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FK-MULTIPLIER SPACES

D. J. FLEMING AND J. C. MAGEE

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Abstract. In 1992 Grosse-Erdmann posed the problem of characterizing those FK-spaces containing the finitely nonzero sequences whose β-duals are themselves FK. Here we consider the more general problem of characterizing FK-spaces containing the finitely nonzero sequences with the property that the multipliers into a BK-sum space admit an FK-topology.

1. Introduction

Let ω denote the linear space of all scalar sequences. By a sequence space we shall mean a linear subspace of ω. If E and F are sequence spaces, then the multipliers from E into F are given by $M(E,F) = \{x \in \omega : xy \in F \ \forall y \in E\}$ where xy is the coordinatewise product. If $E = F$ we write $M(E,E) = M(E)$. Occasionally, it is convenient to use the notation $M(E,F) = E^F$. Then by E^{FF} we shall mean $M(M(E,F),F)$. A sequence space E is called F-perfect if $E^{FF} = E$. Let $l_1 = \{x \in \omega : \sum_n |x_n| < \infty\}$, $cs = \{x \in \omega : \sum_n x_n$ is convergent$\}$ and $bs = \{x \in \omega : \sup_n |\sum_{k=1}^n x_k| < \infty\}$. The α-, β- and γ-duals of a sequence space E are given by $E^\alpha = M(E,l_1)$, $E^\beta = M(E,cs)$ and $E^\gamma = M(E,bs)$. Let $\varphi = sp\{e_n\}$ where e_n is the nth coordinate sequence. If E is an FK-space containing φ then the f-dual of E is defined by $E^f = \{(g(e_k)) : g \in E'\}$ where E' is the topological dual of E. An FK-space E containing φ is called a sum space if $M(E) = E^f$. Sum spaces were introduced by Ruckle in [10] and subsequently studied by him in [12, 13].

In [7] Grosse-Erdmann characterized those FK-spaces containing φ whose f-duals are themselves FK and he posed the analogous problem for β-duals. A similar problem could be posed for the α-dual or the γ-dual. Since l_1, cs and bs are BK-sum spaces one could pose the more general problem of characterizing those FK-spaces E containing φ for which $M(E,F)$ is FK where F is a BK-sum space. It is this more general problem that we address in this note. The authors would like to express their appreciation to W. H. Ruckle for several helpful conversations during the preparation of this work.

2. Notation and terminology

We will assume throughout familiarity with basic FK-space theory, the standard FK-spaces and their natural topologies (see e.g., [14, 15]). If E is an FK-space

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containing φ, then E° will denote the closure of φ in E. If $E = E^\circ$, then E is called an AD-space. For E, F FK-spaces the FK product $E\widehat{\otimes}F$ was introduced by Buntinas and Goes in [4] where it was shown to be the smallest FK-space containing $EF = \{xy : x \in E, y \in F\}$. An alternate characterization was given by Buntinas in [3]. If F is a BK-AD space, the biorthogonal system (e_i, E_i) is said to be norming where E_i is the ith coordinate functional if there exists $A \subseteq sp\{E_i\}$ such that $\|x\|_A = \{\sup|f(x)| : f \in A\}$ yields an equivalent norm on F. This will be the case [12, 5.3] if and only if $F^f = F^{fff}$. If F is an FK-space containing φ, then $H \in F'$ is called a sum on F if $H(e_i) = 1\ \forall i \in \mathbf{N}$. Equivalently, H is a sum on F if $H \in F'$ and $H(x) = \sum_k x_k$ for all $x \in \varphi$. If F is an FK-sum space, then $e = (1, 1, \dots, 1, \dots) \in M(F) = F^f$ and thus every FK-sum space has a sum. If an FK-AD space admits a sum it is unique.

In [8] Magee solved a conjecture of Wilansky by showing that the β-bidual of an FK-space containing φ is itself FK. The key was his observation that E^β is an LBK-space, i.e. a countable increasing union of BK-spaces. Recently, Benholz [1] has shown, more generally, that if E is an FK-space containing φ and F is a BK-space, then $M(E, F)$ is an LBK-space. Specifically he observed that if (V_n) is a base for the neighborhood system at the origin in E with $V_{n+1} \subseteq V_n\ \forall n \in \mathbf{N}$, then $M(E, F) = \bigcup_{n=1}^\infty T_n$ where $T_n = \{y \in M(E, F) : \sup\{\|xy\|_F : x \in V_n\} < \infty\}$ is BK $\forall n \in \mathbf{N}$. From a result of Zeller [16, Satz 4.6] it follows that $M(E, F)$ is FK $\Leftrightarrow M(E, F)$ is BK $\Leftrightarrow M(E, F) = T_{n_0}$ for some $n_0 \in \mathbf{N}$.

3. The main results

Let F be a BK-AD sum space with (e_i, E_i) norming. Then [12, 5.3] $F^{fff} = F^f$ and hence $M(F)^{ff} = F^{fff} = F^f = M(F)$ and so by [12, Theorem 7.2] (e_i, E_i) is strongly series summable. Thus there exists $(u^{(n)}) \subseteq \varphi$ such that

(1) $\lim_n u^{(n)}(j) = 1\ \forall j \in \mathbf{N}$,
(2) $(u^{(n)})$ is bounded in $M(F) = F^f$,
(3) $u^{(n)}x \to x$ in $F\ \forall x \in F$.

Let H denote the unique sum on F. Then given any sequence space E containing φ, $(E, M(E, F))$ is a dual pair under the bilinear form $\langle x, y\rangle = H(xy), x \in E$, $y \in M(E, F)$. Let $q_0 = (F^f)^0$ and let $q = q_0 \oplus e$. Since F is a sum space $e \in F^f$ and consequently $q \subseteq F^f$. Also F a sum space implies $FF^f \subseteq F$ and thus $F^f M(E, F) \subseteq M(E, F)$.

Lemma 3.1. *Let E be a sequence space containing φ, F a BK-AD sum space with (e_i, E_i) norming, and let $B \subseteq E$. If B is $\sigma(E, M(E, F))$ bounded, then for every $t \in M(E, F)$ there exists $A_t > 0$ such that*

$$\sup\{|\langle x, tu\rangle| : x \in B, u \in D_1(q)\} \le A_t$$

where $D_1(q) = \{u \in q : \|u\| \le 1\}$.

Proof. For $t \in M(E, F)$ and $x \in B$ define $f_{xt} : q \to \mathbf{C}$ by $f_{xt}(u) = \langle x, tu\rangle = H(xtu) = H \circ T_{xt}(u)$. Since the diagonal map $T_{xt} : q \to F$ given by $T_{xt}(u) = xtu$ is continuous it follows that f_{xt} is continuous.

Now B is $\sigma(E, M(E, F))$ bounded, so $\sup_{x\in B} |H(xy)| < \infty$ for $y \in M(E, F)$ and hence $\sup_{x\in B} |H(xut)| < \infty$ for all $u \in q, t \in M(E, F)$ as $qM(E, F) \subseteq F^f M(E, F) \subseteq M(E, F)$. So, $f_{xt} : q \to \mathbf{C}$ is pointwise bounded for $x \in B$, hence uniformly bounded for each t, which implies the result. $\square$

The following result establishes necessary and sufficient conditions for $M(E,F)$ to be BK with no topological condition on E.

Theorem 3.2. *Let E be a sequence space containing φ, and let F be a BK-AD sum space with (e_i, E_i) norming. The following are equivalent*:

(i) *$M(E,F)$ is a BK-space.*

(ii) *There exists $B \subseteq E$ such that B absorbs E and B is $\sigma(E, M(E,F))$ bounded.*

(iii) *There exists a K-norm $\|\cdot\|$ on E such that for each $t \in M(E,F)$, $g_t \in (E, \|\cdot\|)'$ where, for $x \in E, g_t(x) = \langle x, t\rangle$.*

Proof. ((i) $\Rightarrow$ (ii)) Let $M(E,F) = E^F$ be a BK-space with norm $\|\cdot\|_{E^F}$. Then

$$|||x||| = \sup\{\|xy\|_F : y \in M(E,F), \|y\|_{E^F} \leq 1\}$$

defines a seminorm on E^{FF}. However, since $\varphi \subseteq F$ it follows that $\varphi \subseteq M(E,F)$. Consequently, the unit ball of $M(E,F)$ contains a multiple of e_i for each $i \in \mathbf{N}$ and thus $|||\cdot|||$ is a norm on E^{FF}. Let $B = \{x \in E : |||\hat{x}||| \leq 1\}$, where $\hat{}: E \to E^{FF}$ is the natural embedding. Then for each $y \in E^F$ we have

$$\begin{aligned}\sup_{x\in B} |\langle x, y\rangle| = \sup_{x\in B} |H(xy)| &\leq \|H\| \sup_{x\in B} \|xy\|_F = \|H\| \sup_{x\in B} \|\hat{x}y\|_F \\ &\leq \|H\| \sup_{x\in B} |||\hat{x}||| \, \|y\|_{E^F} \leq \|H\| \, \|y\|_{E^F} < \infty.\end{aligned}$$

Hence B is $\sigma(E, M(E,F))$ bounded and clearly B absorbs E.

((ii) $\Rightarrow$ (i)) Suppose $B \subseteq E$ is such that B absorbs E and B is $\sigma(E, M(E,F))$ bounded.

Let $t \in M(E,F)$. By Lemma 3.1 there exists $A_t > 0$ such that

$$\sup_{x\in B, u\in D_1(q)} |\langle x, tu\rangle| \leq A_t.$$

Let $P = \sup_n \|u^{(n)}\|_q$ and let $f \in F'$ with $z = (f(e_k))$. Then $\frac{u^{(n)}}{P} \frac{z}{\|z\|_{F^f}} \in D_1(q)$ $\forall n \in \mathbf{N}$ as F^f is a BK-algebra [9, 3.3] and hence

$$\sup_{x\in B, n\in\mathbf{N}} |\langle x, u^{(n)} zt\rangle| \leq PA_t \|z\|_{F^f}.$$

Define $p\colon \omega \to \mathbf{R}$ by $p(t) = \sup_{x\in B, n\in\mathbf{N}} \|u^{(n)}xt\|_F$. This is clearly an extended seminorm and indeed an extended norm as B absorbs each coordinate vector e_j and $u^{(n)}(j) \xrightarrow{n} 1$ $\forall j \in \mathbf{N}$. Let $S_p = \{t \in \omega : p(t) < \infty\}$. It is routine to verify that S_p is a BK-space under the norm p. Let $t \in M(E,F), x \in B$,

$$\begin{aligned}\|u^{(n)}xt\|_F &= \sup_{\|f\|\leq 1} |f(u^{(n)}xt)| = \sup_{\|f\|\leq 1} \left|\sum_k u^{(n)}(k) x_k t_k f(e_k)\right| \\ &= \sup_{\|f\|\leq 1} |H(u^{(n)}xtz_f)| \quad (\text{where } z_f = (f(e_k)) \in F^f) \\ &= \sup_{\|f\|\leq 1} |\langle x, u^{(n)} z_f t\rangle| \leq PA_t.\end{aligned}$$

Therefore $t \in S_p$ and so $M(E,F) \subseteq S_p$.

Let $(t^{(k)}) \subseteq M(E,F)$ and suppose $t^{(k)} \to t$ in S_p. Let $\varepsilon > 0$ be a given and choose k_0 such that $p(t^{(k)} - t) < \varepsilon/3$ for $k \geq k_0$.Let $x \in B$, then $(u^{(n)}t^{(k_0)}x)_n$ is

Cauchy in F, so choose n_0 such that $\|(u^{(n)} - u^{(m)})xt^{(k_0)}\|_F < \varepsilon/3$ for $m, n \geq n_0$. Then

$$\begin{aligned}
&\|(u^{(n)} - u^{(m)})xt\|_F \\
&\quad \leq \|(u^{(n)} - u^{(m)})x(t - t^{(k_0)})\|_F + \|(u^{(n)} - u^{(m)})xt^{(k_0)}\|_F \\
&\quad \leq \|u^{(n)}x(t - t^{(k_0)})\|_F + \|u^{(m)}x(t - t^{(k_0)})\|_F + \|(u^{(n)} - u^{(m)})xt^{(k_0)}\|_F \\
&\quad \leq p(t - t^{(k_0)}) + p(t - t^{(k_0)}) + \|(u^{(n)} - u^{(m)})xt^{(k_0)}\|_F \\
&\quad < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Thus for each $x \in B$, $(u^{(n)}xt)$ is Cauchy and hence convergent in F. From the continuity of the coordinate functionals and the fact that $u^{(n)}(j) \xrightarrow{n} 1$ $\forall j \in \mathbf{N}$ it follows that $u^{(n)}xt \to xt$. Thus $xt \in F$ $\forall x \in B$, and since B absorbs E, $xt \in F$ $\forall x \in E$. Thus $t \in M(E, F)$, so $M(E, F)$ is closed in S_p and hence is a BK-space.

((ii) $\Rightarrow$ (iii)) Let $B \subseteq E$ be such that B absorbs E and B is $\sigma(E, M(E, F))$ bounded. Then $\widehat{B}$, the absolutely convex hull of B, enjoys the same properties. For $x \in E$ define

$$\|x\| = \inf\{r : r > 0, x \in r\widehat{B}\}.$$

It is clear that $\|\cdot\|$ defines a K-norm on E. Moreover, for $t \in M(E, F)$,

$$\|g_t\| = \sup\{|\langle x, t\rangle| : x \in E, \|x\| \leq 1\} = \sup\{|\langle x, t\rangle| : x \in \widehat{B}\} < \infty$$

as $\widehat{B}$ is $\sigma(E, M(E, F))$ bounded.

((iii) $\Rightarrow$ (ii)) Let $\|\cdot\|$ be a K-norm on E with $g_t \in (E, \|\ \|)'$ for each $t \in M(E, F)$. Then B, the unit ball with respect to $\|\cdot\|$, is clearly absorbing and $\sigma(E, M(E, F))$ bounded. □

In particular, if E is a sequence space containing φ, then E^β will be a BK-space if and only if there is a K-norm $\|\cdot\|$ on E such that $E^\beta \subset (E, \|\cdot\|)'$ in the sense of the standard duality between a sequence space and its β-dual.

Corollary 3.3. *Let E be an FK-space containing φ, F a BK-AD sum space with (e_i, E_i) norming and let $\{V_n\}$ be a base for the neighborhoods of zero in E satisfying $V_{n+1} \subseteq V_n$ $\forall n \in \mathbf{N}$. Then $M(E, F)$ is FK if and only if there exists $n_0 \in \mathbf{N}$ such that V_{n_0} is $\sigma(E, M(E, F))$ bounded.*

Proof. ($\Rightarrow$) By [1] if $M(E, F)$ is FK, then it is BK and there exists $n_0 \in \mathbf{N}$ such that $M(E, F) = T_{n_0} = \{y \in M(E, F) : \sup_{x \in V_{n_0}} \|xy\|_F < \infty\}$. Thus

$$\sup_{x \in V_{n_0}} |\langle x, y\rangle| = \sup_{x \in V_{n_0}} |H(xy)| \leq \sup_{x \in V_{n_0}} \|H\|\, \|xy\|_F < \infty.$$

So V_{n_0} is $\sigma(E, M(E, F))$ bounded.

The converse is immediate from Theorem 3.2. □

The following affords a characterization of when $M(E, F)$ is BK assuming only that F is a BK-space containing φ.

Theorem 3.4. *Let E be a sequence space containing φ, and let F be a BK-space containing φ. Then $M(E, F)$ is a BK-space if and only if there exists a norm $\|\cdot\|$ on E such that for every $y \in M(E, F)$ the diagonal map $T_y : E \to F, T_y(x) = xy$, is continuous with respect to this norm.*

Proof. ($\Rightarrow$) For $x \in E$ define

$$\|x\| = \sup\{\|xy\|_F : y \in M(E,F), \|y\|_{E^F} \leq 1\}.$$

As noted in Theorem 3.2, $\| \ \|$ is a norm on E. For $y \in M(E,F), y \neq 0$,

$$\begin{aligned}\|T_y\| &= \sup_{\|x\|\leq 1} \|xy\|_F = \|y\|_{E^F} \sup_{\|x\|\leq 1} \|x\frac{y}{\|y\|_{E^F}}\|_F \\ &\leq \|y\|_{E^F}\|x\| \leq \|y\|_{E^F}.\end{aligned}$$

($\Leftarrow$) Since E is normed, $L(E,F)$ is a Banach space. Let $(y^{(n)}) \subseteq M(E,F)$ be such that $T_{y^{(n)}} \to T$ in $L(E,F)$. Then $y^{(n)}x \to Tx$ in F for each $x \in E$. In particular, for each $i \in \mathbf{N}$, $(y^{(n)}e_i) = (y^{(n)}(i)e_i)$ is Cauchy in F and thus $(y^{(n)}(i))_n$ is Cauchy in $\mathbf{C}$ as F is a K-space. Let $y = (y_i)$ where $y_i = \lim_n y^{(n)}(i)$. Thus for $x \in E$, $(Tx)_i = \lim_n y^{(n)}(i)x_i = y_i x_i$, so $Tx = xy$. Thus $M(E,F)$ is closed in $L(E,F)$ and hence $M(E,F)$ is a Banach space. A similar argument shows that the induced norm on $M(E,F)$ is a K-norm. □

Theorem 3.5. *Let E be an FK-space containing φ and let F be a BK-AD sum space with (e_i, E_i) norming. Then $M(E,F)$ is an FK-space if and only if $q_0\widehat{\otimes}E$ is a BK-space.*

Proof. ($\Rightarrow$) If $M(E,F)$ is FK, then it is BK so $M(q_0, M(E,F))$ is BK. By [4, 5.6 and 2.3(i)] $M(q_0, M(E,F)) = M(q_0\widehat{\otimes}E, F)$. Since F is $AD, M(F) = M(F^f)$ [5, 3.4] and thus $F^f = M(F^f)$. Let $t \in M(F^f), x \in q_0E$ say $x = yz, y \in q_0, z \in E$. Then since $T_t : q_0 \to F^f$ is continuous $T_t(y) = ty \in F^{f_0} = q_0$ so $tx = (ty)z \in q_0E \subseteq q_0\widehat{\otimes}E$. Now by [4, 2.3(i)] $M(q_0E, q_0\widehat{\otimes}E) = M(q_0\widehat{\otimes}E)$ and hence $F^f = M(F^f) \subseteq M(q_0\widehat{\otimes}E)$. Since $q_0\widehat{\otimes}E$ is an AD-space and F admits a sum it follows from [11, 3.2] that $M(q_0\widehat{\otimes}E, F) = (q_0\widehat{\otimes}E)^f$. Thus by [7, Theorem 2] $q_0\widehat{\otimes}E$ is BK.

($\Leftarrow$) If $q_0\widehat{\otimes}E$ is BK, then $M(q_0\widehat{\otimes}E, F)$ is BK. Thus by Corollary 3.3 there is a basic neighborhood of zero in $q_0\widehat{\otimes}E$ which is $\sigma(q_0\widehat{\otimes}E, M(q_0\widehat{\otimes}E, F))$ bounded. Therefore by [3, Theorem 5] there is $k > 0$ and V a basic neighborhood of zero in E such that for every $t \in M(q_0\widehat{\otimes}E, F)$ there is $A_t > 0$ such that $\sup_{w\in\widehat{UV}} |\langle w, t\rangle| \leq A_t$ where $U = D_k(q_0)$ and $\widehat{UV}$ denotes the closed convex hull of UV in $q_0\widehat{\otimes}E$. Let $P = \sup_n \|u^{(n)}\|_{q_0}$ then $\frac{k}{P}u^{(n)} \in D_k(q_0)$ so $\sup_{n\in\mathbf{N}, x\in\mathbf{V}} |\langle u^{(n)}x, t\rangle| \leq \frac{PA_t}{k}$. For $u \in q$ write $u = u_0 + \lambda_u e$ where $u_0 \in q_0, \lambda_u \in \mathbf{C}$. If we define $\|u\|_* = \|u_0\|_{q_0} + |\lambda_u|$, then $(q, \| \ \|_*)$ is a BK-space so $\| \ \|_*$ is equivalent to the norm of q as a closed subspace of F^f. Let $u \in D_k(q)$. Then $u = u_0 + \lambda_u e$ with $u_0 \in q_0, \lambda_u \in \mathbf{C}$ and $\|u_0\|_{q_0} \leq k, |\lambda_u| \leq k$. Let $x \in V$ and $t \in M(q\widehat{\otimes}E, F) \subseteq M(q_0\widehat{\otimes}E, F)$. Also note that since $e \in q, E \subseteq q\widehat{\otimes}E$. Then

$$\begin{aligned}|\langle ux, t\rangle| &\leq |\langle u_0x, t\rangle| + |\lambda_u|\,|\langle x, t\rangle| \\ &\leq A_t + k|H(xt)| \\ &= A_t + k\lim_n |H(u^{(n)}xt)| \\ &\leq A_t + k\sup_n |\langle u^{(n)}x, t\rangle| \\ &\leq A_t + \frac{kPA_t}{k} \\ &= (1+P)A_t.\end{aligned}$$

Therefore UV and hence $\widehat{UV}$ is $\sigma(q\widehat{\otimes}E, M(q\widehat{\otimes}E, F))$ bounded. Thus by Corollary 3.3 $M(q\widehat{\otimes}E, F)$ is FK and hence BK. Now [4, 5.6] $M(q\widehat{\otimes}E, F) = M(q, M(E, F)) \subseteq M(E, F)$ as $e \in q$. Let $x \in M(E, F)$ and let $z \in q \subseteq F^f$. Since $FF^f \subseteq F$ we have $xyz \in F\ \forall z \in q, y \in E$. Therefore $xz \in M(E, F)\ \forall z \in q$ so $x \in M(q, M(E, F))$. Thus $M(E, F) = M(q\widehat{\otimes}E, F)$ and hence $M(E, F)$ is BK. □

Proposition 3.6. *Let E be an FK-space containing φ and let F, G be BK-spaces containing φ.*

(i) *If G is a closed subspace of F, then $M(E, F)$ BK implies $M(E, G)$ is BK.*

(ii) *If F is G-perfect, then $M(E, G)$ BK implies that $M(E, F)$ is BK.*

Proof. (i) Let $\{V_n\}$ be a base for the neighborhood system at the origin in E with $V_{n+1} \subseteq V_n\ \forall n \in \mathbf{N}$. By [1] $M(E, F) = \bigcup_{n=1}^{\infty} T_n$, $M(E, G) = \bigcup_{n=1}^{\infty} S_n$ where S_n, T_n are the BK-spaces given by

$$S_n = \{y \in M(E, G) : \sup_{x \in V_n} \|xy\|_G < \infty\},$$
$$T_n = \{y \in M(E, F) : \sup_{x \in V_n} \|xy\|_F < \infty\}.$$

If $M(E, F)$ is BK, then [16, Satz 4.6] $\exists n_0$ such that $M(E, F) = T_{n_0}$. Let $x \in M(E, G)$, then $x \in M(E, F) = T_{n_0}$. Therefore $x \in M(E, G) \cap T_{n_0} = S_{n_0}$ and thus $M(E, G) = S_{n_0}$ and hence is a BK-space.

(ii) Suppose F is G-perfect and E^G is BK. Then by [4, 5.1] $E^F = M(E^{GG}, F)$. But E^G BK implies E^{GG} is BK and hence $M(E^{GG}, F) = M(E, F)$ is BK. □

Since l_1 is cs-perfect it follows from 3.6(ii) that E^β BK implies E^α is BK. This raises the question of whether there exists an FK-space whose α-dual is BK while its β-dual is not?

Examples. 1. Let E be an FK-space containing φ. Since cs is closed in bs and bs is cs-perfect, we have E^γ is FK $\Leftrightarrow$ E^β is FK $\Leftrightarrow$ $bv_0\widehat{\otimes}E$ is BK. By [4, 4.1(i)] $bv_0\widehat{\otimes}E$ is the closure of φ in $bv\widehat{\otimes}E$. From [4, 4.1(ii) and 3, Theorem 3] this latter space is the smallest FK-AB-space containing E. Let E be the FK-space given in [7, Example A] which is due to W. H. Ruckle. As noted in [7] E is an FK-non BK-space whose β-dual is BK. So by Theorem 3.5, $bv_0\widehat{\otimes}E$ is BK. But Ruckle's example is clearly an AB-space so $bvE \subseteq E$ [15, 10.3, Lemma 17], and thus [3, Theorem 3] $bv\widehat{\otimes}E = E$. Consequently, E affords an example where $bv_0\widehat{\otimes}E$ is BK but $bv\widehat{\otimes}E$ is FK-non BK.

2. Let E be an FK-space containing φ. Then E^α is FK if and only if $c_0\widehat{\otimes}E$ is BK. The latter space being the closure of φ in $c\widehat{\otimes}E$ the smallest FK-UAB space containing E.

3. Similar statements can be made for the Toeplitz duals $E^{\alpha_T}, E^{\beta_T}$ and E^{γ_T} with the appropriate hypotheses on T (see e.g., [2, 6]).

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Department of Mathematics, St. Lawrence University, Canton, New York 13617

Department of Mathematics, SUNY at Potsdam, Potsdam, New York 13676