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SEMICONSTANT MEASURES ON HYPERBOLIC LOGICS

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Abstract. We characterize the set of all semiconstant measures on the hyperbolic logics of projections in indefinite metric spaces and describe the set of all probability measures on these logics.

1. Introduction

One of the basic problems related to the propositional calculus approach to the foundations of quantum mechanics [1] is the description of probability measures (called states in physical terminology) on the set of experimentally verifiable propositions regarding a physical system. The set of propositions form an orthomodular partially ordered set, where the order is induced by a relation of implication, and is called a *quantum logic*.

An important intepretation of a quantum logic is the set Π of all orthogonal projections of a Hilbert space H. The remarkable Gleason theorem [2] asserts: *Let H be a Hilbert space, $dimH \geq 3$ and let $\mu : \Pi \to R$ be a probability measure. There exists a positive trace class operator T such that* $\mu(p) = tr(Tp)$, $\forall p \in \Pi$.

The problem of the construction of a quantum field theory leads to the indefinite metric spaces [3]. In this case, the set $\mathcal{P}$ of all J-orthogonal projections serves to be an analog to the logic Π. There is an indefinite analog to the Gleason theorem [4]. *Let H be a J-space, $dim \geq 3$ and let $\mu : \mathcal{P} \to R$ be an indefinite measure. There exist a J-selfadjoint trace class operator T and a semiconstant measure μ_0 such that $\mu(p) = tr(Tp) + \mu_0$, $\forall p \in \mathcal{P}$.*

2. Preliminary notions

We present the necessary definitions and notations. Let H be a space with an indefinite metric $[.,.]$, a canonical decomposition $H = H^+[\dot{+}]H^-$, and a canonical symmetry J. Following the terminology of [5], H is a Krein space (sometimes H is called a *J-space*). H is a Hilbert space with respect to the inner product (see [5]) $(z, y) = [Jz, y]$. There exist orthogonal projections P^+ and P^- such that $P^+ + P^- = I$, $J = P^+ - P^-$ and $P^+H = H^+$, $P^-H = H^-$, $[z, y] = (Jz, y)$, $\forall z, y \in H$. Let $p \in B(H)$. It is easy to see that $[pz, y] = [z, py]$, $\forall z, y \in H \Leftrightarrow p = Jp^*J$.

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A vector $z \in H$ is said to be *positive* (*negative*) if $[z,z] > 0$ ($[z,z] < 0$). The set $\Gamma \equiv \{f \in H : [f,f]^2 = 1\}$ (=$\{f \in H : [f,f] = 1$ or $[f,f] = -1\}$) is an analog to the unit sphere $S = \{f \in H : (f,f) = 1\}$. For any operator p we denote by e_p the orthogonal projection onto the subspace $\overline{pH}$.

A W^*-factor $\mathcal{A}$ in H is called a *W^*J-factor*, if $J \in \mathcal{A}$. Let Π be the set of all orthogonal projections in $\mathcal{A}$. Let $\mathcal{P}$ (=$\mathcal{P}(\mathcal{A})$) be the set of all J-selfadjoint projections in $\mathcal{A}$, i.e., $\mathcal{P} = \{p \in \mathcal{A}\colon\ p^2 = p,\ [pz,y] = [z,py],\ \forall z,y \in H\}$.

Denote by $\mathcal{P}^+$ ($\mathcal{P}^-$) the set of all projections $p \in \mathcal{P}$, for which the subspace pH is positive, i.e., ($\forall z \in pH,\ z \neq 0,\ [z,z] > 0$) (respectively, negative, i.e., $\forall z \in pH$, $z \neq 0, [z,z] < 0$). Any projection $e \in \mathcal{P}$ is representable in the form $e = e_+ + e_-$, where $e_+ \in \mathcal{P}^+$, $e_- \in \mathcal{P}^-$. Note that $p \in \mathcal{P}^+ \Rightarrow pJ \geq 0$ and $p \in \mathcal{P}^- \Rightarrow pJ \leq 0$. Let p be a projection. Then $p = e_p p = e_p + e_p p e_p^\perp$. Hence

$$pp^* = e_p + e_p p e_p^\perp p^* e_p \geq e_p.$$

Thus $(pp^*)^{1/2} \geq e_p^{1/2} = e_p$. Now it is clear that if $p \in \mathcal{P}^+$ then $e_p \leq (pp^*)^{1/2} = (pJJp^*)^{1/2} = (pJpJ)^{1/2} = pJ$ and if $p \in \mathcal{P}^-$ then $e_p \leq -pJ$. Let $x \in \mathcal{A}$ be such that $x \geq e_x$ and $e_x \leq P^+$, and let v be an arbitrary isometry with the initial projection e_x and the final one $vv^* \leq P^-$. Define an operator

$$p = x + v(x^2 - x)^{1/2} - (x^2 - x)^{1/2} v^* - v(x - e_x)v^*. \tag{1}$$

It can be easily verified that $p^2 = p$ and $Jp^*J = p$. Hence $p \in \mathcal{P}$. Since $p^*Jp \geq 0$, it follows that $[pz,pz] = (Jpz,pz) = (p^*Jpz,z) \geq 0$, $\forall z \in H$. Thus $p \in \mathcal{P}^+$. (It can be proved [6] that any projectin $p \in \mathcal{P}^+$ has the form (1) where $x = P^+pP^+$ and v is the partial isometry from the polar decomposition $P^-pP^+ = v|P^-pP^+|$.) In order to emphasize the fact that p from (1) depends on x and v, we shall use the notation $p(x,v)$ as well.

A projection $p \in \mathcal{P}^+$ ($p \in \mathcal{P}^-$) is said to be *maximal positive* (*maximal negative*) if the subspace pH is the maximal positive (maximal negative) subspace. Let $\mathcal{P}_m^+$ ($\mathcal{P}_m^-$) be the set of all maximal positive (maximal negative) projections in $\mathcal{P}$. By [5, Theorem 45, §4], $p \in \mathcal{P}_m^+ \Leftrightarrow p \in \mathcal{P}^+$ and $P^+pH = H^+$ ($p \in \mathcal{P}_m^- \Leftrightarrow p \in \mathcal{P}^-$ and $P^-pH = H^-$). It is clear that $p \in \mathcal{P}_m^+ \Leftrightarrow p^\perp \in \mathcal{P}_m^-$. Let $r \in \mathcal{P}_m^+$. By the definitions (see Definitions 2.1 and 2.2 in [5, §2, Chapter I]), the operator $J_r = r - r^\perp$ is a canonical symmetry with respect to the canonical decomposition $H = H_r^+ [\dot{+}] H_r^-$, where $H_r^+ \equiv rH$ and $H_r^- \equiv r^\perp H$. Let $\mathcal{A}$ be a W^*J-factor. Then $\mathcal{A}$ is a W^*J-factor of the same type in the Hilbert space H with the inner product $(z,y)_r \equiv [J_r z, y]$.

With respect to the standard relations, an ordering $p \leq q \Leftrightarrow pq = qp = p$, the orthocomplementation $p \to p^\perp \equiv I - p$, and the orthogonal relation $p \perp q \Leftrightarrow pq = 0$, the set $\mathcal{P}$ is a *quantum logic*. Any one-dimensional projection from $\mathcal{P}(B(H))$ can be represented in the form $p_f = [f,f][.,f]f$, $f \in \Gamma$. Evidently, p_f is an orthogonal projection $\Leftrightarrow f \in S \cap \Gamma$ (=$S \cap (H^+ \cup H^-)$).

The logic $\mathcal{P}$ is not a σ-logic.

We will construct a sequence of mutually orthogonal projections $\{e_n\}_1^\infty \subset \mathcal{P}$ such that the supremum $\sum e_n$ does not exist in $\mathcal{P}$.

Let H be such that P^+H and P^-H are infinite-dimensional separable Hilbert spaces. Let $\{\phi_n^+\}_1^\infty$, $\{\phi_n^-\}_1^\infty$ be orthonormal bases in P^+H, P^-H. Put $f_n \equiv (n+1)^{\frac{1}{2}}\phi_n^+ + n^{\frac{1}{2}}\phi_n^-$. Then $[f_n,f_n] = (Jf_n,f_n) = 1$, $p_{f_n} \in \mathcal{P}^+(B(H))$, and $\{p_{f_n}\}_1^\infty$ is an orthogonal sequence. $lin\{f_n, n \in N\}$ and $L = \overline{lin\{f_n, n \in N\}}$ are positive

linear subspaces (see Definition 1.6 in [5, §1, Chapter I]). By [5, Theorem 4.7, §4, Chapter I], L is a maximal positive subspace. Since $[f_n, f_n](f_n, f_n)^{-1} \to 0$ by [5, Theorem 5.7, §5, Chapter I], L is not a Krein space, with respect to the product $[.,.]$. This means that there is no projection $p \in \mathcal{P}$ such that $pH = L$.

Assume now that the supremum $\sum\limits_1^\infty p_{f_n} \in \mathcal{P}$ exists. Then $L \subseteq \sum\limits_1^\infty p_{f_n} H$. Put

$$p_m \equiv \sum_1^m p_{f_n} + \sum_{m+1}^\infty [(., \phi_n^+)\phi_n^+ + (., \phi_n^-)\phi_n^-].$$

It is clear that $p_m \in \mathcal{P}$, $p_m \geq p_{m+1}$, $\forall m$, and $p_m \geq p_{f_n}$, $\forall m, n$. Hence $p_m \geq \sum\limits_1^\infty p_{f_n}$, i.e. $p_m H \supseteq (\sum\limits_1^\infty p_{f_n})H$, $\forall m$, and $L = \cap_1^\infty p_m H \supseteq (\sum\limits_1^\infty p_{f_n})H$. Thus $L = (\sum\limits_1^\infty p_{f_n})H$. This is a contradiction.

Any sum $e = \sum e_\iota$, for $e_\iota \in \mathcal{P}$, $e_\iota \perp e_j$ $(i \neq j)$ is said to be decomposition of e (the sum is understood in the strong sense). A mapping $\mu : \mathcal{P} \to R$ is called a *measure* if $\mu(e) = \sum \mu(e_\iota)$ for every decomposition $e = \sum e_\iota$

Here, the convergence of an uncountable family of summands means that there exists only a countable set of nonzero terms in the family and the usual series with these summands converges absolutely.

A nonnegative measure μ is said to be a *probability* if $\mu(I) = 1$. A measure μ is said to be *indefinite* if $\mu/\mathcal{P}^+ \geq 0$ and $\mu/\mathcal{P}^- \leq 0$; *semiconstant* if there exist a faithful normal semifinite trace τ on $\mathcal{A}$ and a number t such that $\mu(e) = t\tau(e_+)$, $\forall e \in \mathcal{P}$ or $\mu(e) = t\tau(e_-)$, $\forall e \in \mathcal{P}$.

3. The main results

Let $e, f \in \Pi$. We write $e \sim f$ if there exists a partial isometry $v \in \mathcal{A}$ with the initial projection e and the final one f. We denote by e_p^+ the orthogonal projection onto the subspace P^+pH $(= P^+e_pH)$, $\forall p \in \mathcal{P}^+$.

Theorem 1. *Any measure $\mu : \mathcal{P} \to R$ in the W^*J- factor $\mathcal{A}$ is constant on the set of all maximal positive (negative) projections if and only if the measure μ is representable as a sum of semiconstant measures.*

Proof. Let a measure μ be a sum of semiconstant measures. Then, obviously, μ is constant on the set of all maximal positive projections.

Conversely, let $\mu : \mathcal{P} \to R$ be constant on the set $\mathcal{P}_m^+$. Hence μ is constant on the set $\mathcal{P}_m^-$. For any $p \in \mathcal{P}^+$ the projections P^+ and $P^+ - e_p^+ + p$ are maximal positive. Hence

$$\mu(P^+ - e_p^+) + \mu(e_p^+) = \mu(P^+)$$

$$= \mu(P^+ - e_p^+ + p) = \mu((P^+ - e_p^+) + \mu(p).$$

Therefore,

$$\mu(e_p^+) = \mu(p), \forall p \in \mathcal{P}^+. \tag{2}$$

1. For the algebra $\mathcal{A}$ first assume that $\mathcal{A} = B(H)$. If $dimH = 2$, then Theorem 1 is trivial. Now assume that $dimH > 2$. For any vector $z \in S \cap H^-$ we set $L_z^- = \{p_f \in \mathcal{P}\colon (P^- - p_z)f = 0\}$. In a similar way, for any vector $y \in S \cap H^+$ we can set $L_y^+ = \{p_f \in \mathcal{P}\colon (P^+ - p_y)f = 0\}$. The equality (2) gives that $\mu \equiv const$

($\equiv c_+(z)$) on the set $L_y^+ \cap \mathcal{P}^+$ and $\mu \equiv const$ ($\equiv c_-(z)$) on the set $L_z^- \cap \mathcal{P}^-$. Now let the vector $r \in S \cap H^-$ be such that $(r,z) \neq 0$ and $p_r \neq p_z$. We can assume that $(r,z) \in R\backslash\{0\}$. Choose a negative vector $f \in lin_R\{r,z,y\}$ such that $p_f \in L_r^-$ and the real line $lin_R\{f,z\}$ contains at least one positive vector. Then there exists a positive vector $g \in lin_R\{f,z\}$ such that $p_g \in L_y^+$ and $(P^-g,z) = 0$. This means that $p_g p_z = p_z p_g = 0$. Then $p_z + p_g \in \mathcal{P}$ and $p_z + p_g - p_f \in L_y^+$. We have

$$c_+(y) = \mu(p_z + p_g - p_f) = \mu(p_z) + \mu(p_g) - \mu(p_f)$$

$$= c_-(z) + c_+(y) - c_-(r).$$

Hence $c_-(z) = c_-(r)$ and, therefore, the function $z \to c_-(z)$, $z \in S \cap H^-$, is constant. Similarly, the function $y \to c_+(y)$, $y \in S \cap H^+$, is constant, too. Thus the measure μ is constant $\equiv c_+$ ($\equiv c_-$) on the set of all one-dimensional positive (negative) projections. This means that

$$\mu(e) = c_+ tr(e_+) + c_- tr(e_-), \forall e \in \mathcal{P}, \text{ where } (0 \cdot \infty \equiv 0).$$

2. Now let $\mathcal{A}$ be a continuous W^*J-factor. Choose e and $f \in \Pi$ such that $e + f \leq P^+$ and $e \sim f$. Let $v \in \mathcal{A}$ be a partial isometry with the initial projection e and the final one f. Assume there is a partial isometry $w \in \mathcal{A}$ with the initial projection e and the final one $ww^* \leq P^-$. The minimal *-algebra $\mathcal{A}(e,w,v)$ generated by the operators e, v, w is a W^*J-factor of type I_3 (acting in the Hilbert space $(e + p + vv^*)H$ with the canonical symmetry $J' = e + f - vv^*$). For any positive projection p maximal in $\mathcal{A}(e,w,v)$, we have $p + P^+ - e - f \in \mathcal{P}_m^+$. Because μ is constant on $\mathcal{P}_m^+$, it follows that μ is also constant ($= \mu(e) + \mu(f)$) on the set of all maximal in $\mathcal{A}(e,w,v)$ positive projections. By Section 1, $\mu(e) = \mu(f)$. Therefore, if the projection P^+ is an infinite projection with respect to $\mathcal{A}$, then $\mu(e) = 0$ for any projection $e \in \mathcal{P}$, $e \leq P^+$. Due to the equality (2) we conclude that $\mu/\mathcal{P}^+$ identically equals zero. In a similar way, if the projection P^- is infinite then $\mu/\mathcal{P}^- \equiv 0$.

Now let the projection P^+ be a finite one, with respect to $\mathcal{A}$. Let $e, p \in \mathcal{P} \cap \Pi$, $e \vee p \leq P^+$ and $e \sim p$. From the continuity of the factor $\mathcal{A}$ it follows that there exist decompositions $e = e_1 + e_2$ and $p = p_1 + p_2$, where e_1 and p_1 are projections from $\mathcal{A}$ such that $e_\iota \perp p_\iota$ and $e_\iota \sim p_\iota$ ($\iota = 1, 2$). Again, by the step 1, $\mu(e_\iota) = \mu(p_\iota)$. Hence $\mu(e) = \mu(p)$. Therefore, the restriction of μ to $\{e \in \mathcal{P}\colon\ e \leq P^+\}$ is unitary invariant (i.e., $e \sim f \Rightarrow \mu(e) = \mu(f)$). Let τ be a faithful normal semifinite trace on $\mathcal{A}$. Then there exists a unique constant $c \in R$ such that $\mu(p) = \mu(e_p) = c\tau(p)$, $\forall p \in \mathcal{P}^+$.

By analogy, if the projection P^- is finite then there exists a constant $t \in R$ such that $\mu(p) = t\tau(p)$, $\forall p \in \mathcal{P}^-$.

In the general case, $\mu(p) = c\tau(p_+) + t\tau(p_-)$, $\forall p \in \mathcal{P}$ $(0\infty \equiv 0)$. □

A W^*J-factor $\mathcal{A}$ is said to be a *W^*P-factor* if at least one of the projections P^+ or P^- is finite with respect to $\mathcal{A}$.

Theorem 2. *Let $\mu : \mathcal{P} \to R^+$ be a probability measure on a W^*J-factor $\mathcal{A}$ (the type of $\mathcal{A}$ is different from I_2). Then $\mathcal{A}$ is a W^*P-factor and μ is a sum of semiconstant measures.*

Proof. Let μ be a nonnegative measure. The proof will consist of several steps.

1) Suppose first $\mathcal{A} = B(H)$, where $3 \leq dimH < \infty$. The function $\nu(p) \equiv tr(\mu(I)Jp) - \mu(p)$, $\forall p \in \mathcal{P}$ is an indefinite measure on $\mathcal{P}$ because if $p \in \mathcal{P}^+$ $(p \neq 0)$ then

$$\nu(p) = \mu(I)tr(pJ) - \mu(p) \geq \mu(I)tr(e_p) - \mu(p) \geq \mu(I) - \mu(p) \geq 0,$$

and if $p \in \mathcal{P}^-$ $(p \neq 0)$ then $\nu(p) \leq -\mu(I) - \mu(p) \leq 0$. By Theorem [4], there exist a J-selfadjoint operator T (i.e., $T = JT^*J$) and a number c such that $\nu(p) = tr(Tp) - cdim(p_+H)$. Hence $\mu(p) = tr((\mu(I)J - T)p) + cdim(p_+H)$, $\forall p \in \mathcal{P}$. It is obvious that $\mu(p) = tr((\mu(I)J - T + cI)p) - cdim(p_-H)$, $\forall p \in \mathcal{P}$.

Let $z \in H^+ \cap S$, $y \in H^- \cap S$ and $v = (.,z)y$. Put $B \equiv \mu(I)J - T$. Since $B = JB^*J$, we get

$$(Bz, y) = (BJz, y) = (JB^*z, y) = (B^*z, Jy)$$

$$= -(B^*z, y) = -(z, By) = -\overline{(By, z)}.$$

Hence

$$0 \leq \mu(p(\beta p_z, e^{\iota\theta}v)) = tr(Bp(\beta p_z, e^{\iota\theta}v)) + c$$

$$= \beta tr(Bp_z) + (\beta^2 - \beta)^{1/2}(tr(e^{\iota\theta}Bv) - tr(e^{-\iota\theta}Bv^*)) - (\beta - 1)tr(Bp_y) + c$$

$$= \beta(Bz, z) + (\beta^2 - \beta)^{1/2}(e^{\iota\theta}(By, z) - e^{-\iota\theta}(Bz, y)) - (\beta - 1)(By, y) + c$$

$$= \beta((Bz, z) - (By, y)) + 2(\beta^2 - \beta)^{1/2}Re(e^{\iota\theta}(By, z)) + (Bz, z) + c \leq 1,$$

$$\forall \beta > 1, \forall \theta \in R.$$

Therefore, $(Bz, z) = (By, y)$ and $(By, z) = 0$, $\forall z \in H^+ \cap S$, $\forall y \in H^- \cap S$. Hence $B = aI$, where $a \equiv (Bz, z)$. Therefore,

$$\mu(p) = tr(Bp) + cdim(p_+H) = atr(p) + cdim(p_+H)$$

$$= (a + c)dim(p_+H) + adim(p_-H), \forall p \in \mathcal{P}.$$

Thus μ is a sum of semiconstant measures.

2) Let $dimH = \infty$. It follows from the step 1) that $\mu(p_y) \equiv const$ $(= a \geq 0)$ $\{\mu(p_z) \equiv const$ $(= c + a \geq 0)\}$ on the set of all one-dimensional orthogonal negative $\{$positive$\}$ projections. It is clear that if $dimH^- = \infty$ then $a = 0$. This means that $cdimH^+ = \mu(P^+) = \mu(I) < \infty$. Hence $dimH^+ < \infty$ and $\mu(p) = cdimp_+H$, $\forall p \in \mathcal{P}$. By analogy if $dimH^+ = \infty$, then $a + c = 0$, $dimH^- < \infty$ and $\mu(p) = adimp_-H$, $\forall p \in \mathcal{P}$.

3) Now let $\mathcal{A}$ be a continuous W^*J-factor. By analogy with the proof of Theorem 1 and from the step 1), we have

$$e \sim f \Rightarrow \mu(e) = \mu(f) \tag{3}$$

where $e, f \leq P^+$ or $e, f \leq P^-$. Let $r \in \mathcal{P}_m^+$. Then $\mathcal{A}$ is a continuous W^*J-factor in H with the Hilbert product $[J_r., .]$. By analogy, if $e, f \in \mathcal{P}$ and $e, f \leq r$ or $e, f \leq r^\perp$, and, in addition, $e \sim f$ with respect to the product $[J_r., .]$ then $\mu(e) = \mu(f)$.

a) Assume that P^+ is the infinite projection with respect to $\mathcal{A}$. Then every $p \in \mathcal{P}_m^+$ is infinite, too. We can represent P^+ as $P^+ = \sum_{i=1}^{\infty} e_i$, where $\forall e_i \in \mathcal{P}^+$ and $e_i \sim e_j$ $(i \neq j)$. By (3),

$$0 \leq \mu(P^+) = \sum_{i=1}^{\infty} \mu(e_i) = +\infty \mu(e_1) \leq \mu(I) < +\infty.$$

Hence $\mu(P^+) = 0$. It follows that $\mu/\mathcal{P}_m^+ \equiv 0$. By Theorem 1, μ is a sum of semiconstant measures.

b) Now, assume that P^+ and P^- are finite with respect to $\mathcal{A}$. Let $r \in \mathcal{P}_m^+$. We can write $r = p+q+g$, where $p, q, g \in \mathcal{P}^+$ and $p \sim q \sim g$ in H with the Hilbert product $[J_r., .]$. Denote by p_i the orthogonal projection onto $H \ominus \{x, px = 0\}$. Then $e_p \sim p_i$. By the choice of p, there exists a decomposition $P^+ = e_p^+ + e + f$, where $e, f \in \mathcal{P}$ and $e_p^+ \sim e \sim f$, such that $e \perp p_i$. Then $e + p \in \mathcal{P}^+$. There exists $d \in \mathcal{P}_m^+$ such that $e + p \leq d$. Because of $e \sim e_p^+$ and $e_p^+ \sim e_p$, we have $e \sim p$ in H with the Hilbert product $[J_d., .]$. By (3), $\mu(p) = \mu(e)$. Since $\mu(e) = \mu(e_p^+)$, it follows that $\mu(p) = \mu(e_p^+)$. Hence

$$\mu(P^+) = 3\mu(e_p^+) = 3\mu(p) = \mu(p + q + g) = \mu(r).$$

Thus $\mu/\mathcal{P}_m^+ \equiv const$. Again, by Theorem 1, μ is a sum of semiconstant measures. □

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