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INVARIANCE OF SPECTRUM FOR REPRESENTATIONS OF C*-ALGEBRAS ON BANACH SPACES

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Abstract. Let $\mathcal{K}$ be a Banach space, $\mathcal{B}$ a unital C*-algebra, and $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{K})$ an injective, unital homomorphism. Suppose that there exists a function $\gamma : \mathcal{K} \times \mathcal{K} \to \mathbb{R}^+$ such that, for all $k, k_1, k_2 \in \mathcal{K}$, and all $b \in \mathcal{B}$,

(a) $\gamma(k, k) = \|k\|^2$,

(b) $\gamma(k_1, k_2) \leq \|k_1\| \, \|k_2\|$,

(c) $\gamma(\pi_b k_1, k_2) = \gamma(k_1, \pi_{b^*} k_2)$.

Then for all $b \in \mathcal{B}$, the spectrum of b in $\mathcal{B}$ equals the spectrum of π_b as a bounded linear operator on $\mathcal{K}$. If γ satisfies an additional requirement and $\mathcal{B}$ is a W*-algebra, then the Taylor spectrum of a commuting n-tuple $b = (b_1, \ldots, b_n)$ of elements of $\mathcal{B}$ equals the Taylor spectrum of the n-tuple π_b in the algebra of bounded operators on $\mathcal{K}$. Special cases of these results are (i) if $\mathcal{K}$ is a closed subspace of a unital C*-algebra which contains $\mathcal{B}$ as a unital C*-subalgebra such that $\mathcal{B}\mathcal{K} \subseteq \mathcal{K}$, and $b\mathcal{K} = \{0\}$ only if $b = 0$, then for each $b \in \mathcal{B}$, the spectrum of b in $\mathcal{B}$ is the same as the spectrum of left multiplication by b on $\mathcal{K}$; (ii) if $\mathcal{A}$ is a unital C*-algebra and $\mathcal{J}$ is an essential closed left ideal in $\mathcal{A}$, then an element a of $\mathcal{A}$ is invertible if and only if left multiplication by a on $\mathcal{J}$ is bijective; and (iii) if $\mathcal{A}$ is a C*-algebra, $\mathcal{E}$ is a Hilbert $\mathcal{A}$-module, and T is an adjointable module map on $\mathcal{E}$, then the spectrum of T in the C*-algebra of adjointable operators on $\mathcal{E}$ is the same as the spectrum of T as a bounded operator on $\mathcal{E}$. If the algebra of adjointable operators on $\mathcal{E}$ is a W*-algebra, then the Taylor spectrum of a commuting n-tuple of adjointable operators on $\mathcal{E}$ is the same relative to the algebra of adjointable operators and relative to the algebra of all bounded operators on $\mathcal{E}$.

1. Introduction and basic definitions

Let $\mathcal{B}$ be a C*-algebra, $\mathcal{K}$ a Banach space, and π a representation of $\mathcal{B}$ as an algebra of bounded operators on $\mathcal{K}$. We obtain a sufficient condition, suggested by the notion of adjointable map in the theory of Hilbert modules, for the identification of the spectrum of an element $b \in \mathcal{B}$ with the spectrum of the operator π_b in $\mathcal{L}(\mathcal{K})$, the Banach algebra of all bounded operators on $\mathcal{K}$. When $\mathcal{B}$ is a W*-algebra, we show that in the presence of a strengthened form of this condition, the Taylor spectrum of a commuting n-tuple $(b_1, \ldots, b_n)$ of elements of $\mathcal{B}$ equals the Taylor

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spectrum of the n-tuple $(\pi_{b_1}, \ldots, \pi_{b_n})$ in $\mathcal{L}(\mathcal{K})$. This is true, in particular, if $\mathcal{K}$ is a Hilbert $\mathcal{A}$-module over a unital C*-algebra $\mathcal{A}, \mathcal{B}$ is the algebra $\mathcal{B}(\mathcal{K})$ of adjointable operators on $\mathcal{K}$, and π is the inclusion of $\mathcal{B}(\mathcal{K})$ in $\mathcal{L}(\mathcal{K})$, whenever $\mathcal{B}(\mathcal{K})$ is a W*-algebra. Our Taylor spectrum results are based on arguments of Curto [Cu1].

In the case of the left regular representation of $\mathcal{B}$ we show that if $\mathcal{K}$ is a closed subspace of a unital C*-algebra $\mathcal{A}$ which contains $\mathcal{B}$ as a unital C*-subalgebra, such that (i) $\mathcal{B}\mathcal{K} \subseteq \mathcal{K}$ and (ii) $b\mathcal{K} = \{0\}$ only if $b = 0$, then for each $b \in \mathcal{B}$ the spectrum of b in $\mathcal{B}$ equals the spectrum of the operator of left multiplication by b in $\mathcal{L}(\mathcal{K})$. This result can be viewed as a generalization of the standard spectral permanence result for C*-algebras, which, in effect, is the case $\mathcal{K} = \mathcal{A}$. It also follows as a special case that if $\mathcal{A}$ is a unital C*-algebra and $\mathcal{J}$ is an essential closed left ideal in $\mathcal{A}$, then an element a of $\mathcal{A}$ is invertible if and only if left multiplication by a on $\mathcal{J}$ is bijective, which was a question left open in [DLW]. We also include an independent proof of this last result (for the case of an essential closed, two-sided ideal) using the multiplier algebra of $\mathcal{J}$.

Our primary motivation arose from the study, begun in [DLW], of conditional expectation operators on a C*-algebra. If Φ is a conditional expectation on the C*-algebra $\mathcal{A}$ with range $\mathcal{B}$ and kernel $\mathcal{K}$, then $\mathcal{B}\mathcal{K} \subseteq \mathcal{K}$, so defining π_b to be left multiplication by b on $\mathcal{K}$ gives a representation π of $\mathcal{B}$ as an algebra of bounded operators on $\mathcal{K}$. In [DLW] the case that π is injective was our principal focus, and it was shown that if, in addition, $\mathcal{B}$ is abelian and π and $\mathcal{B}$ are unital, then $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{L}(\mathcal{K})}(\pi_b)$. We used an old result of Rickart [R] on spectral permanence:

Theorem 0 ([R]). *Let $\mathcal{A}$ be a Banach algebra, and $\mathcal{B}$ a C*-algebra which is algebraically embedded in $\mathcal{A}$. Then, for all $b \in \mathcal{B}$, $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b) \cup \overline{\sigma_{\mathcal{A}}(b^*)}$ (where the overbar denotes complex conjugation).*

The present work also uses Rickart's result as its starting point, and shows that the hypothesis that $\mathcal{B}$ is abelian is not needed.

Much of [DLW] is devoted to the study of the algebra $\mathcal{D}$ of operators D_a on $\mathcal{A}$ defined by $D_a x = \Phi(a)x - a\Phi(x)$. These operators map $\mathcal{A}$ to $\mathcal{K}$, and may be useful in studying the structure of $\mathcal{K}$. In [DLW] it is shown, as a consequence of the result quoted above, that when $\mathcal{B}$ is abelian and π and $\mathcal{B}$ are unital, the spectrum of the operator D_a is $\{0\} \cup \sigma_{\mathcal{B}}\Phi(a)$. (This generalized a result of [LW] in a measure-theoretic setting concerning certain operators D_f defined on $L^2(X, \mathcal{S}, \mu)$ by the same formal identity, where $(X, \mathcal{S}, \mu)$ is a complete σ-finite measure space, $\mathcal{T}$ is a sub-σ-algebra of $\mathcal{S}$, and Φ is the classical conditional expectation from $L^2(X, \mathcal{S})$ to $L^2(X, \mathcal{T})$, namely that the spectrum of D_f is $\{0\} \cup \text{ess. range } \Phi(f)$.) Again, the present work shows that the abelian hypothesis is unnecessary.

We begin with the following definitions.

Definition 1. Let $\mathcal{K}$ be a normed vector space. A Cauchy-Schwarz function for $\mathcal{K}$ is a function $\gamma : \mathcal{K} \times \mathcal{K} \to \mathbb{R}^+$ such that, for all $k, k_1, k_2 \in \mathcal{K}$,

(a) $\gamma(k, k) = \|k\|^2$,

(b) $\gamma(k_1, k_2) \leq \|k_1\| \, \|k_2\|$.

For some of our results we will need a more specialized type of Cauchy-Schwarz function.

Definition 2. Let $\mathcal{K}$ be a normed linear space. A Cauchy-Schwarz function of semilinear type on $\mathcal{K}$ is a Cauchy-Schwarz function γ such that there exist a vector space $\mathcal{Y}$; a function $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \to \mathcal{Y}$ satisfying $\langle \lambda k_1, k_2 \rangle = \langle k_1, \overline{\lambda} k_2 \rangle$ and

$\langle k_1 + k_2, k_3\rangle = \langle k_1, k_3\rangle + \langle k_2, k_3\rangle$ for all $k_1, k_2, k_3 \in \mathcal{K}$ and $\lambda \in \mathbb{C}$; and a function $\delta : \mathcal{Y} \to \mathbb{R}^+$, with $\gamma(k_1, k_2) = \delta(\langle k_1, k_2\rangle)$. If in addition $\langle k_1, k_2+k_3\rangle = \langle k_1, k_2\rangle+\langle k_1, k_3\rangle$, and $\sum_{i=1}^n \langle k_i, k_i\rangle = 0$ implies that each $k_i = 0$, then γ is called a Cauchy-Schwarz function of positive definite type.

Remark. Note that if $\gamma(\cdot, \cdot) = \delta(\langle\cdot, \cdot\rangle)$ is a Cauchy-Schwarz function of semilinear type, then $\langle\cdot, 0\rangle = 0$, since for each $k \in \mathcal{K}$,

$$\langle k, 0\rangle = \langle k, 0(0)\rangle = \langle 0k, 0\rangle = \langle 0, 0\rangle = \langle k - k, 0\rangle = \langle k, 0\rangle + \langle -k, 0\rangle = 2\langle k, 0\rangle.$$

Thus, if γ is of positive definite type, then also $\langle 0, \cdot\rangle = 0$.

Example 1. Let $\mathcal{A}$ be a C*-algeba and $\mathcal{K}$ a closed subspace of $\mathcal{A}$. If $\gamma_1(k_1, k_2) = \|k_1^* k_2\|$ for all $k_1, k_2 \in \mathcal{K}$, then γ_1 is a Cauchy-Schwarz function for $\mathcal{K}$ of positive definite type.

Example 2. Let $\mathcal{A}$ be a C*-algebra and $\mathcal{E}$ a Hilbert $\mathcal{A}$-module, i.e., an $\mathcal{A}$-module which is also a complex vector space equipped with an $\mathcal{A}$-valued inner product $(\cdot|\cdot)$ that is sesquilinear, positive definite, and respects the module action, and which is complete with respect to the norm defined by $\|x\|^2 = \|(x|x)\|$. (See [W-O] for a discussion of Hilbert modules.) If γ_2 is defined by $\gamma_2(x, y) = \|(x|y)\|$ for $x, y \in \mathcal{E}$, then γ_2 is a Cauchy-Schwarz function for $\mathcal{E}$ of positive definite type.

Example 3. Let $\mathcal{H}$ be a Hilbert space, X a compact Hausdorff space, and $\mathcal{K} = C(X, \mathcal{H})$ with the norm defined by $\|f\| = \sup\{\|f(x)\| : x \in X\}$. Let γ_3 be defined by $\gamma_3(f, g) = \sup\{|\langle f(x), g(x)\rangle| : x \in X\}$. Then γ_3 is a Cauchy-Schwarz function on $\mathcal{K}$.

Definition 3. Let γ be a Cauchy-Schwarz function on a normed linear space $\mathcal{K}$. If $\mathcal{C}$ is a subalgebra of $\mathcal{L}(\mathcal{K})$ with an involution $*$, then $\mathcal{C}$ is called γ-symmetric if, for all $T \in \mathcal{C}$ and $k_1, k_2 \in \mathcal{K}$, $\gamma(Tk_1, k_2) = \gamma(k_1, T^*k_2)$.

If $\mathcal{A}$ is a Banach algebra, γ a Cauchy-Schwarz function for $\mathcal{A}$, and $\mathcal{B}$ a subalgebra of $\mathcal{A}$ with an involution such that $\gamma(ba_1, a_2) = \gamma(a_1, b^*a_2)$ for all $b \in \mathcal{B}$, $a_1, a_2 \in \mathcal{A}$, then $\mathcal{B}$ is called a γ-symmetric subalgebra of $\mathcal{A}$.

Example 4. In Example 1 above, let $\mathcal{K} = \mathcal{A}$. Let $\mathcal{B}$ be a C*-subalgebra of $\mathcal{A}$ and let $\mathcal{C}$ be the image of the left regular representation of $\mathcal{B}$ on $\mathcal{A}$ with the involution induced by $\mathcal{B}$. Then $\mathcal{C}$ is γ_1-symmetric.

Example 5. Let $\mathcal{E}$ be a Hilbert module and $\mathcal{C}$ a self-adjoint subalgebra of the algebra of adjointable maps, i.e., the maps $T : \mathcal{E} \to \mathcal{E}$ for which there exists a map $T^* : \mathcal{E} \to \mathcal{E}$ such that $(Tx|y) = (x|T^*y)$ for all $x, y \in \mathcal{E}$. Then $\mathcal{C}$ is γ_2-symmetric, where γ_2 is defined as in Example 2.

Example 6. Let $\mathcal{H}$ be a Hilbert space, X a compact Hausdorff space, and let $\mathcal{K} = C(X, \mathcal{H})$. Let $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ be defined by $\pi_A(f) = A \circ f$. If $(\pi_A)^*$ is defined to be π_{A^*}, then $\pi(\mathcal{L}(\mathcal{H}))$ is γ_3-symmetric where γ_3 is as in Example 3.

2. Main result

Theorem 1. *Let $\mathcal{K}$ be a Banach space, γ a Cauchy-Schwarz function for $\mathcal{K}$, $\mathcal{B}$ a unital C*-algebra and $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{K})$ an injective, unital homomorphism such that $\pi(\mathcal{B})$, with the involution defined by $(\pi_b)^* = \pi_{b^*}$, is γ-symmetric. Then, for all $b \in \mathcal{B}$, $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{L}(\mathcal{K})}(\pi_b)$.*

For convenience, the proof of this theorem is divided into two propositions.

Proposition 1. *Let $\mathcal{K}$ be a Banach space, γ a Cauchy-Schwarz function for $\mathcal{K}$, and $\mathcal{C}$ a subalgebra of $\mathcal{L}(\mathcal{K})$ containing I which is a C^*-algebra with respect to some involution and some norm. If $\mathcal{C}$ is γ-symmetric, then, for all $T \in \mathcal{C}$, $\sigma_{\mathcal{C}}(T) = \sigma_{\mathcal{L}(\mathcal{K})}(T)$.*

Proof. By Theorem 0, it suffices to show that for all $T \in \mathcal{C}$,

$$\sigma_{\mathcal{L}(\mathcal{K})}(T) = \overline{\sigma_{\mathcal{L}(\mathcal{K})}(T^*)}.$$

Choose λ such that $T - \lambda I$ is invertible. Let $S = (T - \lambda I)^{-1}$. For all $k \in \mathcal{K}$, we have $k = (T - \lambda I)Sk$. Therefore

$$\|k\|^2 = \gamma(k, k) = \gamma(Sk, (T^* - \overline{\lambda} I)k),$$

so that $T^* - \overline{\lambda} I$ is injective. Also,

$$\|k\|^2 \leq \|S\| \, \|(T^* - \overline{\lambda} I)k\| \, \|k\|,$$

so that $T^* - \overline{\lambda} I$ has closed range. It remains to show that the range of $T^* - \overline{\lambda} I$ is dense.

For all positive integers n, $(T^* - \overline{\lambda} I)(T - \lambda I) + n^{-1} I$ is invertible in the C*-algebra $\mathcal{C}$. Hence, given $k \in \mathcal{K}$, for each n there exists $k_n \in \mathcal{K}$ such that

$$(*) \qquad (T^* - \overline{\lambda} I)(T - \lambda I)k_n = k - n^{-1} k_n.$$

If $\{k_n\}$ is not bounded, there is a subsequence, denoted again by $\{k_n\}$, such that $\|k_n\| \to \infty$. Dividing $(*)$ by $\|k_n\|$ and using the fact that $T^* - \overline{\lambda} I$ is bounded below, one sees that

$$(T - \lambda I)(\|k_n\|^{-1} k_n) \to 0 \quad \text{as } n \to \infty.$$

But $T - \lambda I$ is invertible, so this is impossible. Thus $\{k_n\}$ is bounded. Therefore, the right side of $(*)$ converges to k as $n \to \infty$ so that k is in the closure of the range of $T^* - \overline{\lambda} I$.

This shows that $\overline{\sigma_{\mathcal{L}(\mathcal{K})}(T^*)} \subseteq \sigma_{\mathcal{L}(\mathcal{K})}(T)$. Replacing T by T^* gives the reverse inequality. □

Proposition 2. *Let $\mathcal{K}$ be a Banach space, γ a Cauchy-Schwarz function for $\mathcal{K}, \mathcal{B}$ a unital C^*-algebra, and π an injective, unital homomorphism from $\mathcal{B}$ to $\mathcal{L}(\mathcal{K})$. If $*$ is the involution defined on $\pi(\mathcal{B})$ by $\pi_b^* = \pi_{b^*}$, and if with this involution $\pi(\mathcal{B})$ is γ-symmetric, then π is an isometry. (In particular, $\pi(\mathcal{B})$ is a unital C^*-algebra which is isometrically $*$-isomorphic to $\mathcal{B}$.)*

Proof. For each $b \in \mathcal{B}$ and $k \in \mathcal{K}$,

$$(**) \qquad \|\pi_b k\|^2 = \gamma(\pi_b k, \pi_b k) = \gamma(k, \pi_{b^*} \pi_b k) \leq \|\pi_{b^* b}\| \, \|k\|^2 \leq \|\pi_{b^*}\| \, \|\pi_b\| \, \|k\|^2.$$

Therefore, $\|\pi_b\|^2 \leq \|\pi_{b^*}\| \, \|\pi_b\|$, so that $\|\pi_b\| \leq \|\pi_{b^*}\|$. By symmetry, this last inequality is actually an equality. Also, by taking the supremum over vectors k of norm 1 in $(**)$ we see that $\|\pi_b\|^2 \leq \|\pi_{b^* b}\| \leq \|\pi_{b^*}\| \, \|\pi_b\| = \|\pi_b\|^2$. Since π is a homomorphism, this shows that $\pi(\mathcal{B})$, with the involution defined in the statement of the proposition, is an algebra whose norm satisfies the C*-algebra property. Since π is injective, it follows that defining $|b|$ to be $\|\pi_b\|$ gives a second norm on $\mathcal{B}$ which satisfies the C*-algebra property. Thus, the identity map is an injective $*$-homomorphism from $\mathcal{B}$ to the C*-algebra which is the completion of $(\mathcal{B}, |\,|)$, hence the identity map is an isometry, hence for each $b \in \mathcal{B}$, $\|b\| = \|\pi_b\|$. □

3. Some corollaries

Our first corollary is a result on invertibility of adjointable operators on Hilbert modules.

Corollary 1. *Let $\mathcal{E}$ be a Hilbert module and $\mathcal{B}(\mathcal{E})$ the C^*-algebra of adjointable maps on $\mathcal{E}$. If $T \in \mathcal{C}$, then $\sigma_{\mathcal{B}(\mathcal{E})}(T) = \sigma_{\mathcal{L}(\mathcal{E})}(T)$.*

Proof. In view of Example 2, this corollary follows immediately from Proposition 1. □

Corollary 2. *Let $\mathcal{K}, \mathcal{B}$ and π be as in Theorem 1, with the exception of the requirement that π be injective. Let $S = \{b : \pi_b k = 0$ for all $k \in \mathcal{K}\}$. Then $\mathcal{S}$ is a closed, two sided ideal in $\mathcal{B}$ and if $[b]$ denotes the coset of b in $\mathcal{B}/\mathcal{S}$, then*

$$\sigma_{\mathcal{B}/\mathcal{S}}([b]) = \sigma_{\mathcal{L}(\mathcal{K})}(\pi_b).$$

Proof. If ρ is the canonical homomorphism of $\mathcal{B}$ onto $\mathcal{B}/\mathcal{S}$, then $\pi = \rho \circ \tilde{\pi}$ where $\tilde{\pi} : \mathcal{B}/\mathcal{S} \to \mathcal{L}(\mathcal{K})$. The homomorphism $\tilde{\pi}$ satisfies all the hypotheses of Theorem 1. □

Corollary 3. *Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{K}$ a closed subspace of $\mathcal{A}$, and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{A}$ such that $1_{\mathcal{A}} \in \mathcal{B}$. Suppose* (i) *$\mathcal{B}\mathcal{K} \subseteq \mathcal{K}$ and* (ii) *if $b \in \mathcal{B}$ and $b\mathcal{K} = \{0\}$ then $b = 0$. Then for all $b \in \mathcal{B}$, the spectrum of b in $\mathcal{B}$ equals the spectrum of left multiplication by b in $\mathcal{L}(\mathcal{K})$.*

Proof. If $\gamma(k_1, k_2) = \|k_1^* k_2\|$ and $\pi_b k = bk$ for all $k_1, k_2, k \in \mathcal{K}$ and $b \in \mathcal{B}$, then γ is a Cauchy-Schwarz function on $\mathcal{K}$, π is an injective, unital homomorphism, and $\pi(\mathcal{B})$ is γ-symmetric. □

Recall that an essential ideal in a C*-algebra is an ideal whose only left annihilator is 0. Combining Corollary 2 and Corollary 3, we obtain the following result.

Corollary 4. *If $\mathcal{J}$ is an essential closed left ideal in a unital C^*-algebra $\mathcal{A}$, then $a \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if left multiplication by a is a bijection of $\mathcal{J}$. More generally, if $\mathcal{J}$ is a closed left ideal in $\mathcal{A}$ and $\mathcal{S} = \{a \in \mathcal{A} : a\mathcal{J} = (0)\}$, then a is invertible* mod *$\mathcal{S}$ if and only if left multiplication by a is a bijection of $\mathcal{J}$.*

If $\mathcal{A}$ is a C*-algebra and $\mathcal{B}$ is a C* subalgebra of $\mathcal{A}$, then $\mathcal{B}$ is a γ-symmetric subalgebra if γ is the Cauchy-Schwarz function defined by $\gamma(a_1, a_2) = \|a_1^* a_2\|$. Thus the following result should be compared with the usual spectral permanence theorem for C*-algebras on the one hand and Rickart's theorem (Theorem 0) on the other hand.

Corollary 5. *If $\mathcal{A}$ is a unital Banach algebra, γ a Cauchy-Schwarz function for $\mathcal{A}$, and $\mathcal{B}$ a C^*-algebra which is algebraically embedded as a γ-symmetric subalgebra of $\mathcal{A}$ with $1_{\mathcal{A}} \in \mathcal{B}$, then for all $b \in \mathcal{B}$, $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$.*

4. Invariance of Taylor spectrum

Let $\mathcal{B}$ be a unital C*-algebra and $b = (b_1, \ldots, b_n)$ an n-tuple of elements of $\mathcal{B}$. In ([Cu1], [Cu2]) Curto defines a transformation $b \to \hat{b}$ which associates to b a $2^{n-1} \times 2^{n-1}$ matrix $\hat{b}$ over $\mathcal{B}$. He proves that if b is a commuting n-tuple (i.e., $b_i b_j = b_j b_i$, $1 \le i, j \le n$) and $\lambda \in \mathbb{C}^n$, then λ belongs to $Sp(b, \mathcal{B})$, the Taylor spectrum of b relative to $\mathcal{B}$, if and only if $(b - \lambda)\hat{}$ is invertible in $M_{2^{n-1}}(\mathcal{B})$. He also proves that if $\mathcal{A}$ is a unital C*-algebra containing $\mathcal{B}$ as a unital subalgebra,

then $Sp(b, \mathcal{B}) = Sp(b, \mathcal{A})$. Using Theorem 1 together with Curto's arguments we obtain the following generalizations of these results of Curto.

Theorem 2. *Let $\mathcal{K}$ be a Banach space, γ a Cauchy-Schwarz function of positive definite type for $\mathcal{K}$, and $\mathcal{C}$ a subalgebra of $\mathcal{L}(\mathcal{K})$ containing I which is a W^*-algebra with respect to some involution and some norm. If $\mathcal{C}$ is γ-symmetric, then a commuting n-tuple $T = (T_1, \ldots, T_n)$ of elements of $\mathcal{C}$ is Taylor invertible in $\mathcal{L}(\mathcal{K})$ if and only if $\widehat{T}$ is invertible in $\mathcal{L}(\mathcal{K})$.*

Theorem 3. *Let $\mathcal{K}$ be a Banach space, γ a Cauchy-Schwarz function of positive definite type for $\mathcal{K}, \mathcal{B}$ a unital W^*-algebra, and $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{K})$ an injective, unital homomorphism such that $\pi(\mathcal{B})$, with the involution defined by $(\pi_b)^* = \pi_{b^*}$, is γ-symmetric. Then, if $b = (b_1, \ldots, b_n)$ is a commuting n-tuple, $Sp(b, \mathcal{B}) = Sp((\pi_{b_1}, \ldots, \pi_{b_n}), \mathcal{L}(\mathcal{K}))$.*

By virtue of Examples 2 and 5, Theorem 3 applies in particular when $\mathcal{K}$ is a Hilbert $\mathcal{A}$-module over a unital C*-algebra $\mathcal{A}$, $\mathcal{B}$ is the algebra $\mathcal{B}(\mathcal{K})$ of adjointable operators on $\mathcal{K}$ which is assumed to be a W*-algebra, and π is the inclusion of $\mathcal{B}(\mathcal{K})$ in $\mathcal{L}(\mathcal{K})$. For example, $\mathcal{B}(\mathcal{K})$ is a W*-algebra whenever $\mathcal{A}$ is a W*-algebra and $\mathcal{K}$ is a self-dual $\mathcal{A}$-module [Pas].

The proof of Theorem 2 follows the outline used by Curto, and requires the following lemmas.

Lemma 1. *Let $\mathcal{K}$ be a Banach space, γ a Cauchy-Schwarz function of positive definite type for $\mathcal{K}$, and $\mathcal{C}$ a subalgebra of $\mathcal{L}(\mathcal{K})$ containing I which is a C^*-algebra with respect to some involution and some norm. Let $D = (D_{ij})$ be an $n \times k$ matrix of elements of $\mathcal{C}$ and let D^* be the matrix adjoint of D relative to the involution on $\mathcal{C}$. (Thus $(D^*)_{ij} = (D_{ji})^*$.) Then, considering D as an operator from $\mathcal{K}^k$ to $\mathcal{K}^n$, $\ker\ D \cap \operatorname{ran}\ D^* = \{0\}$. In particular, if $k = n$ and $D = D^*$, then D is onto if and only if D is invertible.*

Proof. Let $\gamma(z, w) = \delta(\langle z, w\rangle)$ be a Cauchy-Schwarz function of positive definite type for $\mathcal{K}$. Let $z = (z_1, \ldots, z_k) \in \mathcal{K}^k$. Suppose that $Dz = 0$ and that $z = D^*w$ for some $w \in \mathcal{K}^n$. Then for each m,

$$\begin{aligned}\sum_m \langle z_m, z_m\rangle &= \sum_m \left\langle z_m, \sum_i D^*_{im} w_i \right\rangle \\ &= \sum_m \sum_i \langle z_m, D^*_{im} w_i\rangle = \sum_m \sum_i \langle D_{im} z_m, w_i\rangle \\ &= \sum_i \left\langle \sum_m D_{im} z_m, w_i \right\rangle = \sum_i \langle 0, w_i\rangle = 0\end{aligned}$$

by the remark following Definition 2. Thus $z = 0$. □

Lemma 2. *Let $\mathcal{C}$ be a W^*-algebra. Let D be an $n \times k$ matrix of elements of $\mathcal{C}$. Then there exists an $n \times k$ matrix V of elements of $\mathcal{C}$ such that $D = (DD^*)^{1/2}V$, where D^* is the matrix adjoint to D.*

Proof. Use the polar decomposition of the element $\left(\begin{smallmatrix} D^* & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in the W*-algebra $M_{n+k}(\mathcal{C})$. □

Using Lemma 1 and Lemma 2 we observe that Curto's argument in [Cu1] yields the following generalization of his Proposition 3.4.

Proposition 3. *Let $\mathcal{K}$ be a Banach space, γ a positive definite Cauchy-Schwarz function for $\mathcal{K}$, and $\mathcal{C}$ a subalgebra of $\mathcal{L}(\mathcal{K})$ containing I which is a W^*-algebra with respect to some involution and some norm. Let $\{n_k\}$ be a sequence of nonnegative integers, and D_k an $n_{k-1} \times n_k$ matrix of elements of $\mathcal{C}$, such that $D_k D_{k=1} = 0$ for all k. Then, considering D_k as an element of $\mathcal{L}(\mathcal{K}^{n_k}, \mathcal{K}^{n_{k-1}})$, $\ker\ D_k = \operatorname{ran}\ D_{k+1}$ for all k if and only if $L_k = D_k^* D_k + D_{k+1} D_{k+1}^*$ is invertible in $\mathcal{L}(\mathcal{K}^{n_k})$ for all k.*

Now we can proceed to prove Theorem 2.

Proof of Theorem 2. Suppose T is an n-tuple of elements of $\mathcal{C}$, and that T is Taylor invertible in $\mathcal{L}(\mathcal{K})$. Let $n_k = \binom{n}{k}$ and let $\{D_k\}$ be the operators of the Koszul complex used in the definition of Taylor spectrum. Then, viewed as matrices over $\mathcal{L}(\mathcal{K})$, each D_k is a matrix whose entries lie in $\mathcal{C}$. Thus by Proposition 3, $L_k = D_k^* D_k + D_{k+1} D_{k+1}^*$ is invertible in $\mathcal{L}(\mathcal{K}^{n_k})$ for all k. If $\mathcal{K}^{n_k}$ is given the norm defined by $\|K\|^2 = \sum_i \|K_i\|^2$ and we define γ_k by $\gamma_k(K, L) = \sum_i \gamma(K_i, L_i)$, then γ_k is a Cauchy-Schwarz function on $\mathcal{K}^{n_k}$, and $M_{n_k}(\mathcal{C})$ is a γ_k-symmetric unital subalgebra of $\mathcal{L}(\mathcal{K}^{n_k})$. Hence by Proposition 1, L_k is invertible in $M_{n_k}(\mathcal{C})$ for all k. Since the matrices $\widehat{T}^*\widehat{T}$ and $\widehat{T}\widehat{T}^*$ are block-diagonal matrices whose diagonal entries are, respectively, the L_k's with k even and the L_k's with k odd, $\widehat{T}^*\widehat{T}$ and$\widehat{T}\widehat{T}^*$ are invertible in $\mathcal{L}(\mathcal{K}^{2^{n-1}})$, hence $\widehat{T}$ is invertible in $\mathcal{L}(\mathcal{K}^{2^{n-1}})$. Conversely, if $\widehat{T}$ is invertible in $\mathcal{L}(\mathcal{K}^{2^{n-1}})$, then so are $\widehat{T}^*\widehat{T}$ and $\widehat{T}\widehat{T}^*$, hence so is each L_k. Thus by Proposition 3, T is Taylor invertible. $\square$

Remark. A careful reading of Curto's proof of his Proposition 3.4 shows that the implication ($L_k = D_k^* D_k + D_{k+1} D_{k+1}^*$ *is invertible in* $\mathcal{L}(\mathcal{K}^{n_k})$ *for all* k) $\Rightarrow$ ($\ker\ D_k = \operatorname{ran}\ K_{k+1}$ *for all* k) in Proposition 3 does not use the hypothesis that $\mathcal{C}$ is a W*-algebra, hence neither does the implication ($\widehat{T}$ *is invertible in* $\mathcal{L}(\mathcal{K})$) $\Rightarrow$ (T *is invertible in* $\mathcal{L}(\mathcal{K})$) in Theorem 2.

Proof of Theorem 3. For each positive integer d, π induces an injective, unital homomorphism $\pi_d : M_d(\mathcal{B}) \to M_d(\mathcal{L}(\mathcal{K}))$ defined by $\pi_d = \pi \otimes \mathrm{id}$. If $M_d(\mathcal{L}(\mathcal{K}))$ is given the norm defined by $\|K\|^2 = \sum_{1 \le i,j \le d} \|K_{ij}\|^2$ and we define γ_d by $\gamma_d(K, L) = \sum_{1 \le i,j \le d} \gamma(K_{ij}, L_{ij})$, then γ_d is a Cauchy-Schwarz function on $M_d(\mathcal{L}(\mathcal{K}))$ and $\pi_d(\mathcal{B})$ is γ_d-symmetric. If $\lambda \in \mathbb{C}^n$, then by Theorem 1 and Theorem 4,

$$\begin{aligned}
\lambda \in Sp(b, \mathcal{B}) &\Leftrightarrow (b - \lambda)\hat{}\ \text{ is invertible in } M_{2^{n-1}}(\mathcal{B}) \\
&\Leftrightarrow ((\pi_{b_1}, \ldots, \pi_{b_n}) - \lambda)\hat{}\ \text{ is invertible in } M_{2^{n-1}}(\mathcal{L}(\mathcal{K})) \\
&\Leftrightarrow \lambda \in Sp(\pi_b, \mathcal{L}(\mathcal{K})). \quad \square
\end{aligned}$$

Remark. We have not succeeded in eliminating the W*-algebra hypothesis in Theorems 2 and 3, even though, as shown in [Cu2], Curto's results do not require this assumption. We conjecture that Theorems 2 and 3 remain true for arbitrary unital C*-algebras.

5. The non-unital case

When $\mathcal{K}$ is a Banach space with a Cauchy-Schwarz function of semilinear type, Theorem 1 can be generalized to the case that either $\mathcal{B}$ is not unital or π is not unital.

If $\mathcal{B}$ is a C*-algebra, we denote by $\mathcal{B}^+$ the set $\mathcal{B} \times \mathbb{C}$ equipped with pointwise sum and adjoint and the product defined by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$. It

is well-known that $\mathcal{B}^+$ can be normed so as to become a C*-algebra. (We refer to [W-O] for details.) We usually write the elements $(b,\mu),(b,0)$, and $(0,\mu)$ as $b+\mu, b$, and μ, respectively. In particular, $\mathcal{B}^+$ is a unital algebra with unit 1.

Lemma 3. *If $\mathcal{B}$ is a C^*-algebra, $\mathcal{K}$ is a Banach space, and π is an algebra homomorphism from $\mathcal{B}$ to the Banach algebra $\mathcal{L}(\mathcal{K})$, then there exists a unital homomorphism $\pi^+ : \mathcal{B}^+ \to \mathcal{L}(\mathcal{K})$ which extends π.*

Proof. Define π^+ by the formula $\pi^+_{(b+\lambda)}k = \pi_b k + \lambda k$. Then $\pi^+(1) = I$ and a routine computation shows that π^+ is an algebra homomorphism. □

When $\mathcal{B}$ is not unital and $b \in \mathcal{B}$ we define $\sigma_{\mathcal{B}}(b)$ to be $\sigma_{\mathcal{B}^+}(b)$. For all $b \in \mathcal{B}$ we then have $0 \in \sigma_{\mathcal{B}}(b)$. When $\mathcal{B}$ is unital, with unit e, we have the following known result, whose proof we provide for the convenience of the reader.

Lemma 4. *If $\mathcal{B}$ is a unital C^*-algebra and $\mathcal{B}^+$ is defined as above, then for all $b \in \mathcal{B}$, $\sigma_{\mathcal{B}^+}(b) = \sigma_{\mathcal{B}}(b) \cup \{0\}$.*

Proof. Let e denote the unit of $\mathcal{B}$. If $\lambda \notin \sigma_{\mathcal{B}^+}(b)$ there exist $c \in \mathcal{B}$ and $\mu \in \mathbb{C}$ such that $(b-\lambda)(c+\mu) = 1 = (c+\mu)(b-\lambda)$. Multiplying this identity by e^2 yields the identity $(b-\lambda e)(c+\mu e) = e = (c+\mu e)(b-\lambda e)$, which shows that $\lambda \notin \sigma_{\mathcal{B}}(b)$. Moreover, if $0 \notin \sigma_{\mathcal{B}}(b)$, then there exists $c \in \mathcal{B}$ such that $bc = e = cb$. If also $0 \notin \sigma_{\mathcal{B}^+}(b)$, then there exist $d \in \mathcal{B}$ and $\mu \in \mathbb{C}$ such that $b(d+\mu) = 1 = (d+\mu)b$, so that

$$c = cb(d+\mu) = e(d+\mu) = d+\mu.$$

Thus $d = c - \mu e$, so that

$$1 = b(c-\mu e+\mu) = bc - \mu b + \mu b = bc = e.$$

This contradiction shows that $0 \in \sigma_{\mathcal{B}^+}(b)$, hence $\sigma_{\mathcal{B}}(b) \cup \{0\} \subseteq \sigma_{\mathcal{B}^+}(b)$.

Now suppose that $\lambda \notin \sigma_{\mathcal{B}}(b) \cup \{0\}$. Then there exists $c \in \mathcal{B}$ such that $(b-\lambda e)c = e = c(b-\lambda e)$. A simple computation shows that if $d = \lambda^{-1}bc$ and $\mu = -\lambda^{-1}$, then $(b-\lambda)(d-\mu) = 1 = (d-\mu)(b-\lambda)$, which shows that $\lambda \notin \sigma_{\mathcal{B}^+}(b)$. Therefore, $\sigma_{\mathcal{B}}(b) \cup \{0\} = \sigma_{\mathcal{B}^+}(b)$. □

Theorem 4. *Let $\mathcal{K}$ be a Banach space, γ a Cauchy-Schwarz function of semilinear type for $\mathcal{K}$, $\mathcal{B}$ a C^*-algebra and $\pi : \mathcal{B} \to \mathcal{L}(\mathcal{K})$ an injective homomorphism such that $\pi(\mathcal{B})$, with the involution defined on $\pi(b)^* = \pi_{b^*}$, is γ-symmetric. If either $\mathcal{B}$ has no unit, or $\mathcal{B}$ has a unit e with $\pi(e) \neq I$, then $\sigma_{\mathcal{B}}(b) \cup \{0\} = \sigma_{\mathcal{L}(\mathcal{K})}(\pi_b)$.*

Proof. Define $\pi^+ : \mathcal{B}^+ \to \mathcal{L}(\mathcal{K})$ as above. Then π^+ is an algebra homomorphism. To see that π^+ is injective, observe that if $\pi_b k + \lambda k = 0$ for all $k \in \mathcal{K}$, then either $\lambda = 0$, in which case $b = 0$ since by hypothesis π is injective, or $\lambda \neq 0$, in which case $\pi_{(-\lambda^{-1}bc)} = \pi_c$ for all $c \in \mathcal{B}$, so that $-\lambda^{-1}b$ is a left unit for $\mathcal{B}$. Similarly, $-\overline{\lambda}^{-1}b^*$ is a right unit for $\mathcal{B}$. Hence $-\lambda^{-1}b$ is a unit for $\mathcal{B}$ and $\pi(-\lambda^{-1}b)$ is the identity operator on $\mathcal{K}$, which contradicts our hypotheses.

The identity $\gamma(\pi^+_{(b+\lambda)}k_1, k_2) = \gamma(k_1, \pi^+_{(b+\lambda)^*}k_2)$ follows immediately from the semilinearity of γ and the fact that $(b+\lambda)^* = (b^* + \overline{\lambda})$. Applying Theorem 1 to π^+ and $\mathcal{B}^+$ we conclude that for all $b \in \mathcal{B}$, $\sigma_{\mathcal{B}}(b) \cup \{0\} = \sigma_{\mathcal{B}^+}(b) = \sigma_{\mathcal{L}(\mathcal{K})}(\pi_b)$. □

6. A multiplier algebra proof of Corollary 4

Finally, we give a second proof of Corollary 4, for the case that $\mathcal{J}$ is a closed, essential two-sided ideal, which is independent of the notion of Cauchy-Schwarz function. This result may be known to the experts, but we were unable to find any reference to it in the literature. The properties of the multiplier algebra which we use can be found in [W-O].

Let $\mathcal{M}(\mathcal{J})$ be the multiplier algebra of $\mathcal{J}$. Then $\mathcal{M}(\mathcal{J})$ is a unital C*-algebra containing $\mathcal{J}$ as an essential ideal, and there is a unique, injective, $*$-homomorphism $\mu : \mathcal{A} \to \mathcal{M}(\mathcal{J})$ such that μ restricted to $\mathcal{J}$ is the identity. If we consider $\mathcal{J}$ as an algebra of bounded operators on a Hilbert space $\mathcal{H}$, and arrange matters so that the linear span, $[\mathcal{J}\mathcal{H}]$, of $\mathcal{J}\mathcal{H}$ is dense in $\mathcal{H}$, then $\mathcal{M}(\mathcal{J})$ can be identified with $\{t \in \mathcal{L}(\mathcal{H}) : t\mathcal{J} \subseteq \mathcal{J}, \mathcal{J}t \subseteq \mathcal{J}\}$. In particular we may suppose that $\mathcal{J} \subseteq \mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$.

Suppose now that $a \in \mathcal{A}$ and that L_a, the operator of left multiplication by a on $\mathcal{J}$, is a bijection of $\mathcal{J}$. Then L_a^{-1} is a bounded operator on $\mathcal{J}$, and it is easy to verify that $L_a^{-1}(jk) = (L_a^{-1}j)k$ for all $j, k \in \mathcal{J}$. We first show that there exists $t \in \mathcal{L}(\mathcal{H})$ such that $taj = j$ for all $j \in \mathcal{J}$. If $\xi_1, \ldots, \xi_n \in \mathcal{H}$ and $\xi = \sum j_\nu(\xi_\nu)$, let $t(\xi) = \sum (L_a^{-1} j_\nu)(\xi_\nu)$. (In particular, for each $j, tj = L_a^{-1}j$ so that $taj = j$.) To see that t defines a bounded operator, let $\{e_\lambda\}$ be an approximate unit for $\mathcal{J}$. Then

$$L_a^{-1}j = \lim L_a^{-1}(e_\lambda j) = \lim (L_a^{-1}e_\lambda)j.$$

Let $w_\lambda = L_a^{-1}(e_\lambda)$. We have $L_a^{-1}j = \lim w_\lambda j$ and $\|w_\lambda\| \leq \|L_a^{-1}\|$. Therefore,

$$t(\xi) = \sum \lim w_\lambda j_\nu(\xi_\nu) = \lim w_\lambda \left(\sum j_\nu(\xi_\nu)\right) = \lim w_\lambda(\xi),$$

and $\|t(\xi)\| \leq \|L_a^{-1}\| \, \|\xi\|$. Since $[\mathcal{J}\mathcal{H}]$ is dense in $\mathcal{H}$, t extends to a bounded operator on $\mathcal{H}$.

Since L_a is surjective and $taj = j$ for all $j \in \mathcal{J}$, it follows that $t\mathcal{J} \subseteq \mathcal{J}$. We next show that $jt \in \mathcal{J}$ for each $j \in \mathcal{J}$. Let $k \in \mathcal{J}$. Then $jtk = j \lim w_\lambda k = \lim (jw_\lambda)k$. Let $m = \lim(jw_\lambda)$. If $\xi = \sum k_\nu(\xi_\nu)$, then

$$jt(\xi) = \sum m k_\nu(\xi_\nu) = \left(\sum k_\nu(\xi_\nu)\right) = m(\xi).$$

Thus $jt = m$, so that $jt \in \mathcal{J}$. It follows that $t \in \mathcal{M}(\mathcal{J})$, and that for all $j \in \mathcal{J}$, $taj = j$ and $atj = aL_a^{-1}j = j$. Thus a is invertible in the unital C*-algebra $\mathcal{M}(\mathcal{J})$, hence a is invertible in $\mathcal{A}$.

References

[Cu1] R. Curto, *Fredholm and invertible n-tuples of operators*, Trans. Amer. Math. Soc. **266** (1981), 129–159. MR **82g:**47010

[Cu2] ———, *Spectral permanence for joint spectra*, Trans. Amer. Math. Soc. **270** (1982), 659–665. MR **83i:**46061

[DLW] J. Daughtry, A. Lambert, and B. Weinstock, *Operators on C^*-algebras induced by conditional expectations*, Rocky Mountain Journal of Mathematics **25** (1995), 1243–1275.

[LW] A. Lambert and B. Weinstock, *A class of operator algebras induced by probabilistic conditional expectations*, Michigan Math. J. **40** (1993), 359–376. CMP 93:14

[Pas] W. Paschke, *Inner product modules over B^*-algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468. MR **50:**8087

[R] C. E. Rickart, *Spectral permanence for certain Banach algebras*, Proc. Amer. Math. Soc. **4** (1953), 191–196. MR **14:**660e

[W-O] N. E. Wegge-Olsen, *K-Theory and C^*-Algebras: A Friendly Approach*, Oxford University Press, 1993. MR **95c:**46116

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