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THE OPTIMALITY OF JAMES'S DISTORTION THEOREMS

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ABSTRACT. A renorming of ℓ_1, explored here in detail, shows that the copies of ℓ_1 produced in the proof of the Kadec-Pełczyński theorem inside nonreflexive subspaces of $L_1[0,1]$ cannot be produced inside general nonreflexive spaces that contain copies of ℓ_1. Put differently, James's distortion theorem producing one-plus-epsilon-isomorphic copies of ℓ_1 inside any isomorphic copy of ℓ_1 is, in a certain sense, optimal. A similar renorming of c_0 shows that James's distortion theorem for c_0 is likewise optimal.

James's distortion theorems for ℓ_1, the space of absolutely summable sequences of scalars, and c_0, the space of null sequences of scalars, are well-known [J]. The former states that, whenever a Banach space contains a subspace isomorphic to ℓ_1, the Banach space contains subspaces that are almost isometric to ℓ_1. Several of the authors of this article, individually and in concert, have tried to use this feature of ℓ_1 to determine if all (equivalent) renormings of ℓ_1 fail to have the fixed point property for nonexpansive mappings (the FPP); i.e. if, in any renorming of ℓ_1, there exist a nonempty, closed, bounded and convex subset C and a nonexpansive self-map T of C without a fixed point. The basis of these attempts was to use the fact that ℓ_1 in its usual norm fails to have the fixed point property and, since each renorming of ℓ_1 contains subspaces almost isometric to ℓ_1, a perturbation of the usual example would hopefully produce a nonexpansive self-map of a nonempty, closed, bounded, convex set in any renorming of ℓ_1. Similar attempts in c_0 were also made. What appeared to be needed in these attempts were strengthened versions of James's distortion theorems.

To be specific, James's theorem for ℓ_1 states that if a Banach space X with norm $\|\cdot\|$ contains an isomorphic copy of ℓ_1, then, for each $\epsilon > 0$, there exists a sequence (x_k) in the unit sphere of X such that $(1-\epsilon)\sum_{k=1}^{\infty}|t_k| \le \|\sum_{k=1}^{\infty} t_k x_k\| \le \sum_{k=1}^{\infty}|t_k|$ for all $(t_k) \in \ell_1$. The proof of the theorem shows even more than the statement indicates. The sequence (x_k) may be chosen to have the additional property that, if (ϵ_n) is a sequence of positive numbers decreasing to 0, then for each n, $(1-\epsilon_n)\sum_{k=n}^{\infty}|t_k| \le \|\sum_{k=n}^{\infty} t_k x_k\| \le \sum_{k=n}^{\infty}|t_k|$, for all $(t_k) \in \ell_1$. That is, for each $\delta > 0$, by ignoring a finite number of terms at the beginning of the sequence (x_k), one obtains copies of ℓ_1 which are $(1+\delta)$-isomorphic to ℓ_1. This

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leads one to ask if James's distortion theorem can be strengthened in the following sense:

Question. *If X is a Banach space that contains an isomorphic copy of ℓ_1 and (ϵ_n) is a sequence of positive numbers that decreases to 0, does there exist a sequence (x_k) in the unit sphere of X such that $\sum_{k=1}^{\infty}(1-\epsilon_k)|t_k| \leq \|\sum_{k=1}^{\infty} t_k x_k\| \leq \sum_{k=1}^{\infty}|t_k|$ for all $(t_k) \in \ell_1$?*

The closed linear span of such a sequence (x_n) in the above question is called an *asymptotically isometric copy* of ℓ_1. As noted in [DL], the proof of the Kadec-Pełczyński theorem [KP] shows that nonreflexive subspaces of $L_1[0,1]$ contain such "good" copies of ℓ_1 and, in this case, there exist nonexpansive self-maps on closed, bounded and convex sets without fixed points. (This provides a converse to a theorem of Maurey [M] that every reflexive subspace of $L_1[0,1]$ has the FPP.) Thus, if every renorming of ℓ_1 were to contain an asymptotically isometric copy of ℓ_1, then every renorming of ℓ_1 would fail the fixed point property. One purpose of this article is to present a renorming of ℓ_1 which contains no asymptotically isometric copy of ℓ_1. Thus James's distortion theorem for ℓ_1 is, in this sense, optimal and the question of whether ℓ_1 can be given an equivalent norm with the fixed point property remains open. Using the predual of this renorming of ℓ_1, it will be seen that James's distortion theorem for c_0 is similarly optimal and the question as to whether c_0 can be given an equivalent norm with the fixed point property likewise remains open.

Recent papers ([CDL, DLT]) have extended the classes of spaces known to contain asymptotically isometric copies of ℓ_1. In a related paper, Smyth [S] showed that the dual of every space $C(\Omega)$, where Ω is an infinite compact Hausdorff space, fails the weak-star fixed point property with an affine contraction.

In the ensuing discussion, $\mathbb{K}$ will denote the scalar field (the real or the complex numbers) and $\mathbb{N}$ will denote the positive integers. The Banach space ℓ_1 is as usual the space of absolutely summable scalar sequences with its usual norm $\|x\|_1 := \sum_{n=1}^{\infty}|\xi_n|$, for all $x = (\xi_n) \in \ell_1$. More generally, for $p \geq 1$, the Banach space of p-summable sequences of scalars is denoted by ℓ_p and is normed by $\|x\|_p := \left(\sum_{n=1}^{\infty}|\xi_n|^p\right)^{1/p}$ for all $x = (\xi_n) \in \ell_p$. The sequence (e_n) will always denote the canonical unit vector basis in ℓ_p. Recall that $\|x\|_p \leq \|x\|_1$ for $p \geq 1$ and $x \in \ell_1$.

The space to be defined is, on the surface, quite simple. It is a countable sum of $\mathbb{K}$'s and is akin to the classical ℓ_p-spaces. There are two significant features to notice: the varying values of the exponents (similar to spaces of Nakano) and the placement of the parentheses in defining the norm. Fix a sequence $p = (p_n)$ of real numbers in $(1, \infty)$ converging to 1. Then the space we wish to define is:

$$\mathbb{K} \oplus_{p_1} (\mathbb{K} \oplus_{p_2} (\mathbb{K} \oplus_{p_3} (\mathbb{K} \oplus_{p_4} \ldots))) .$$

Let $X := \mathbb{K}^{\mathbb{N}}$. For $x = (\xi_n) \in X$, define:

$$\nu_1(p,x) := |\xi_1| \ ,$$

$$\nu_2(p,x) := \left(|\xi_1|^{p_1} + |\xi_2|^{p_1}\right)^{1/p_1} \ ,$$

$$\nu_3(p,x) := \left(|\xi_1|^{p_1} + \left(|\xi_2|^{p_2} + |\xi_3|^{p_2}\right)^{p_1/p_2}\right)^{1/p_1} ,$$

$$\nu_4(p,x) := \left(|\xi_1|^{p_1} + \left(|\xi_2|^{p_2} + (|\xi_3|^{p_3} + |\xi_4|^{p_3})^{p_2/p_3}\right)^{p_1/p_2}\right)^{1/p_1} .$$

To proceed further with this inductive construction, some notation is useful. Define the shift operator $S : X \to X$ by $Sz := (z_2, z_3, \dots, z_k, \dots)$ for all $z \in X$. For p and x as above define, for each $n \in \mathbb{N}$,

$$\nu_{n+1}(p,x) := (|\xi_1|^{p_1} + \nu_n(Sp,Sx)^{p_1})^{1/p_1} .$$

Each $\nu_n(p,\cdot)$ is a seminorm on X and, for each $x \in X$, the sequence $(\nu_n(p,x))_{n=1}^\infty$ increases to a limit $\nu_p(x)$. Clearly, for all $x \in X$, $\nu_n(p,x) \le |\xi_1| + \cdots + |\xi_n|$ for every n. Thus $\nu_p(x) \le \|x\|_1$ for each $x \in \ell_1$.

In seeking lower estimates for $\nu_p(x)$, first note that since all two-dimensional normed linear spaces are equivalent, two-dimensional ℓ_q^2 is equivalent to ℓ_1^2 and in fact, for $q \ge 1$,

$$\|(\xi_1,\xi_2)\|_q \ge 2^{-1+1/q}\|(\xi_1,\xi_2)\|_1.$$

Then, with $K_j = 2^{-1+1/p_j}$,

$$\begin{aligned}
\nu_n(p,x) &= (|\xi_1|^{p_1} + \nu_{n-1}(Sp,Sx)^{p_1})^{1/p_1} \\
&\ge K_1\,(|\xi_1| + \nu_{n-1}(Sp,Sx)) \\
&= K_1\left(|\xi_1| + \left(|\xi_2|^{p_2} + \nu_{n-2}(S^2p,S^2x)^{p_2}\right)^{1/p_2}\right) \\
&\ge K_1\left(|\xi_1| + K_2\left(|\xi_2| + \nu_{n-2}(S^2p,S^2x)\right)\right) \\
&\ge K_1\,K_2\left(|\xi_1| + |\xi_2| + \nu_{n-2}(S^2p,S^2x)\right) \\
&\;\vdots \\
&\ge K_1\,K_2\cdots K_n \sum_{j=1}^n |\xi_j|.
\end{aligned}$$

Specializing to the sequence $p = (p_j)$ where $p_j = \frac{2^j}{2^j-1}$ yields:

$$\frac{1}{2}\,\|x\|_1 \le \nu_p(x) \le \|x\|_1 \qquad \text{for all } x \in \ell_1.$$

Thus, for this specific choice of p, $\nu_p(\cdot)$ is an equivalent norm on ℓ_1.

It is clear that $\nu_p(\cdot)$ is equivalent to the ℓ_1 norm whenever p_n converges to 1 sufficiently quickly. Moreover, it is easy to determine for which sequences p the norm $\nu_p(\cdot)$ is equivalent to the ℓ_1 norm. The characterization is in terms of the dual norm $\nu_q(\cdot)$, where $q = (q_1, q_2, \dots)$ satisfies $\dfrac{1}{p_n} + \dfrac{1}{q_n} = 1$ for each $n \in \mathbb{N}$.

Proposition 1. *Let p be a sequence in $(1,\infty)$ and let q be the sequence of conjugate exponents of p. Then the following are equivalent.*

(i) $\nu_p(\cdot)$ *is equivalent to the ℓ_1 norm.*
(ii) $\nu_q(\cdot)$ *is equivalent to the c_0 norm.*
(iii) $\lim_{n\to\infty} \nu_q(1_{[1,n]}) < \infty$.
(iv) *There exists $\delta > 0$ so that for all n, $q_n^\# \ge \delta \log n$, where $(q_n^\#)$ is the increasing rearrangement of q.*

Proof. We will use the notation 1_E to denote the characteristic function of a subset E of $\mathbb{N}$. The equivalence of the first three conditions is well-known in a general context. That implication (iii) implies (iv) is straightforward. Indeed, since for each n, there are at least n values of k for which $q_k \leq q_n^{\#}$, we have for sufficiently large N that

$$\nu_q(1_{[1,N]}) \geq n^{1/q_n^{\#}}.$$

So, if we set $C := \lim_{N\to\infty} \nu_q(1_{[1,N]})$, then for all n,

$$q_n^{\#} \geq \frac{\log n}{\log C} \quad .$$

For (iv) implies (iii), let $C > 3^{1/\delta}$. Then

$$\sum_{n=1}^{\infty} C^{-q_n} = \sum_{n=1}^{\infty} C^{-q_n^{\#}} < \infty.$$

Now for $x > 0$, $s > 1$, $(1 + x^{-1})^{1/s} \leq 1 + s^{-1}x^{-1}$, and therefore $(1 + x^s)^{1/s} \leq x + s^{-1}x^{1-s}$. Hence, for each $k \leq N$

$$\nu_q(1_{[k,N]}) \leq \nu_q(1_{[k+1,N]}) + q_k^{-1}\nu_q(1_{[k+1,N]})^{1-q_k}.$$

If $\nu_q(1_{[1,N]}) \leq C$ for all $N \in \mathbb{N}$, then we are done. If $\nu_q(1_{[1,N]}) > C$, choose m so that $\nu_q(1_{[m,N]}) \geq C > \nu_q(1_{[m+1,N]})$. Then

$$\nu_q(1_{[1,N]}) \leq C + 1 + \sum_{k=1}^{m-1} q_k^{-1}\nu_q(1_{[k+1,N]})^{1-q_k} \leq C + 1 + C\sum_{k=1}^{\infty} q_k^{-1}C^{-q_k} < \infty.$$

□

The next result shows the optimality of James's theorem by proving that the above renormings of ℓ_1 fail to contain any asymptotically isometric copies of ℓ_1.

Theorem 1. *Let $p = (p_n)$ be a sequence in $(1,\infty)$, converging to 1 and such that ν_p is an equivalent norm on ℓ_1; and let (ϵ_n) be a null sequence in $(0,1)$. Then there does not exist a ν_p-normalized sequence (x_k) in ℓ_1 such that, for all $t = (t_j) \in \ell_1$,*

$$\sum_{j=1}^{\infty}(1-\epsilon_j)|t_j| \leq \nu_p\left(\sum_{j=1}^{\infty} t_j x_j\right) \leq \sum_{j=1}^{\infty}|t_j| \ .$$

Proof. Without loss of generality, assume p strictly decreases to 1. In order to obtain a contradiction, assume that there exists a null sequence (ϵ_n) in $(0,1)$ and a ν_p-normalized sequence (x_k) in ℓ_1 such that

$$(*) \qquad \sum_{j=1}^{\infty}(1-\epsilon_j)|t_j| \leq \nu_p\left(\sum_{j=1}^{\infty} t_j x_j\right) \leq \sum_{j=1}^{\infty}|t_j| \qquad \text{for all } t = (t_j) \in \ell_1 \ .$$

By passing to a subsequence of (x_n) if necessary, there is no loss of generality in assuming that

$$(**) \qquad \sum_{n=1}^{\infty} \epsilon_n < 1 \ .$$

Note also that there is no loss of generality in assuming additionally that the sequence (x_n) is disjointly supported, i.e., that the support of x_m is disjoint from the support of x_n if $m \neq n$. Indeed this is a classical gliding hump argument. Since the closed unit ball of ℓ_1 is weak-star sequentially compact with respect to the predual c_0, by passing to a subsequence, we may suppose that (x_n) converges weak-star (and so pointwise with respect to the usual basis (e_n) of ℓ_1) to some $y \in \ell_1$. By replacing (x_n) by the ν_p-normalization of the sequence $\left(\frac{x_{2j} - x_{2j+1}}{2}\right)$, we may assume that $y = 0$. As in the proof of the Bessaga-Pełczyński theorem [BP] (or see, for example [D]), by passing to a subsequence of (x_n) which is essentially disjointly supported, truncating to obtain a disjointly supported sequence, and then normalizing, yield a block basis (b_k) of (e_n) which satisfies (*). Consequently, we henceforth assume that (x_n) is disjointly supported.

Let $(m(k))_{k=0}^{\infty}$ be a strictly increasing sequence in $\mathbb{N} \cup \{0\}$ with $m(0) = 0$ and $(\xi_j)_{j=1}^{\infty}$ a sequence of scalars such that, for each $k \in \mathbb{N}$,

$$x_k = \sum_{j=m(k-1)+1}^{m(k)} \xi_j e_j \ .$$

Let N be in $\mathbb{N}$ and, in (*), set $t_j = 1$ for $j = 1, \cdots, N$ and 0 otherwise. Then, for $N \geq m(1)$:

$$\begin{aligned}
N - \sum_{j=1}^{N} \epsilon_j \leq \nu_p\left(\sum_{k=1}^{N} x_k\right)
&= \left(|\xi_1|^{p_1} + \nu_p\left(\sum_{j=2}^{m(N)} \xi_j e_j\right)^{p_1}\right)^{1/p_1} \\
&\leq \left(|\xi_1|^{p_2} + \nu_p\left(\sum_{j=2}^{m(N)} \xi_j e_j\right)^{p_2}\right)^{1/p_2} \\
&= \left(|\xi_1|^{p_2} + |\xi_2|^{p_2} + \nu_p\left(\sum_{j=3}^{m(N)} \xi_j e_j\right)^{p_2}\right)^{1/p_2} \\
&\leq \left(|\xi_1|^{p_3} + |\xi_2|^{p_3} + \nu_p\left(\sum_{j=3}^{m(N)} \xi_j e_j\right)^{p_3}\right)^{1/p_3} \\
&\ \vdots \\
&\leq \left(|\xi_1|^{p_{m(1)}} + \cdots + |\xi_{m(1)}|^{p_{m(1)}} + \nu_p\left(\sum_{j=m(1)+1}^{m(N)} \xi_j e_j\right)^{p_{m(1)}}\right)^{1/p_{m(1)}} \\
&= \left(\|x_1\|_{p_{m(1)}}^{p_{m(1)}} + \nu_p\left(\sum_{k=2}^{N} x_k\right)^{p_{m(1)}}\right)^{1/p_{m(1)}} \\
&\leq \left(\|x_1\|_{p_{m(1)}}^{p_{m(1)}} + (N-1)^{p_{m(1)}}\right)^{1/p_{m(1)}} .
\end{aligned}$$

Thus, for $N \geq m(1)$,

$$\left(N - \sum_{j=1}^{N} \epsilon_j\right)^{p_{m(1)}} - (N-1)^{p_{m(1)}} \leq \|x_1\|_{p_{m(1)}}^{p_{m(1)}} .$$

By (**), the left-hand side of the inequality tends to ∞ with N. This yields a contradiction which finishes the proof. □

In the previous proof, choosing vectors of the form $x_1 + Mx_N$, instead of $x_1 + \cdots + x_N$, also leads to a contradiction (by letting N and then M become arbitarily large).

We note here that the proof of James's distortion theorem for c_0 gives us that if a Banach space $(X, \|\cdot\|)$ contains an isomorphic copy of c_0, then for each sequence (ϵ_n) of positive numbers decreasing to 0, there exists a sequence (x_n) in the unit sphere of X such that for each n, $(1-\epsilon_n)\max_{k\geq n}|t_k| \leq \|\sum_{k=n}^{\infty} t_k x_k\| \leq (1+\epsilon_n)\max_{k\geq n}|t_k|$, for all $(t_k) \in c_0$. In order to show that James's distortion theorem for c_0 is also optimal, the construction introduced for ℓ_1 can be used as long as the sequence $p = (p_j)$ is chosen to increase sufficiently quickly to infinity. For example, with $p_j = 2^j$,

$$\|x\|_\infty \leq \nu_p(x) \leq 2\,\|x\|_\infty \qquad \text{for all } x \in c_0.$$

Thus, with this choice of p, $\nu_p(\cdot)$ is an equivalent norm on c_0. (For other choices of p, we may apply Proposition 1.)

A Banach space is said to contain an *asymptotically isometric copy of* c_0 if, for every sequence of positive numbers (ϵ_n) decreasing to 0, there exists a sequence (x_n) in the Banach space such that $\max_{n\in F}(1-\epsilon_n)|\alpha_n| \leq \|\sum_{n\in F}\alpha_n x_n\| \leq \max_{n\in F}(1+\epsilon_n)|\alpha_n|$ for all choices of scalars (α_n) and for all finite subsets F of natural numbers. (Note that $(1+\epsilon_n)$ may be replaced by 1 in this definition.) The next result provides a useful connection between the two asymptotically isometric properties.

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space that contains an asymptotically isometric copy of c_0. Then X^*, with the dual norm, contains an asymptotically isometric copy of ℓ_1.*

Proof. By hypothesis, given any null sequence (ϵ_n) in $(0,1)$, there is a sequence (x_n) in X such that for all finite sequences of scalars $(\alpha_n)_{n=1}^N$,

$$\max_{1\leq n\leq N}(1-\epsilon_n)|\alpha_n| \leq \left\|\sum_{n=1}^{N} \alpha_n x_n\right\| \leq \max_{1\leq n\leq N}|\alpha_n| \quad .$$

Let (x_n^*) be a sequence of Hahn-Banach extensions to elements of X^* of the linear functionals on the span of (x_n) that are biorthogonal to (x_n). Consider x_m^*, for some $m \in \mathbb{N}$. Then, for all vectors x of the form $\sum_{n=1}^N \alpha_n x_n$ with $N \geq m$, we have

$$\begin{aligned} |x_m^*(x)| &= |\alpha_m| = (1-\epsilon_m)^{-1}\ (1-\epsilon_m)\ |\alpha_m| \\ &\leq (1-\epsilon_m)^{-1} \max_{1\leq n\leq N}(1-\epsilon_n)\ |\alpha_n| \leq (1-\epsilon_m)^{-1}\ \|x\| \quad ; \end{aligned}$$

and hence it follows that $||x_m^*|| \leq (1-\epsilon_m)^{-1}$.

Set $x_n' := ||x_n^*||^{-1}x_n^*$ for each $n \in \mathbb{N}$. Fix a sequence $(\alpha_n)_{n=1}^N$ of scalars and let $\beta_n = \text{sign}\ \alpha_n$ for all n. Then, since $\|\sum_{n=1}^N \beta_n x_n\| \leq \max_{1\leq n\leq N}|\beta_n| = 1$, we have that

$$\left\| \sum_{n=1}^{N} \alpha_n x_n' \right\| \geq \left\langle \sum_{n=1}^{N} \alpha_n x_n', \sum_{k=1}^{N} \beta_k x_k \right\rangle = \sum_{n=1}^{N} ||x_n^*||^{-1} |\alpha_n| \geq \sum_{n=1}^{N} (1-\epsilon_n)|\alpha_n| \quad .$$

Thus, X^* contains an asymptotically isometric copy of ℓ_1. □

Combining Theorems 1 and 2, we immediately get that not every renorming of c_0 contains an asymptotically isometric copy of c_0.

Theorem 3. *Let $q = (q_n)$ be a sequence in $(1, \infty)$, diverging to ∞ and such that ν_q is an equivalent norm on c_0; and let (ϵ_n) be a null sequence in $(0, 1)$. Then there does not exist a sequence (x_k) in c_0 such that, for all $\alpha = (\alpha_j) \in c_0$,*

$$\max_{n \in F} (1-\epsilon_n)|\alpha_n| \leq \nu_q \left(\sum_{n \in F} \alpha_n x_n \right) \leq \max_{n \in F} (1+\epsilon_n)|\alpha_n|$$

for all finite subsets F of natural numbers.

Proof. It is enough to apply Theorems 1 and 2, after noting that the dual of (c_0, ν_q) is (ℓ_1, ν_p), where p is the sequence of conjugate exponents of q. □

In closing, note that other renormings of ℓ_1 exist that fail to contain asymptotically isometric copies of ℓ_1. One such norm is:

$$\|x\|' := \sup_{n \in \mathbb{N}} \gamma_n \sum_{k=n}^{\infty} |\xi_k| \ , \text{ for all } x = (\xi_n) \in \ell_1 \quad ,$$

where (γ_n) is a fixed sequence in $(0, 1)$ that strictly increases to 1. The details needed to show that ℓ_1 with this norm fails to contain asymptotically isometric copies of ℓ_1 are similar to those given for the ν_p-norm. (Although, when checking the analogue of the proof of Theorem 1 for the $\| \cdot \|'$-norm, (after assuming, without loss of generality, that (ϵ_n) decreased to 0 sufficiently fast), we used the sequence $(x_1 + N x_N)_N$ instead of $(x_1 + \cdots + x_N)_N$.)

Whether ℓ_1 endowed with either of the norms $\| \cdot \|'$ or ν_p has the fixed point property is unknown. The norm $\| \cdot \|'$, suggested to us by the referee of another paper, is interesting because of its link to the strengthening of James's distortion theorem described earlier.

Finally, let us consider c_0. It is shown in [DLT] that whenever a Banach space $(X, \|\cdot\|)$ contains an asymptotically isometric copy of c_0, it must fail the FPP. We remark that the spaces (c_0, ν_q) of Theorem 3 *also* fail the FPP. Indeed, without loss of generality, assume (q_n) increases to ∞. Then a fixed point free ν_q-nonexpansive map T on a closed, bounded and convex set C is provided by the usual c_0 example: i.e. let $C := \{x = (\xi_n) \in c_0 : 0 \leq \xi_n \leq 1 \text{ for all } n\}$ and define $T(x) := (1, \xi_1, \xi_2, \xi_3, \dots)$.

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