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THE FUNDAMENTAL LEMMA FOR $Sp(4)$

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ABSTRACT. The fundamental lemma is a conjectural identity between the orbital integrals on two reductive groups. The fundamental lemma is required for the stabilization of the trace formula and for various applications to automorphic forms. This paper proves the fundamental lemma for the group $Sp(4)$ and its endoscopic groups.

Let F be a p-adic field of characteristic zero with ring of integers O_F. Let G be an unramified reductive group defined over O_F, and let H be a standard (i.e., untwisted) unramified endoscopic group of G (also defined over O_F). Fix an embedding $\xi : {}^L H \to {}^L G$ of L-groups, which satisfies the hypotheses for unramified endoscopic data in [H6, 1]. The embedding ξ, by the Satake transform, determines a map $b : \mathcal{H}_G \to \mathcal{H}_H$ between the spherical Hecke algebras $\mathcal{H}_G$ and $\mathcal{H}_H$ on G and H. We may assume that $\mathcal{H}_G$ and $\mathcal{H}_H$ are defined relative to the hyperspecial maximal compact subgroups $G(O_F)$ and $H(O_F)$.

If $f \in C_c^\infty(G)$ and γ_G is a strongly regular element of G, we form the orbital integral $\Phi(\gamma_G, f)$ over the conjugacy class of γ_G. Similarly, for $f^H \in C_c^\infty(H)$ and strongly G-regular elements $\gamma_H \in H(F)$, we form the orbital integral $\Phi(\gamma_H, f^H)$. Various normalizations of measures enter into the definition of orbital integrals. Fixing invariant differential forms ω_G on G and ω_T on $T = C_G(\gamma_G)$ (the centralizer of γ_G) that are defined over F, we may form the orbital integrals $\Phi(\gamma_G, f)$ with respect to the measure $|\omega_G/\omega_T|$ on the orbit of γ_G. We assume that the choices ω_T and $\omega_{T'}$ for various Cartan subgroups T, T' are conjugate over the algebraic closure $\bar{F}$ of F. We make similar selections of measures on H.

Let $\Delta(\gamma_H, \gamma_G)$ be the Langlands-Shelstad transfer factor [LS2] with the canonical normalization of [H4, 7]. We form the expression

$$\Lambda(\gamma_H, f) = \sum_{\gamma_G} \Delta(\gamma_H, \gamma_G)\Phi(\gamma_G, f) - \sum_{\gamma'_H} \Phi(\gamma'_H, b(f)). \tag{1}$$

The first sum runs over representatives of all regular semisimple conjugacy classes in G and the second sum runs over representatives of the conjugacy classes in H that are stably conjugate to the class of γ_H. Both sums have only finitely many nonzero terms. Both b and Δ depend on the choice of embedding ξ.

The measures on G and H must be compatible. This is achieved by fixing a strongly G-regular element $\gamma_H \in H(F)$ such that both terms of the sum defining

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$\Lambda(\gamma_H, f)$ are nonzero for some $f_0 \in \mathcal{H}_G$. Rescale $|\omega_H|$, so as to modify the second sum of (1) by a scalar so that $\Lambda(\gamma_H, f_0) = 0$. Equivalently, we may use the normalizations of [H4, 14.2].

The following result, known as the fundamental lemma, is conjectured to hold for any unramified reductive group G in the setting described above. It is indispensable to various applications of the trace formula.

Theorem. *Suppose $G = Sp(4)$. Then $\Lambda(\gamma_H, f) = 0$ for all $f \in \mathcal{H}_G$ and all strongly G-regular elements $\gamma_H \in H(F)$.*

Remark. The fundamental lemma for $GSp(4)$ follows from this theorem together with a series of reductions made in [H6, 3.6]. The fundamental lemma for $GSp(4)$ was described in a lecture I gave at Luminy in 1992 as a consequence of the results of J.-L. Waldspurger on the homogeneity of Shalika germs and my earlier work on $GSp(4)$ [H1, Wa1, Wa2]. We require the more recent matching results of [H5] for $Sp(4)$. A double coset argument discovered by M. Schröder makes it possible to give another proof of the fundamental lemma for $GSp(4)$ [S1, S2]. The details of this approach have been carried out in a series of preprints by R. Weissauer [We].

A slightly stronger form of the theorem actually holds. A general argument of Langlands and Shelstad shows that γ_H can be taken to be any (G, H)-regular element [LS3, 2.4].

Proof. We will begin with a general reductive group G and will introduce as hypotheses the parts of the proof of the fundamental lemma that have been verified only in special cases. Our approach to the fundamental lemma is by induction on the dimension of the group. It is therefore natural to assume that the fundamental lemma is known for groups of smaller dimension.

Let K be any number field that has a completion at some place v isomorphic to F. Let G' be any reductive group over K with endoscopic group H' such that at v the pair (G', H') is equivalent to a proper Levi factor of G together with the endoscopic group obtained from H by descent, or to the connected centralizer of a noncentral absolutely semisimple element in G together with the endoscopic group obtained from H by descent. Absolutely semisimple elements, also called $\mathbb{F}_q$-semisimple elements, are defined in [K]. See also [H4, 3].

Hypothesis 1. For any of the reductive groups G' and endoscopic groups H' obtained as above, the fundamental lemma is true at almost all places of K.

A narrower formulation of this hypothesis is possible. It is sufficient to verify the hypothesis for a single carefully chosen number field K (depending on F, G, and H). Further, the fundamental lemma can be reduced to the adjoint group of G', provided unramified quasicharacters of G'_{adj} are introduced. See [H6, 3.6]. In the case of $Sp(4)$ the centralizers and Levi factors all have ranks at most one, so the adjoint group, if not trivial, is $PGL(2)$. This special case of the fundamental lemma has several proofs, for example, [K].

Hypothesis 1 allows us to apply the results of [H6]. In particular, we may assume that the residual characteristic of F is as large as we please. It also follows from [H6] that it is sufficient to prove that $\Lambda(\gamma_H, 1_G) = 0$, where 1_G is the unit element of the Hecke algebra $\mathcal{H}_G$. Of course, $b(1_G)$ is then the unit element of $\mathcal{H}_H$.

By applying descent to Levi factors, we may assume that H is elliptic [H4, 12].

Consider a strongly G-regular element $\gamma_H \in G(F)$. If γ_H is not topologically unipotent, Kazhdan's lemma may be invoked to reduce to a lower-rank case. The proof of this in [H4, 13] makes the assumption that G_{der} is simply connected. This restriction is unnecessary: the result is proved in [Ko, 7.1] without that restriction. Thus, we assume that γ_H is topologically unipotent.

Let $\mathfrak{g}_{tn}$ be the set of topologically nilpotent elements of the Lie algebra of G. Let G_{tu} be the set of topologically unipotent elements of G. Similarly, we consider $\mathfrak{h}_{tn} = \mathrm{Lie}(H)_{tn}$ and H_{tu}. If the residual characteristic is sufficiently large, then the exponential map gives an analytic isomorphism between $\mathfrak{g}_{tn}$ and G_{tu}.

Let $\mathfrak{h}' \subset \mathfrak{h}$ be the set of strongly G-regular elements in $\mathfrak{h}$. In everything that follows, it is understood that $X_H \in \mathfrak{h}'$. If $X_H \in \mathfrak{h}'_{tn} = \mathfrak{h}' \cap \mathfrak{h}_{tn}$, the transfer factor $\Delta(\exp(X_H), \gamma_G)$ is zero unless γ_G is also topologically unipotent. So we may pass to the Lie algebras of G and H.

Hypothesis 2. Suppose that the residual characteristic is sufficiently large. For $|t| \leq 1$, and $X_H \in \mathfrak{h}'_{tn}$ fixed, $\Lambda(\exp(t^2 X_H), 1_G)$ is a polynomial in $|t|$.

It is known that for X_H fixed, the transfer factor for $\exp(t^2 X_H)$ is a constant times a power of $|t|$, if the residual characteristic is not two [H4]. Hypothesis 2 is then true for the classical groups by the results of Waldspurger, which express orbital integrals of the unit element on the topologically unipotent set as a sum of terms, each of which is homogeneous, that is, a monomial in $|t|$. See [Wa1], [Wa2, V] for a precise description of the class of groups treated and the restrictions on the residual characteristic. As a very special case of Waldspurger's results, we obtain Hypothesis 2 for $Sp(4)$.

Any polynomial in $|t|$ is zero if it is zero for $|t|$ sufficiently small. Hypothesis 2 allows us to restrict our attention to a small neighborhood of the identity where the Shalika germ expansion is valid. Consider the Shalika germ expansion of the first sum defining Λ when X_H is sufficiently small in $\mathfrak{h}'$. Set $D(X_H) = |\prod_\alpha \alpha(X_H)|^{1/2}$, with the sum running over the roots of $\mathfrak{h}$ with respect to a Cartan subalgebra containing X_H. We have, for $f \in C_c^\infty(G)$,

$$D(X_H) \sum_{X_G} \Delta(\exp(X_H), \exp(X_G)) \Phi(\exp(X_G), f) = \sum_O \Gamma_O^G(X_H) \mu_O(f),$$

for some collection of functions $\Gamma_O^G(X_H)$ defined in a suitable neighborhood of 0 in $\mathfrak{h}'_{tn}$. Up to the factor D, the left-hand side is the first sum in (1) defining Λ. The functions Γ_O^G are combinations of Shalika germs on G. The sum on the right runs over unipotent classes in G. We consider this as an identity which defines Γ_O^G in a small neighborhood of 0 in $\mathfrak{h}'$, and then we extend the functions Γ_O^G to all of $\mathfrak{h}'$ by homogeneity.

Similarly, we consider a *stable* Shalika germ expansion on $\mathfrak{h}'_{tn}$. For X_H sufficiently small, and $f^H \in C_c^\infty(H)$, write the second sum in (1) in the form

$$D(X_H) \sum_{X'_H} \Phi(\exp(X'_H), f^H) = \sum_{O'} \Gamma_{O'}^H(X_H) \mu_{O'}(f^H),$$

where the sum on the right now runs over unipotent conjugacy classes of H. The functions $\Gamma_{O'}^H$ are stable versions of Shalika germs. We extend these functions by homogeneity to $\mathfrak{h}'$.

Hypothesis 3. There exists a linear map $b_0 : C_c^\infty(G) \to C_c^\infty(H)$, $f \mapsto b_0(f)$ satisfying

$$\sum_O \Gamma_O^G(X_H) \mu_O(f) = \sum_{O'} \Gamma_{O'}^H(X_H) \mu_{O'}(b_0(f)),$$

for all $X_H \in \mathfrak{h}'$.

This hypothesis is called *local Δ-transfer at the identity* in [LS3]. It has been verified for $GSp(4)$ in [H1] and for $Sp(4)$ in [H5] when the residual characteristic is odd. Hypothesis 3 is stronger than what is required for the fundamental lemma.

For each X_H, the right-hand side of the identity of Hypothesis 3 defines an invariant distribution on $C_c^\infty(H)$, which is supported on the unipotent set, namely

$$f^H \mapsto \sum_{O'} \Gamma^H_{O'}(X_H)\mu_{O'}(f^H). \tag{2}$$

Let $\mathcal{L}$ be the span of these distributions. The linear map b_0 gives, by duality, a linear map $\mu \mapsto \mu^G$ from $\mathcal{L}$ to invariant distributions supported on the unipotent set of G. Specifically, for $\mu \in \mathcal{L}$, define μ^G by $\mu^G(f) = \mu(b_0(f))$, for $f \in C_c^\infty(G)$.

Hypothesis 4. $\mu^G(1_G) = \mu(b(1_G))$, for all $\mu \in \mathcal{L}$.

If X_H is not elliptic, the unipotent distribution μ on $C_c^\infty(H)$ defined by (2) satisfies the condition of Hypothesis 4 by descent and induction (Hypothesis 1). See [H4, 12]. Call these distributions *stably induced.*

Proposition. *If Hypotheses 1, 2, 3, and 4 hold, then the fundamental lemma is true for G and the endoscopic group H.*

We emphasize that all four hypotheses are assertions about what happens almost everywhere for various number fields K. In Hypothesis 1, this is explicit. After Hypothesis 1, we shifted notation, so that F is no longer a fixed p-adic field for which we wish to prove the fundamental lemma. It becomes an indefinite p-adic field of sufficiently large residual characteristic, obtained by completing various number fields K.

Proof. It is clear by the preceding remarks that the first two hypotheses reduce the fundamental lemma to a statement in a small neighborhood of the identity. We must verify that $b_0(1_G)$ has the same stable germ expansion as $b(1_G)$, that is, that $\mu(b_0(1_G)) = \mu(b(1_G))$ for all $\mu \in \mathcal{L}$. Hypothesis 4 asserts this, and the result follows. □

A unipotent class O is *r-regular* if $2r = \dim(C_G(u)) - \mathrm{rank}(G)$, for $u \in O$. The 0-regular classes are regular, and the 1-regular classes are subregular. We call the partial sum

$$\sum_{O} \Gamma^G_O(X_H)\mu_O(1_G),$$

where the sum ranges over all r-regular unipotent classes O of G, the *r-regular term* on G. Similarly, the corresponding partial sum

$$\sum_{O'} \Gamma^H_{O'}(X_H)\mu_{O'}(b(1_G))$$

over r-regular orbits O' will be called the *stable r-regular term* on H. This distinction between *term* and *germ* is crucial. To verify Hypothesis 4, we must show that the r-regular terms on G and H coincide for a finite set of elements $\{X_H\} \subset \mathfrak{h}'$ whose distributions (2) span $\mathcal{L}$.

Now we restrict our attention to the group $G = Sp(4)$. We may assume that the residual characteristic is odd. The elliptic unramified endoscopic groups of G are $SO(4)$, the quasisplit form $SO^*(4)$ of $SO(4)$ that splits over an unramified quadratic extension E/F, and the product $SL(2) \times U_E(1)$, where $U_E(1)$ denotes a one-dimensional nonsplit torus that is split by E, again with E/F an unramified quadratic extension. When $H = SO(4)$, the regular and both subregular classes give stably induced distributions μ, so that, by our earlier comment, Hypothesis 4 is satisfied for them. The remaining unipotent conjugacy class of $SO(4)$ is treated in Lemma 1. When $H = SO^*(4)$, the regular class is stably induced. There is no subregular class with rational points. Again, it is enough to consider the two-regular class. If $H = SL(2) \times U_E(1)$, the regular class is stably induced. It is necessary to consider the subregular unipotent class. (In this case, the two-regular germs vanish [H5].)

We take an image of X_H in $Sp(4)$ and let $\pm t_1$ and $\pm t_2$ in $\bar{F}$ be the eigenvalues of the image in $Sp(4) \subset GL(4)$.

The two-regular term in $GSp(4)$ is a product of three factors:

(i) The constant $A_1(M)$ of Langlands [L, page 470]

(ii) The Shalika germ. Formulas appear in [H1]–[H6]

(iii) The unipotent orbital integral $\mu_O(1_G)$ computed relative to the measure $|\omega| = |z\, dz\, dy_1\, dy_2\, dy_3|$ with coordinates on an open set of O:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ y_1 & 1 & 0 & 0 \\ y_2 & 0 & 1 & 0 \\ y_3 & y_2 & -y_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -y_1 & 1 & 0 & 0 \\ -y_2 & 0 & 1 & 0 \\ -y_3 & -y_2 & y_1 & 1 \end{pmatrix} \subseteq O. \tag{3}$$

Here $Sp(4)$ is represented as 4×4 matrices preserving the form $x_1 \wedge x_4 + x_2 \wedge x_3$.

The constant $A_1(M)$ evaluates, by the formula of Langlands, to $1/2$. (This factor of $1/2$ stems from the fact that the irreducible divisor that gives the contribution to the two-regular term has multiplicity 2 in the Igusa variety.)

We define a constant $[G]$, for any reductive group G over a finite field k, to be the cardinality of $G(k)$ divided by q^d, where $d = \dim(G)$, and q is the cardinality of k. For example, the multiplicative group $\mathbb{G}_m$ gives $[\mathbb{G}_m] = (1 - 1/q)$, $[SO(4)] = (1 - 1/q^2)^2$, and $[Sp(4)] = (1 - 1/q^2)(1 - 1/q^4)$. If G is defined over F, we also let $[G]$ denote the corresponding constant over the residue field k of F. As in [H4, 14], we normalize the unit element of the Hecke algebra of (any) G to be $ch/[G]$, where ch is the characteristic function of $G(O_F)$.

If $G = SL(2)$, then the stable subregular term is the product of the Shalika germ and the subregular orbital integral. The subregular orbital integral is merely $f \mapsto f(1)$. Let $t = t(X)$ be an eigenvalue of $X \in \mathfrak{sl}(2)$. By [LS1], the subregular term evaluated at X is then

$$|t| I_F / [SL(2)],$$

where the constant I_F is expressed as a principal-value integral

$$I_F = \int_{Q(F)} |\omega_Q|.$$

The surface Q is a twisted form of a product of two projective lines. The volume form ω_Q is $da \wedge db/(a - b)^2$, for an appropriate choice of coordinate a, b. See [LS1] for details. The surface Q and integral depend on a choice of Cartan subalgebra containing X, although the notation does not show this.

Since $\mathfrak{so}(4)$ is isomorphic to a direct sum of two copies of $\mathfrak{sl}(2)$, we may also conclude from the $SL(2)$ calculation that the stable two-regular term of $SO(4)$, evaluated at $X_H \in \mathfrak{so}(4)'$, is

$$|t_1^2 - t_2^2| I_F^2 / [SO(4)]. \tag{4}$$

Similarly, since $\mathfrak{so}^*(4)$ is a restriction of scalars of $\mathfrak{sl}(2)$ over E, we conclude that the stable two-regular term of $SO^*(4)$, evaluated at $X_H \in \mathfrak{so}^*(4)'$, is

$$|t_1^2 - t_2^2| I_E/[SO^*(4)]. \tag{5}$$

($t_1 - t_2$ and $t_1 + t_2$ are to be identified with the positive roots of $\mathfrak{so}(4)$.) For similar reasons the stable subregular term of $SL(2) \times U_E(1)$ at X_H is

$$|t_2| I_F/[SL(2) \times U_E(1)]. \tag{6}$$

($2t_2$ is to be identified with the positive root of $\mathfrak{sl}(2) \times \mathfrak{u}_E(1)$.)

To prove Hypothesis 4 we must show that these expressions coincide with the corresponding terms in $Sp(4)$. We begin with the endoscopic group $SO(4)$.

Lemma 1. *If $H = SO(4)$, then the two-regular term of G, for X_H elliptic, is $|t_1^2 - t_2^2| I_F^2/[SO(4)]$.*

Proof. For this endoscopic group we may rely on the results of [H1]. In $GSp(4)$ there is a single two-regular unipotent conjugacy class.

It follows from Hypothesis 3 and Expression 4 that the formula is correct up to a nonzero scalar. To check the scalar, we pick any convenient elliptic element X_H. Consider the Cartan subgroup $(U_E(1) \times U_E(1))/\{\pm 1\}$ in H, with E/F unramified. Fix X_H in its Lie algebra.

In the notation of [H1, H6], we set $w' = w_2 \xi_1 \epsilon / f$, where ϵ is a unit in E of trace zero. Also set $\sigma = \sigma_0 \sigma_\alpha \sigma_\beta \sigma_\alpha \sigma_\beta$. Let η be the unramified quadratic character of $F^\times$. Equation 6.8 of [H1] gives the formula

$$|t_1^2 - t_2^2| \int |dw'/w'||d\ell_2/\ell_2| \eta(1 - \ell_2^2) \int_{Q(F)} |d\xi_1 d\xi_2/(\xi_1 - \xi_2)^2|,$$

for the Shalika germ, where σ acts on the coordinates by $\sigma(w') = 1/w'$, $\sigma(\ell_2) = \ell_2$, $\sigma(\xi_1) = -1/\xi_2$, and $\sigma(\xi_2) = -1/\xi_1$. All integrals extend over $\mathbb{P}^1$ unless indicated otherwise. The second integral is I_F. Section 11 of [H2] shows that the first integral is equal to $2I_F$. The germ is then $2I_F^2|t_1^2 - t_2^2|$.

The integral $\mu_O(1_G)$ is computed by the method of [H2, 12]. Details of closely related calculations are also found in [H3, 3.9], so we will simply state the result. We find

$$\mu_O(1_G) = (1 - q^{-2})^{-2} = 1/[SO(4)].$$

The product of the factors (i), (ii), and (iii) is then

$$|t_1^2 - t_2^2| I_F^2/[SO(4)],$$

as desired. This completes Lemma 1. Since this term coincides with the stable two-regular term (4) of H, this also completes the proof of the fundamental lemma for the endoscopic group $SO(4)$. $\square$

Now we turn to the two-regular term in the nonsplit case $H = SO^*(4)$. Again it is the product of three factors: the constant $A_1(M) = 1/2$, the Shalika germ, and a unipotent orbital integral $\mu_O^\kappa(1_G)$. The germ is given by [H5, 5.2]. It is $2|t_1^2 - t_2^2| I_E$. Again, by the methods of [H2], the unipotent orbital integral evaluates to

$$\mu_O^\kappa(1_G) = (1 - 1/q^2)/[Sp(4)] = 1/[SO^*(4)].$$

(As in [H5], the relevant measure to take is

$$\mu_O^\kappa(1_G) = \int_O \eta(z)|\omega|,$$

where z and $|\omega|$ are as in (3).) This is a κ-orbital integral on the two-regular conjugacy class in $Sp(4)$. The result is two-regular term on G

$$|t_1^2 - t_2^2| I_E/[SO^*(4)].$$

This is precisely the stable two-regular term (5) on $SO^*(4)$.

Finally, we turn to the endoscopic group $H = SL(2) \times U_E(1)$. We must compare the subregular terms of G and H. The subregular term on $Sp(4)$ comes from a single $GSp(4)$-orbit (which breaks into two orbits in $Sp(4)$). Recall that the subregular unipotent orbits in $GSp(4)$ are parametrized by quadratic extensions. The $GSp(4)$-orbit that enters here is the one that corresponds to the unramified quadratic extension E/F. We label the two orbits in $Sp(4)$ as O_+ and O_-. We consider the germ for the Cartan subgroup $U_E(1) \times U_E(1)$ of $Sp(4)$. In the notation of [H1, page 230], the Shalika germ is

$$\int |d\xi/\xi| \int \eta(w^2/w_B)|dw/w^2|.$$

Under the substitution $w = x/(t_2 x + t_2)$, this becomes

$$|t_2| \int |d\xi/\xi| \int \eta(1 - x^2)|dx/x^2| = 2|t_2| I_F.$$

(The last equality follows from [H2, 11].) The subregular term on G is then

$$2|t_2| I_F[\mu_{O_+}(1_G) - \mu_{O_-}(1_G)].$$

Rather than evaluate these unipotent orbital integrals directly, we note that with a different normalization of measures, they have already been evaluated by Assem [A, 2.3.17]. We add primes to the distributions (and the characteristic function 1_G) to indicate Assem's normalizations. Assem's result is

$$\mu'_{O_+}(1'_G) - \mu'_{O_-}(1'_G) = \frac{1 - q^{-1}}{2(1 + q^{-1})} = \frac{[\mathbb{G}_m]}{2[U_E(1)]}. \tag{7}$$

(The orbits O_+ and O_- are denoted $(\epsilon, 1)$ and $(\epsilon, -1)$ in [A].) To compare the normalizations of measures, we work with a linear combination that is easier to treat. The sum over all the subregular unipotent classes is the Richardson orbit of a parabolic subgroup. The corresponding distribution $\sum \mu_O$ is easy to compute. We have

$$\frac{\sum \mu_O(1_G)}{\sum \mu'_O(1'_G)} = \frac{\mu_{O_+}(1_G) - \mu_{O_-}(1_G)}{\mu'_{O_+}(1'_G) - \mu'_{O_-}(1'_G)}. \tag{8}$$

Assem finds that $\sum \mu'_O(1'_G) = 1$. By parabolic descent we find $\sum \mu_O(f) = \bar{f}^P(1)$, where $\bar{f}^P$ is the usual function on the Levi of $P = MN$ obtained by descent. For the function $f = 1_G$, this gives $\bar{f}^P(1) = 1/[M] = 1/[SL(2) \times \mathbb{G}_m]$. Thus, (7) and (8) give

$$\mu_{O_+}(1_G) - \mu_{O_-}(1_G) = \frac{1}{2} \frac{1}{[SL(2) \times \mathbb{G}_m]} \frac{[\mathbb{G}_m]}{[U_E(1)]} = \frac{1}{2[H]}.$$

The subregular term of G is then $|t_2| I_F/[H]$. This is precisely the stable subregular term (6) on H. The proof of the fundamental lemma for $Sp(4)$ is complete. $\square$

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