COVERING BY COMPLEMENTS OF SUBSPACES, II

W. EDWIN CLARK AND BORIS SHEKHTMAN

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Abstract. Let $V$ be an $n$-dimensional vector space over an algebraically closed field $K$. Define $\gamma(k,n,K)$ to be the least positive integer $t$ for which there exists a family $E_1, E_2, \ldots, E_t$ of $k$-dimensional subspaces of $V$ such that every $(n-k)$-dimensional subspace $F$ of $V$ has at least one complement among the $E_i$'s. Using algebraic geometry we prove that $\gamma(k,n,K) = k(n-k) + 1$.

1. Introduction

Take $V = V(n,K)$ to be an $n$-dimensional vector space over the algebraically closed field $K$. As usual a subspace $F$ of $V$ is a complement of the subspace $E$ of $V$ if $V = E \oplus F$, i.e., if $E + F = V$ and $E \cap F = \{0\}$. We let $c(E)$ denote the set of all complements of $E$ in $V$ and we write $G(k,n)$ for the set of all $k$-subspaces (= $k$-dimensional subspaces) of $V$. If $E \in G(k,n)$ then $c(E) \subseteq G(n-k,n)$. Define $\gamma(k,n,F)$ to be the least positive integer $t$ such that there exist $k$-subspaces $E_1, E_2, \ldots, E_t$ of $V$ satisfying

\[ c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n-k,n). \]

If (1) holds we say that all $(n-k)$-subspaces of $V$ are covered by the $E_i$'s.

In [1] we studied this problem for an arbitrary field $K$. Among other things we showed that in general $\gamma(k,n,K)$ depends on the field $K$. In particular, we showed that $\gamma(2,4,K)$ is 5 if $K$ is quadratically closed and is 4 otherwise. We conjectured that $\gamma(k,n,K) = k(n-k) + 1$ if $K$ is algebraically closed. Here we prove this conjecture using results from algebraic geometry.

2. The lower bound $k(n-k) + 1 \leq \gamma(k,n,K)$

Let $\Lambda^k(V)$ denote the $k$-vectors in the exterior algebra $\Lambda(V)$ of $V$. We let $D(k,n)$ denote the set of all non-zero decomposable $k$-vectors $\alpha = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ where $v_1, v_2, \ldots, v_k$ are linearly independent vectors in $V$. Let $\langle \alpha \rangle$ denote the 1-dimensional subspace of $\Lambda^k(V)$ generated by $\alpha$ and write

\[ D(k,n) = \{ \langle \alpha \rangle | \alpha \in D(k,n) \}. \]

If $v_1, v_2, \ldots, v_k$ is a basis for $E \in G(k,n)$, then the mapping $E \mapsto \langle v_1 \wedge \cdots \wedge v_k \rangle$ is a bijection from $G(k,n)$ to $D(k,n)$. It is well-known that this gives $G(k,n)$ the

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structure of an irreducible projective variety (the Grassmannian) of dimension 
$k(n-k)$ in $\mathbb{P}^N = \mathbb{P}(\Lambda^k(V))$ where $N = \binom{n}{k} - 1$. We identify $G(k, n)$ with $D(k, n)$.

Now for any positive integer $t$ let $G(k, n)^t$ be the product variety of $G(k, n)$ with itself $t$ times. Let $E = (E_1, \ldots, E_t) \in G(k, n)^t$. For each $i$ let $E_i = \langle \epsilon_i \rangle$ for some decomposable $\epsilon_i \in \Lambda^k(V)$. Define the mappings:

$$\varphi_i : \Lambda^{n-k}(V) \to \Lambda^n(V) \quad \text{by} \quad \varphi_i(\xi) = \epsilon_i \wedge \xi$$

for $i = 1, \ldots, t$ and let

$$\mathcal{K}(E) = \ker(\varphi_1) \cap \ker(\varphi_2) \cap \cdots \cap \ker(\varphi_t).$$

Note that $\mathcal{K}(E)$ is a subspace of $\Lambda^{n-k}(V)$.

**Lemma 1.** For $E \in G(k, n)^t$ the following two conditions are equivalent:

(a) $c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n - k, n)$,

(b) $D(n - k, n) \cap \mathcal{K}(E) = \emptyset$.

**Proof.** This is an immediate consequence of the fact that if $F = \langle \alpha \rangle \in G(n - k, n)$ for some $\alpha \in D(n - k, n)$, then $E_i \cap F = \{0\}$ if and only if $\epsilon_i \wedge \alpha \neq 0$. \hfill $\square$

**Lemma 2.** If $\gamma(k, n, K) = t$ and $E = (E_1, \ldots, E_t) \in G(k, n)^t$ satisfies

$$c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n - k, n),$$

then

$$\dim(\mathcal{K}(E)) = \binom{n}{k} - t.$$

**Proof.** Since $\varphi_i$ is a linear mapping from the $\binom{n}{k}$-dimensional vector space $\Lambda^{n-k}(V)$ to the 1-dimensional vector space $\Lambda^n(V)$, it suffices to show that the mappings

$$\varphi_i \in \text{hom}(\Lambda^{n-k}(V), \Lambda^n(V)), \quad i \in \{1, \ldots, t\},$$

are linearly independent. To see this we first note that the elements $\epsilon_i$ are linearly independent in $\Lambda^{n-k}(V)$. Suppose not; then we can assume that $\epsilon_i = \sum_{s=1}^t a_s \epsilon_s$. It follows that $\bigcap_{i=1}^t \ker(\varphi_i) = \bigcap_{i=1}^t \ker(\varphi_i)$ which implies by Lemma 1 that

$$c(E_1) \cup c(E_2) \cup \cdots \cup c(E_t) = G(n - k, n)$$

and hence $\gamma(k, n, K) \leq t - 1$, a contradiction. Now assume that the mappings $\varphi_1, \ldots, \varphi_t$ are linearly dependent. Say, $\sum_{i=1}^t a_i \varphi_i = 0$. This means that for all $\xi \in \Lambda^{n-k}(V)$ we have $0 = \sum_{i=1}^t a_i (\epsilon_i \wedge \xi) = (\sum_{i=1}^t a_i \epsilon_i) \wedge \xi$. So it suffices to observe that if $\delta \in \Lambda^k(V)$ and $\delta \wedge \xi = 0$ for all $\xi \in \Lambda^{n-k}(V)$ then $\delta = 0$. \hfill $\square$

**Lemma 3.** If $K$ is any algebraically closed field, then

$$k(n-k) + 1 \leq \gamma(k, n, K).$$

**Proof.** Suppose $\gamma(k, n, K) = t \leq k(n-k)$. Then there exists $E = (E_1, \ldots, E_t) \in G(k, n)^t$ such that $c(E_1) \cup \cdots \cup c(E_t) = G(n - k, n)$. So by Lemmas 1 and 2 there is a linear subspace $\mathcal{K}(E)$ of $\Lambda^{n-k}(V)$ such that $D(k, n) \cap \mathcal{K}(E) = \emptyset$ and $\mathcal{K}(E)$ has affine dimension $\binom{n}{k} - t$ which is at least $\binom{n}{k} - k(n-k)$. Let $\mathcal{K}'$ denote the
corresponding projective subspace of \( \mathbb{P}(\Lambda^{n-k}(V)) \). Then \( \mathcal{K}' \cap G(n-k,n) = \emptyset \). But using projective dimensions we have [3, Proposition 11.4]

\[
\dim(\mathcal{K}') + \dim(G(n-k,n)) \geq \binom{n}{k} - k(n-k) - 1 + k(n-k) \\
\geq \binom{n}{k} - 1 = \dim(\mathbb{P}(\Lambda^{n-k}(V)))
\]

and it follows that \( \mathcal{K}' \cap G(n-k,n) \neq \emptyset \) which is a contradiction. \( \square \)

3. The upper bound \( \gamma(k,n,K) \leq k(n-k) + 1 \)

**Lemma 4.** If \( K \) is algebraically closed, then \( \gamma(k,n,K) \leq k(n-k) + 1 \).

**Proof.** Let \( \nu = k(n-k) \) denote the dimension of \( G(k,n) \) (and \( G(n-k,n) \)) as a projective variety. Let

\[
A = G(k,n)^{\nu+1}.
\]

Then \( A \) is a projective variety of dimension \( \nu(\nu + 1) \). For every \( F \in G(n-k,n) \) define

\[
B(F) = \{ E \in G(k,n) | E \cap F \neq 0 \}.
\]

Now \( B(F) \) is an irreducible projective variety with

\[
\dim(B(F)) = \nu - 1.
\]

For \( F \in G(n-k,n) \) define

\[
C(F) = B(F)^{\nu+1}.
\]

Then

\[
\dim(C(F)) = (\nu + 1)(\nu - 1) = \nu^2 - 1.
\]

Now set

\[
C = \bigcup_{F \in G(n-k,n)} C(F).
\]

Note that if \( C \) is properly contained in \( A \), then there exists \( E = (E_1, \ldots, E_{\nu+1}) \in A - C \). Then for all \( F \in G(n-k,n) \) we have \( E \notin C(F) \) so there must exist an index \( i \in \{1, \ldots, \nu + 1\} \) such that \( E_i \cap F = 0 \). Hence \( c(E_1) \cup \cdots \cup c(E_{\nu+1}) = G(n-k,n) \) and so \( \gamma(k,n,K) \leq \nu + 1 \), as desired. So it remains only to show that \( C \) is properly contained in \( A \). In fact we claim that \( C \) is a variety of dimension at most \( \dim(A) - 1 = \nu^2 + \nu - 1 \).

To complete the proof we fix \( F_0 \in G(n-k,n) \) and consider the projective variety

\[
D := C(F_0) \times \text{PGL}_n(K).
\]

We note that

\[
\dim(D) = \dim(C(F_0)) + \dim(\text{PGL}(n,K)) = \nu^2 - 1 + n^2 - 1.
\]

An element \( M \) of \( \text{PGL}(n,K) \) induces a linear automorphism of \( \mathbb{P}(\Lambda^k(V)) \) which induces in turn an automorphism of \( G(k,n) \). Abusing notation we write \( U \mapsto MU \).
to indicate the latter automorphism. Now we define \( \varphi : D \to C \) as follows: For \((E, M) \in D\)

\[
\varphi(E, M) = (ME_1, ME_2, \ldots, ME_{\nu+1}).
\]

Clearly \( \varphi \) is a regular surjection. Hence by \([3, \text{Theorem 11.12}]\)

\[
\dim(D) = \dim(C) + \mu
\]

where

\[
\mu = \min\{\dim(\varphi^{-1}(E')), \ E' \in C\}.
\]

This shows that

\[
\dim(C) = \nu^2 - 1 + n^2 - 1 - \mu.
\]

So to prove that \( \dim(C) \leq \nu^2 + \nu - 1 \) it suffices to prove that \( n^2 - \nu - 1 \leq \mu \). To see this consider the subset \( G(F) \) of \( PGL_n(K) \) whose elements map the fixed \((n - k)\)-subspace \( F_0 \) to the \((n - k)\)-space \( F \). It is easy to see that \( \dim(G(F)) = n^2 - \nu - 1 \). Now if \( E' = (E'_1, \ldots, E'_{\nu+1}) \in C(F) \subseteq C \) then for each \( M \in G(F) \) we have

\[
(M^{-1}E', M) = (M^{-1}E'_1, \ldots, M^{-1}E'_{\nu+1}, M) \in \varphi^{-1}(E').
\]

The mapping \( M \mapsto (M^{-1}E', M) \) is a regular injection from \( G(F) \) into the fiber \( \varphi^{-1}(E') \). It follows that each fiber has dimension at least that of \( G(F) \) and this completes the proof.

**Remarks.** 1. The above proof shows that almost all \((E_1, \ldots, E_{\nu+1}) \in G(k, n)^{\nu+1}\)

satisfy

\[
c(E_1) \cup c(E_2) \cup \cdots \cup c(E_{\nu+1}) = G(n - k, n)
\]

since the complement \( C \) of the set of such \((\nu + 1)\)-tuples forms a variety of dimension smaller than \( \dim(G(k, n)^{\nu+1}) \).

2. As shown in \([1]\) \( \gamma(2, 4, K) = 4 \) when \( K \) is not quadratically closed. So the lower bound \( \gamma(k, n, K) \geq k(n - k) + 1 \) proved here for algebraically closed fields will not hold in general. On the other hand, we suspect that the upper bound \( \gamma(k, n, K) \leq k(n - k) + 1 \) does hold for arbitrary fields. In fact we have verified this for finite fields of sufficiently large order using counting arguments \([2]\). However, as the referee pointed out it is slightly worrying that the conjecture fails in the "thin" case, that is, if we replace \( n \)-space by \( n \)-set, \( k \)-subspace by \( k \)-subset and vector space complement by set complement. However, the upper bound of \( \binom{n}{k} \) given in \([1]\) holds in both cases.

**References**

