigroup of G .

Let H be a Hilbert space. We say that a map, $W : x \mapsto W_x$, from G to $B(H)$ is *σ -positive definite* if the map K on $G \times G$ defined by setting $K(x, y) = \sigma(x^{-1}, x)\overline{\sigma(x^{-1}, y)}W_{x^{-1}y}$ is positive definite. We define a (*minimal*) *Kolmogorov decomposition* for W to be a (minimal) Kolmogorov decomposition for K .

The following result is well known in the case that $\sigma \equiv 1$, see [4, Corollary 2.6].

3.1. Theorem. *Let H be a Hilbert space and W a σ -positive definite map on G with values in $B(H)$. Then there exists a Hilbert space H' , an operator $T \in B(H, H')$ and a unitary σ -representation U of G on H' such that $W_x = T^*U_x T$, for all $x \in G$. Moreover, H' is the closed linear span of the set $\bigcup_x U_x T H$.*

Proof. Let V be a minimal Kolmogorov decomposition of W and set $H' = H_V$. Let $x, y, z \in G$. Then it is routine to verify that

$$\sigma(x^{-1}, z^{-1}, zx)\sigma(z, x)\sigma(x^{-1}, y) = \sigma(x^{-1}, z^{-1}, zy)\sigma(z, y)\sigma(x^{-1}, x)$$

and it follows from this that

$$\begin{aligned} V(zx)^*V(zy) &= \sigma(x^{-1}z^{-1}, zx)\overline{\sigma(x^{-1}z^{-1}, zy)}W_{x^{-1}z^{-1}zy} \\ &= \sigma(x^{-1}, x)\overline{\sigma(x^{-1}, y)}\sigma(z, x)\sigma(z, y)W_{x^{-1}y} \\ &= \overline{\sigma(z, x)}\sigma(z, y)V(x)^*V(y). \end{aligned}$$

Hence, the map, $x \mapsto \overline{\sigma(z, x)}V(zx)$ is another minimal Kolmogorov decomposition for W . Consequently, there exists a unique unitary $U_z \in B(H')$ such that $U_z V(x) = \overline{\sigma(z, x)}V(zx)$, for all elements $x \in G$. Since we have

$$\begin{aligned} U_y U_z V(x) &= \overline{\sigma(y, zx)\sigma(z, x)}V(yzx) \\ &= \overline{\sigma(y, z)\sigma(yz, x)}V(yzx) = \overline{\sigma(y, z)}U_{yz}V(x), \end{aligned}$$

and the set $\bigcup_x V(x)H$ has dense linear span in H' (by minimality of V), therefore $U_{yz} = \sigma(y, z)U_y U_z$. Thus, the map, $U : x \mapsto U_x$, is a projective unitary representation of G with σ as associated multiplier. To complete the proof, set $T = V(e)$. Then $T^*U_x T = \overline{\sigma(x, e)}V(e)^*V(xe) = V(e)^*V(x) = W_x$. Also, $U_x T H = V(x)H$ and therefore H' is the closed linear span of the set $\bigcup_x U_x T H$. \square

The projective representation U is called a *dilation* of W . If W is unital, that is, $W_e = 1$, then T is an isometry, since $T^*T = T^*U_e T = W_e = 1$.

3.2. Theorem. *Let H be a Hilbert space and let $W : S \rightarrow B(H)$ be a projective isometric representation with associated multiplier the restriction of σ to S . Then there exists a unique extension W' of W to G having the following properties:*

- (1) $W'_{xs} = \sigma(x, s)W'_x W'_s$ for all $x \in G$ and $s \in S$;
- (2) $W'_x{}^* = \sigma(x^{-1}, x)W'_{x^{-1}}$ for all $x \in G$.

Moreover, W' is σ -positive definite.

Proof. Uniqueness of W' is obvious. To see existence, suppose that $x = s^{-1}t$, where $s, t \in S$, and set $W'_x = \sigma(s^{-1}, t)\overline{\sigma(s^{-1}, s)}W'_s{}^*W'_t$. Of course, it must be checked that this is well defined; the proof that this is so—which we omit—goes along the same lines as the proof of Theorem 2.3, with the extra requirement that the scalar factors involving σ have to be kept track of. Once it is seen that W' is well defined, conditions (1) and (2) follow by routine computations. It remains to show that W' is σ -positive definite. Thus, if x_1, \dots, x_n belong to G , we must show positivity of the operator matrix (V_{ij}) , where

$$V_{ij} = \sigma(x_i^{-1}, x_i)\overline{\sigma(x_i^{-1}, x_j)}W'_{x_i^{-1}x_j}.$$

We claim that there exists an element $s \in S$ such that $sx_1, \dots, sx_n \in S$. To see this, write $x_i = v_i u_i^{-1}$, where $u_i, v_i \in S$. Then, for $s = u_1 \cdots u_n$, we have $sx_i = u_1 \cdots u_i (u_{i+1} \cdots u_n v_i) u_i^{-1}$, so $sx_i \in S$, as required.

Consequently, for some elements s, t_1, \dots, t_n in S we have $x_i = s^{-1}t_i$; hence, since $\sigma(t_i^{-1}s, s^{-1}t_j) = \overline{\sigma(t_i^{-1}, s)}\sigma(t_i^{-1}, t_j)\sigma(s, s^{-1}t_j)$, we have

$$\begin{aligned} V_{ij} &= \sigma(t_i^{-1}s, s^{-1}t_i)\overline{\sigma(t_i^{-1}s, s^{-1}t_j)}W'_{t_i^{-1}t_j} \\ &= \sigma(t_i^{-1}s, s^{-1}t_i)\overline{\sigma(t_i^{-1}s, s^{-1}t_j)}\overline{\sigma(t_i^{-1}, t_j)}\overline{\sigma(t_i^{-1}, t_i)}W'_{t_i}{}^*W'_{t_j} \\ &= \sigma(s, s^{-1}t_i)\overline{\sigma(s, s^{-1}t_j)}W'_{t_i}{}^*W'_{t_j}. \end{aligned}$$

Thus, $V_{ij} = V_i^* V_j$, where $V_i = \overline{\sigma(s, s^{-1}t_i)}W'_{t_i}$. Hence, (V_{ij}) is positive, as required. \square

3.3. Theorem (Laca-Raeburn [5]). *Let H be a Hilbert space and $W : S \rightarrow B(H)$ a projective isometric representation with associated multiplier the restriction of σ to S . Then there exists a Hilbert space H' , an isometry $T : H \rightarrow H'$, and a unitary σ -representation $U : G \rightarrow B(H')$ such that $T^*U_s T = W_s$, for all $s \in S$. Moreover, H' is the closed linear span of the set $\bigcup_{x \in G} U_x T(H)$.*

Proof. This is obtained by applying Theorem 3.1 to the σ -positive definite map W' extending W that is given by Theorem 3.2. \square

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