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ON INVERTIBILITY IN NON-SELFADJOINT OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{L}$ be a complete commutative subspace lattice on a Hilbert space. When $\mathcal{L}$ is purely atomic, we give a necessary and sufficient condition for $\sigma(T) = \sigma_{\mathcal{L}}(T)$ for every T in $alg\mathcal{L}$, where $\sigma_{\mathcal{L}}(T)$ and $\sigma(T)$ denote the spectrum of T in $alg\mathcal{L}$ and $B(H)$ respectively. In addition, we discuss the properties of the spectra and the invertibility conditions for operators in $alg\mathcal{L}$.

1. Introduction

The invertibility of a nest algebra, the typical non-selfadjoint operator algebra, has been discussed by some authors (see [2, 4, 8]). Motivated by the invertibility in selfadjoint operator algebras and the problem of whether the invertible group of a nest algebra is path-connected, we are interested in the invertibility in a commutative subspace lattice (abbr. CSL) algebras.

For C^* algebras $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$, it is well-known that $\sigma_{\mathcal{A}}(T) = \sigma_{\mathcal{B}}(T)$ for all T in $\mathcal{A}$. But the analogue is not valid generally for non-selfadjoint operator algebras. We discuss the conditions on $\mathcal{L}$ for $alg\mathcal{L}$ to be inverse closed for a CSL $\mathcal{L}$, and give a necessary and sufficient condition on $\mathcal{L}$ under which $alg\mathcal{L}$ is inverse closed.

Since CSL algebras are not inverse closed generally, we discuss the properties of the spectra of operators in them and give an invertibility condition similar to that in [4] for CSL algebras.

In the following, let $B(H)$ be the set of all bounded linear operators on a complex separable Hilbert space H. If $\mathcal{L}$ is a commutative subspace(or projection) lattice in $B(H)$ which is complete and contains 0 and I, then we denote by $alg\mathcal{L}$ the set $\{T \in B(H) : PTP = TP, \text{ for every } P \in \mathcal{L}\}$; $alg\mathcal{L}$ is called a CSL algebra corresponding to $\mathcal{L}$. When $\mathcal{L}''$, the von Neumann algebra generated by $\mathcal{L}$, is purely atomic, that is, I equals the sum of minimal projections in $\mathcal{L}''$, we say $\mathcal{L}$ is a purely atomic CSL. For arbitrary P, Q in $\mathcal{L}$ with $P > Q$, $E = P - Q$ is an interval of $\mathcal{L}$. Minimal interval projections are known as the atoms of $\mathcal{L}$. When $\mathcal{L}$ is purely atomic, every projection in $\mathcal{L}$ is the sum of all atoms dominated by it. The intervals of $\mathcal{L}$ are partially ordered by the relation $\prec$, where $E_1 \prec E_2$ if and only if $E_1B(H)E_2 \subseteq alg\mathcal{L}$. The relation $\prec$ is related essentially to the structure of $alg\mathcal{L}$ (see [3]). It is easy to see that if E, F are atoms of $\mathcal{L}$, then $E \prec F$ if and only if

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for every P in $\mathcal{L}$, $P \geq F$ implies $P \geq E$. We use this relation $\prec$ to determine the condition under which $alg\mathcal{L}$ is inverse closed.

2. Inverse closedness

For a CSL $\mathcal{L}$ and every T in $alg\mathcal{L}$, if T invertible in $B(H)$ implies that T^{-1} belongs to $alg\mathcal{L}$, we say that $alg\mathcal{L}$ is inverse closed. In this section, we discuss the conditions under which $alg\mathcal{L}$ is inverse closed. For convenience, we first give several lemmas.

Lemma 2.1. *Suppose that T is invertible in $B(H)$. Then T is invertible in $alg\mathcal{L}$ if and only if $TPH = PH$ for every P in $\mathcal{L}$.*

Proof. Suppose that T is invertible in $alg\mathcal{L}$. For every P in $\mathcal{L}$, $PH = T(T^{-1}PH) \subseteq TPH \subseteq PH$. So $TPH = PH$. The converse is clear. $\square$

Lemma 2.2 [4]**.** *Suppose that T is an operator on H such that the restriction of T to an invariant subspace of finite codimension is invertible. If T has trivial kernel, then T is invertible on H.*

Lemma 2.3. *Let T be an invertible operator in $B(H)$, P and Q be invariant projections of T with $P < Q$.*

(1) If $TQH = QH$ and $(Q-P)T|_{(Q-P)H}$ is injective, then $TPH = PH$. In particular, if $Q-P$ is of finite dimension and $(Q-P)T(Q-P)H = (Q-P)H$, then $TPH = PH$.

(2) If $TPH = PH$ and $(Q-P)T(Q-P)H = (Q-P)H$, then $TQH = QH$.

Proof. (1) Let $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ be the matrix of $T|_{QH}$ with respect to $PH \oplus (Q-P)H$. For any vector y in PH, there exist vectors x_1 and x_2 in PH and $(Q-P)H$ respectively such that $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)\binom{x_1}{x_2} = \binom{y}{0}$, since $TQH = QH$. So, $Ax_1 + Cx_2 = y$ and $Bx_2 = 0$. Since $B = (Q-P)T|_{(Q-P)H}$ is injective, $x_2 = 0$ and $Ax_1 = y$, that is, $T|_{PH}x_1 = y$. By the arbitrariness of y we have $TPH = PH$.

If $(Q-P)$ is of finite dimension, then $(Q-P)T|_{(Q-P)H}$ is injective, because $(Q-P)T(Q-P)H = (Q-P)H$. So the result follows. (2) can be proved similarly. $\square$

Lemma 2.4. *Let $\mathcal{L}$ be a CSL, $T \in alg\mathcal{L}$ an invertible operator and $\{P_i\}_{i=1}^{\alpha}$ a sequence of projections in $\mathcal{L}$ with $TP_iH = P_iH$ for all i. Then $T(\vee_i P_i)H = (\vee_i P_i)H$ and $T(\wedge_i P_i)H = (\wedge P_i)H$.*

Proof. Let $P = \vee_i P_i$. Suppose that $TPH \subset PH$. Take a nonzero vector f in $PH \ominus TPH$. Since $P \geq P_i$ for each i, $TPH \supset TP_iH = P_iH$. Thus we have $f \perp P_iH$ for each i, and hence $f \perp (\vee_i P_i)H$, that is, $f \perp PH$. Therefore f must be zero, a contradiction. So it follows that $T(\vee_i P_i)H = (\vee_i P_i)H$.

Let $Q = \wedge P_i$. Then $QH = \cap P_iH$. If $TQH \subset QH$, then there exists a nonzero vector $g \in QH \ominus TQH$. Since $QH \subseteq P_iH$ and $TP_iH = P_iH$ for each i, there exists a vector f_i in P_iH for each i, such that $Tf_i = g$. By the invertibility of T, $f_1 = f_i$ for all i. So $f_1 \in \cap P_iH$ and $(g,g) = (Tf_1, g) = 0$. This is impossible. Therefore $T(\wedge P_i)H = (\wedge P_i)H$. $\square$

The following theorem is our main result in this section.

Theorem 2.5. *Let $\mathcal{L}$ be a purely atomic CSL on H. Then $alg\mathcal{L}$ is inverse closed if and only if the following conditions are satisfied:*

(1) there do not exist two infinitely dimensional atoms E, F *of* $\mathcal{L}$ *with* $E \prec F$,

(2) there does not exist a sequence $\{E_n\}_{n=0}^{\infty}$ *of atoms of* $\mathcal{L}$ *such that* $E_0 \prec E_n \prec E_{n+1}$ *or* $E_{n+1} \prec E_n \prec E_0$ *for every* $n \geq 1$, $E_n \neq E_m$ *for* $n \neq m$ *and every* E_n, *but* E_0 *is finite dimensional, and*

(3) there does not exist a sequence $\{E_n\}_{n=-\infty}^{\infty}$ *of finitely dimensional atoms of* $\mathcal{L}$ *such that* $E_n \prec E_{n+1}$ *for all* n *and* $E_n \neq E_m$ *for* $n \neq m$.

Proof. Suppose that $alg\mathcal{L}$ is not inverse closed. We prove that at least one of (1), (2) and (3) in the theorem does not hold.

To the contrary, we suppose that (1), (2) and (3) are all true. Take an operator T in $alg\mathcal{L}$ such that T is invertible in $B(H)$ but T^{-1} is not in $alg\mathcal{L}$. Then there exists a projection P_0 in $\mathcal{L}$ such that $TP_0H \subset P_0H$. If there existed a sequence of atoms of $\mathcal{L}$, $\{E_n\}_{n=-\infty}^{\infty}$, such that $E_n \prec E_{n+1}$ for all n and $E_n \neq E_m$ for $n \neq m$, then it would be clear that one of (1), (2) and (3) in the theorem is not true. So, it follows that $\mathcal{L}$ has no such sequence of atoms. We show that there must exist at least one atom E_0 with $E_0TE_0H \subset E_0H$, that is, $E_0H \ominus E_0TE_0H \neq (0)$ if (2) in the theorem is true.

To do this, we suppose that, for any atom E of $\mathcal{L}$, $EH = ETEH$. Then it is clear that P_0 is not an atom. For an atom E of $\mathcal{L}$, if there does not exist a sequence $\{E_n\}_{n=1}^{\infty}$ of mutually different atoms such that $E \prec E_n \prec E_{n+1}$ ($E \succ E_n \succ E_{n+1}$) for $n \geq 1$, we say that E is upper (lower) finite. Let $E_0 < P' \in \mathcal{L}$ be a lower finite atom. We show that $E_0H \subseteq TP'H$ and $E_0T|_{E_0H}$ is injective.

Let $P_{E_0} = \wedge\{P \in \mathcal{L} : \ PE_0 = E_0\}$. It is clear that $P_{E_0} \leq P'$. If $TP_{E_0}H \neq P_{E_0}H$, then it is easy to see that P_{E_0} is not an atom of $\mathcal{L}$. Put $Q_{E_0} = \vee\{P_{E'} : \ E'$ is an atom of $\mathcal{L}$ with $E' < P_{E_0}$ and $E' \neq E_0\}$. It is obvious that $E_0 = P_{E_0} - Q_{E_0}$. By Lemma 2.4 and Lemma 2.3, it follows that there exists at least one atom E_1 of $\mathcal{L}$ such that $E_1 \neq E_0$, $E_1 < P_{E_0}$ and $TP_{E_1}H \neq P_{E_1}H$. So $E_0 \succ E_1$. Inductively, we can choose a sequence $\{E_n\}_{n=1}^{\infty}$ of mutually different atoms such that $E_0 \succ E_n \succ E_{n+1}$ for each n. This contradicts the assumption on E_0. So $TP'H \supseteq TP_{E_0}H = P_{E_0}H \supseteq E_0H$. Similarly, we can prove that $TQ_{E_0}H = Q_{E_0}H$. Then it follows that $E_0T|_{E_0H}$ is injective for $TE_0H \cap Q_{E_0}H = \{0\}$.

Now let K be the set of all the atoms E such that E is lower finite and $E \leq P_0$. Put $Q_0 = \sum_{E\in K} E$. Then $P_0 \geq Q_0 \in \mathcal{L}$. Indeed, let $Q_0' = \vee_{E\in K} P_E$. Then $P_0 \geq Q_0' \geq Q_0$ and $Q_0' \in \mathcal{L}$. If $Q_0' > Q_0$, then there exists some atom E of $\mathcal{L}$ such that $E \leq Q_0' - Q_0$. Hence there is at least one atom E' in K such that $E \leq P_{E'}$. So $E' \succ E$(since $E' \succ P_{E'}$). By the choice of E', it follows that $E \in K$. This is impossible. Hence $Q_0' = Q_0$. By Lemmas 2.4 and 2.3, we have $TP_0 \supseteq TQ_0H = Q_0H$ and $Q_0 < P_0$. Indeed, if $TQ_0H \subset Q_0H$, then there is an atom $E_1 \in K$ such that P_{E_1} satisfies $TP_{E_1}H \subset P_{E_1}H$ by Lemma 2.4. Let $Q_1 \in \mathcal{L}$ be such that $E_1 = P_{E_1} - Q_1$. Since $E_1TE_1H = E_1H$, by Lemma 2.3, $TQ_1H \subset Q_1H$. From Lemma 2.4, there is some atom $E_2 \leq Q_1$ such that $TP_{E_2}H \subset P_{E_2}H$. It is clear that $E_1 \succ E_2$, because $E_1 \succ P_{E_1}$. Inductively, we have a sequence $\{E_n\}_1^{\infty}$ of mutually different atoms of $\mathcal{L}$ such that $E_1 \succ E_2 \succ \cdots$. This contradicts the fact that $E_1 \in K$.

Let K' be the set of all atoms E of $\mathcal{L}$ with either $EP_0 = 0$ and there is some atom E' such that $E' \leq P_0 - Q_0$ and $E \succ E'$, or E is upper finite. Put $P_1 = \wedge\{P \in \mathcal{L}$: $PE = E$ and $P \geq P_0$ for any atom $E \in K'\}$. Then $TP_1H = P_1H$. Indeed, if not, let $f \in P_1H \ominus TP_1H \subset P_1H \ominus Q_0H$ and $g = T^{-1}f$. Then $g \notin P_1H$ and $Q_0f = 0$.

Let $P_1' = \wedge\{P \in \mathcal{L}\colon\ Pg = g \text{ and } P \geq P_1\}$. Since

$$f = (P_1 - Q_0)T(P_1' - P_1)g + (P_1 - Q_0)TP_1 g,$$

we have

$$(P_1 - Q_0)T(P_1' - P_1)g \neq 0.$$

Hence there must exist atoms $E_1 \leq P_1 - Q_0$ and $E_1' \leq P_1' - P_1$ such that $E_1 T E_1' g \neq 0$. So we can deduce that $E_1' \succ E_1$. By the assumption and the choice of K', it is not hard to show that $E_1' \in K'$ and $E_1' < P_1$, a contradiction.

Let G be the set of all atoms E with $E \leq P_1 - P_0$. For an atom $F \in G$, if there exists an atom $E' \leq P_0 - Q_0$ such that $F \succ E'$, then F must be of finite dimension since, for each atom $E \leq P_0 - Q_0$, there is a sequence $\{E_n\}_{n=1}^{\infty}$ of atoms such that $E \succ E_n \succ E_{n+1}$ for all n and (2) in the theorem is true, and consequently $FT|_{FH}$ is injective for $FTFH = FH$; if F is lower finite then $FT|_{FH}$ is injective. Thus, $ET|_{EH}$ is injective for any $E \in G$. By the assumption, G has some atom E such that there does not exist another atom E' of $\mathcal{L}$ with $E' \succ E$ and $E \succ E_1$ for some atom $E_1 \leq P_0 - Q_0$. Let $U_1 = \{E_{11}, E_{12}, \cdots, E_{1\alpha_1}\}$ (α_1 is an ordinal) be the set of all such atoms. Put $Q_{11} = P_1 - E_{11}$; it is not hard to prove that $Q_{11} \in \mathcal{L}$. Since $E_{11}TE_{11}H = E_{11}H$ and $E_{11}T|_{E_{11}H}$ is injective, we have $TQ_{11}H = Q_{11}H$ and $Q_{11} \geq P_0$ by Lemma 2.3. Again, let $Q_{12} = (P_1 - E_{11}) - E_{12} = Q_{11} - E_{12}$. Similarly, it follows that $TQ_{12}H = Q_{12}H$ and $Q_{12} \geq P_0$. Continuing this process, we obtain $Q_{11}, Q_{12}, \cdots, Q_{1\alpha_1} \in \mathcal{L}$ such that $TQ_{1i}H = Q_{1i}H$ and $Q_{1i} > Q_{1\ i+1} \geq P_0$. By Lemma 2.4, $P_2 = \wedge_i Q_{1i}$ satisfies the conditions: $TP_2H = P_2H$ and $P_2 \geq P_0$. If $G - U_1 = \emptyset$, we obtain that $P_0 = P_2$ and the desired contradiction: $TP_0H = P_0H$. If $G - U_1 \neq \emptyset$, as above we know that there is some atom $E \in G - U_1$ such that there exists some atom E' only in U_1 with $E' \succ E$. Let $U_2 = \{E_{21}, E_{22}, \cdots, E_{2\alpha_2}\}$ be the set of all such atoms in G. As above, let $Q_{21} = P_2 - E_{21}$. Then one can deduce similarly that $Q_{21} \in \mathcal{L}$, $TQ_{21}H = Q_{21}H$, $Q_{2\ i+1} = P_2 - (E_{21} + E_{22} + \cdots + E_{2i}) \in \mathcal{L}$ and $TQ_{2\ i+1}H = Q_{2\ i+1}H$ for each i. Put $P_3 = \wedge_i Q_{2i}$. It follows that $TP_3H = P_3H$ and $P_3 \geq P_0$ by Lemma 2.4. If $G - (U_1 \cup U_2) = \emptyset$, then we have $P_3 = P_0$ and $TP_0H = P_0H$, the desired contradiction. If $G - (U_1 \cup U_2) \neq \emptyset$, we can continue this process as above and obtain a sequence $\{U_i\}$. If $G - (\cup_i U_i) = \emptyset$, then $\wedge P_i = P_0$ and we can obtain $TP_0H = P_0H$, the desired contradiction. If $G - (\cup_i U_i) \neq \emptyset$, then there must exist some atom $E' \in G$ such that that there is some atom E only in $\cup U_i$ with $E \succ E'$ by the assumption. In the same way, we can continue the above process to obtain a countable set $\{P_i\} \subseteq \mathcal{L}$ such that $TP_iH = P_iH$ and $P_i \geq P_0$ for every i. Let $\{P_i\}_{i \in \Lambda}$ be the maximal set of such projections in $\mathcal{L}$. So we have $P_0 = \wedge P_i$ and $TP_0H = P_0H$, the desired contradiction. Thus it follows that (2) in the theorem is not true or there is some atom E_0 such that $E_0TE_0H \neq E_0H$.

Suppose that (2) is true, and choose an atom E_0 with $E_0TE_0H \neq E_0H$. Let P_0' be the smallest element in $\mathcal{L}$ which dominates E_0. It follows that $TP_0'H \subset P_0'H$. Indeed, suppose that $TP_0'H = P_0'H$. Let Q be in $\mathcal{L}$ with $E_0 = P_0 - Q$. For a nonzero vector $f \in E_0H \ominus E_0TE_0H$, we can choose a vector $g \in P_0'H$ so that $Tg = f$. Thus, $\|f\|^2 = (f, f) = (Tg, f) = (TE_0g + TQg, f) = (TE_0g, f) + (QTQg, f) = 0$, a contradiction. So $TP_0'H \subset P_0'H$.

Now choose a nonzero vector f_0 in $E_0H \ominus E_0TE_0H$. Then there exists some vector f such that $Tf = f_0$. It is easy to know that $f \notin P_0'H$. Since $\mathcal{L}$ is purely atomic, there is a subset $\{E_k\}$ of atoms of $\mathcal{L}$ such that $f = \sum_k E_k f$, $E_k f \neq 0$ for all

k. Thus $f_0 = Tf = \sum_k TE_kf$. So there is at least one E_{k_1} such that $E_0TE_{k_1}f \neq 0$ and $E_{k_1}P_0' = 0$ for $f \notin P_0'H$. Thus $E_0 \prec E_{k_1}$. Further assume that every atom E of $\mathcal{L}$ with $E_0 \prec E$ is finite dimensional. If $E_{k_1}H \ominus E_{k_1}TE_{k_1}H = (0)$, then $E_{k_1}TE_{k_1}f \neq 0$ since E_{k_1} is of finite dimension, and thus $E_{k_1}T|_{E_k}H$ is invertible. Since $E_{k_1}f_0 = \sum_k E_{k_1}TE_kf = 0$, we have $E_{k_1}TE_{k_1}f = -\sum_{k\neq k_1} E_{k_1}TE_kf$. Therefore there is some k_2 $(\neq k_1)$ such that $E_{k_1}TE_{k_2}f \neq 0$, and then $E_{k_1} \prec E_{k_2}$. If $E_{k_1}H \ominus E_{k_1}TE_{k_1}H \neq (0)$, then we can find another E_{k_2} $(k_1 \neq k_2)$ such that $E_{k_1} \prec E_{k_2}$ as above. So, by induction, we can find some infinite-dimensional atom E_1 of $\mathcal{L}$ such that $E_0 \prec E_1$ or a sequence $\{E_k\}_{k=1}^{\infty}$ of finite dimensional atoms of $\mathcal{L}$ such that $E_k \prec E_{k+1}$ for $k \geq 0$ and $E_n \neq E_m$ for $m \neq n$.

If E_0 is infinite dimensional, then since (2) is true, (1) is not valid by above arguments. So, without loss of generality, assume that E_0 is finite dimensional. Let $P_E = \wedge\{P \in \mathcal{L} : PE = E\}$ for every atom E of $\mathcal{L}$, and $E_0 = P_{E_0} - Q$ for some $Q \in \mathcal{L}$. By the choice of P_{E_0}, we have $Q \prec E_0$. If $TQH = QH$, then we have $T|_{P_{E_0}H}$ is invertible by Lemma 2.2. This is impossible for $TP_{E_0}H \subset P_{E_0}H$. So $TQH \subset QH$. If Q is an atom of $\mathcal{L}$, then Q is of infinite dimension and (1) or (2) in the theorem cannot hold. Hence Q is not an atom. If $TP_{E'}H = P_{E'}H$ for any atom $E' < Q$, then $TQH = QH$ by Lemma 2.4. This is impossible. Hence there must exist an atom $E_1' \leq Q$ such that $TP_{E_1'}H \subset P_{E_1'}H$ and $E_0 \succ E_1'$. Inductively, we can find an infinite dimensional atom E_1' with $E_1' \prec E_0$ or a sequence $\{E_k'\}_{k=1}^{\infty}$ of finite dimensional atoms such that $E_{k+1}' \prec E_k' \prec E_0$ for all $k \geq 1$ and $E_n' \neq E_m'$ for $n \neq m$. Therefore combining the above arguments, it follows that one of (1) or (3) is not valid. Thus we have proved that if $alg\mathcal{L}$ is not inverse closed, then one of (1), (2) and (3) of the theorem does not hold.

For the necessity, suppose that one of (1), (2) and (3) is not satisfied. We show that there is an invertible operator $T \in alg\mathcal{L}$ such that T^{-1} is not in $alg\mathcal{L}$. We only prove this for (3), since the others can be proved similarly and are omitted.

Let $\{E_n\}_{n=-\infty}^{\infty}$ be a sequence of atoms of $\mathcal{L}$ such that $E_n \prec E_{n+1}$ for each n and $E_n \neq E_m$ for $n \neq m$. Choose a unit vector e_n in E_nH for every n. Define an operator T:

$$\begin{aligned} &Te_{n+1} = e_n, \qquad \text{for each } n,\\ &Tx = x, \qquad \text{for every } \ x \in [\vee\{e_n : n \in \mathbb{Z}\}]^{\perp}. \end{aligned}$$

It is clear that T is invertible in $B(H)$. Let P be an arbitrary element in $\mathcal{L}$; if $PE_n = 0$ or $PE_n = E_n$ for all n, then it is easy to see that $TPH = PH$. So suppose that there exists some n such that $E_nP = E_n$ and $E_{n+k}P = 0$, $k \geq 1$. Hence $PE_k = E_k$ for $k \leq n$. For every x in PH, we write $x = x_0 + \sum_n (x, e_n)e_n$, $x_0 \in PH \ominus \sum_{k\leq n} E_kH$. Then

$$Tx = Tx_0 + \sum_{k\geq n+1} (x, e_k)Te_k = x_0 + \sum_{k\geq n+1} (x, e_k)e_{k-1} \in PH.$$

Hence $T \in alg\mathcal{L}$. However, it is obvious that $T^{-1} \notin alg\mathcal{L}$. The proof is complete. □

Remark. Although Theorem 2.5 requires a CSL to be purely atomic, it is easy to see from the above proof that if there is some purely atomic sublattice $\mathcal{L}_1$ of $\mathcal{L}$ such that $\mathcal{L}_1$ does not satisfy the conditions of Theorem 2.5 with respect to the relation $\prec$ determined by $\mathcal{L}$, then $alg\mathcal{L}$ is not inverse closed. Also by Theorem 2.5, it is clear that a CSL algebra on a finite dimensional space is always inverse closed.

Corollary 2.6. *If $alg\mathcal{L}E$ is not inverse closed for some interval E of $\mathcal{L}$, neither is $alg\mathcal{L}$. If there exists a sublattice $\mathcal{L}_1$ of $\mathcal{L}$ such that $\mathcal{L}_1$ does not satisfy the conditions of Theorem 2.3 with respect to the partial order $\prec$ determined by $\mathcal{L}$, then $alg\mathcal{L}$ is not inverse closed.*

Corollary 2.7. *Let $\mathcal{L}$ be a nest. If $\mathcal{L}$ is infinite, then $alg\mathcal{L}$ is inverse closed if and only if every atom of $\mathcal{L}$ is of finite dimension, and has order type $\alpha + \beta^*$, where α and β are ordinals; If $\mathcal{L}$ is finite, then $alg\mathcal{L}$ is inverse closed if and only if every atom but one is finite dimensional.*

3. Spectra of operators in CSL algebras

In this section, we discuss the properties of operators in a CSL algebra. We generalize results in [4] and [8].

Proposition 3.1. *If $A \in alg\mathcal{L}$, then $\sigma_{\mathcal{L}}(A) \subseteq \eta(\sigma(A))$, where $\eta(\sigma(A))$ denotes the full spectrum of A in $B(H)$.*

Proof. Let $\lambda_0 \in \sigma_{\mathcal{L}}(A)$. Suppose that λ_0 is in the unbounded connected component of $\rho(A)$, the resolvent set of A. By Lemma 2.1, there exists an element $P \in \mathcal{L}$ such that $(\lambda_0 - A)PH \subset PH$. So there exists a λ in $\partial\sigma(A|_{PH}) \cap \rho(A)$. This is impossible since

$$\partial\sigma(A|_{PH}) \subset \sigma_\pi(A|_{PH}) \subset \sigma_\pi(A),$$

where σ_π is the approximate point spectrum. □

We first give a proposition which will be needed in the sequel.

Theorem 3.2. *Let $\mathcal{L}$ be a CSL. Then for every A in $alg\mathcal{L}$, $\sigma_{\mathcal{L}}(A) = \sigma(A) \cup \bigcup_{\{E_i\}} \{\bigcup_i \sigma_{\mathcal{L}E_i}(E_iA|_{E_iH}) : \{E_i\}$ is a finite set of intervals of $\mathcal{L}$ with $\sum_i E_i = I\}$.*

In order to prove this proposition, we first give a lemma.

Lemma 3.3. *Suppose that $\{E_i\}_{i=1}^n$ is a finite partition of $\mathcal{L}$, that is, each E_i is an interval of $\mathcal{L}$ and $\sum_{i=1}^n E_i = I$. Then after appropriately arranging $\{E_i\}$, the matrix of each operators in $alg\mathcal{L}$ with respect to the decomposition $H = \sum_{i=1}^n \oplus E_iH$ is upper triangular.*

Proof. Let $\mathcal{L}_1 = \{P_i\}_{i=1}^k$ be a finite sublattice of $\mathcal{L}$ such that each E_i is an interval of $\mathcal{L}_1$. For a fixed operator A in $alg\mathcal{L}$, it is clear that A has the form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with respect to $P_1H \oplus P_1^\perp H$. Since $I = P_1 + P_1^\perp = P_1(P_2 + P_2^\perp) + P_1^\perp(P_2 + P_2^\perp) = P_1P_2 + P_1P_2^\perp + P_1^\perp P_2 + P_1^\perp P_2^\perp$, the matrix of A with respect to $H = P_1P_2H \oplus P_1P_2^\perp H \oplus P_1^\perp P_2 H \oplus P_1^\perp P_2^\perp H$ has the form

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{23} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{pmatrix}.$$

By induction, we can show that the matrix of A with respect to $H = E_1'H \oplus E_2'H \oplus \cdots \oplus E_m'H$ has upper triangular form, where each E_i' is an atom of $\mathcal{L}_1$. Let $\overline{E}_1$ be the interval in $\{E_i\}$ which contains E_1'. Deleting the atoms from $\{E_1', \cdots, E_m'\}$ which are dominated by $\overline{E}_1$, let $\overline{E}_2$ be the interval in $\{E_i\}$ which contains the first remaining atom. By induction, we get a permutation $\{\overline{E}_1, \cdots, \overline{E}_n\}$ of $\{E_i\}$ in a similar way. It is not hard to see that every operator in $alg\mathcal{L}$ has upper triangular form with respect to $\overline{E}_1H \oplus \overline{E}_2H \oplus \cdots \oplus \overline{E}_nH$. □

Proof of Theorem 3.2. It is clear that $\sigma_{\mathcal{L}}(A) \supset \underset{\{E_i\}}{\cup}\{\underset{i}{\cup}\sigma_{\mathcal{L}E_i}(E_iA|_{E_iH}) : \{E_i\}$ is a finite set of intervals of $\mathcal{L}$ with $\sum_i E_i = I\}$. We need to prove the converse. Without loss of generality, we suppose that $0 \in \mathbb{C} - \underset{\{E_i\}}{\cup}\{\underset{i}{\cup}\sigma_{\mathcal{L}E_i}(E_iA|_{E_iH}) : \{E_i\}$ is a finite set of intervals of $\mathcal{L}$ with $\sum_i E_i = I\} \cup \sigma(A)$. Let $\mathcal{L}_1$ be any finite sublattice of $\mathcal{L}$ with atoms set $\{E_i\}_{i=1}^n$. Then A_{ii} is invertible in $alg\mathcal{L}E_i$ for each i. Let its inverse be A_{ii}^{-1}. By Lemma 3.3, we can suppose that the matrix of each operator in $alg\mathcal{L}$ has upper triangular form with respect to $E_1H \oplus E_2H \oplus \cdots \oplus E_nH$. Put $B = A_{11}^{-1} \oplus A_{22}^{-1} \oplus \cdots \oplus A_{nn}^{-1}$. We have $B \in alg\mathcal{L}$. Since $AB - I$ is strictly upper triangular, $N = AB - I$ is nilpotent in $alg\mathcal{L}$, that is, $N^n = 0$. Therefore, $AB = N + I$ is invertible in $alg\mathcal{L}$ and so is A. This shows that $0 \notin \sigma_{\mathcal{L}}(A)$.

A well-known result of Ringrose [8] (or see [2]) is that, for any compact operator K , there is a maximal nest $\mathcal{N}$ such that $K \in alg\mathcal{L}$ and $\sigma(K) = \sigma_{\mathcal{N}}(K) = \{0\} \cup \{\sigma(A_\alpha K|_{A_\alpha H})$: $\{A_\alpha\}$ is the set of all atoms of $\mathcal{N}\}$. The analogue is valid for CSL algebras which is actually implied in [5]. Here we give a direct proof. □

Theorem 3.4. *Let K be a compact operator in $alg\mathcal{L}$ for a CSL $\mathcal{L}$, and let G be the set of all atoms of $\mathcal{L}$. Then $\sigma_{\mathcal{L}}(K) = \sigma(K) = \{0\} \cup \{\underset{E\in G}{\cup}\sigma(EAE)\}$.*

Proof. By proposition 3.1, $\sigma(K) = \sigma_{\mathcal{L}}(K)$. We only need to show the second equality. It is easy to see that $\delta(K) = \sum_k E_kKE_k$ is compact and $\sigma_{\mathcal{L}}(K) \supseteq \underset{E\in G}{\cup}\{\sigma(EK|_{EH})\}$ by Theorem 3.2.

Let G be the set of all atoms of $\mathcal{L}$. It is clear that

$$\delta(K) = \sum_{E\in G} EKE$$

is a compact operator. For a finite sublattice $\mathcal{F}$ of $\mathcal{L}$, let

$$\mathcal{D}_{\mathcal{F}}(K) = \sum_{P\in\mathcal{F}} \triangle(P)K\triangle(P),$$

where $\triangle(P) = P - \sup\{Q \in \mathcal{F} : \ Q < P\}$. By Theorem 5.4 of [5], $\mathcal{D}_{\mathcal{F}}(K) \to \delta(K)$ as $\mathcal{F} \to \mathcal{L}$. For any $\epsilon > 0$, by the compactness of $\delta(K)$, there exist finitely many elements $E_1, \cdots, E_n$ in G such that $\|EKE\| < \frac{\epsilon}{2}$ for every E in $G\backslash\{E_1, \cdots, E_n\}$. Choose a finite sublattice $\mathcal{F}$ such that $\{E_1,\cdots,E_n\} \subset \{\triangle(P) : \ P \in \mathcal{F}\}$ and

$$\|\sum_{P\in\mathcal{F}} \triangle(P)K\triangle(P) - \delta(K)\| \leq \frac{\epsilon}{2}.$$

Hence, for each $P' \in \mathcal{F}$, either P' is an atom of $\mathcal{L}$ or

$$\begin{aligned}
&\|\triangle(P')K\triangle(P')\| \leq \|\sum_{\substack{P\in\mathcal{F}\\ \Delta(P)\neq E_1,\cdots E_n}} \triangle(P)K\triangle(P)\| \\
&\leq \|\sum_{P\in\mathcal{F}} \triangle(P)K\triangle(P) - \delta(K)\| + \|\sum_{\substack{E\in G\\ E\neq E_1,\cdots E_n}} EKE\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\end{aligned}$$

For a fixed $\lambda \neq 0$, let $\epsilon = \frac{|\lambda|}{2}$. So there is a finite sublattice $\mathcal{F}_0 \subset \mathcal{L}$ such that $\|\triangle(P)K\triangle(P)\| < \epsilon$ or $\triangle(P)$ is an atom of $\mathcal{L}$ for each P in $\mathcal{F}_0$. If $\lambda \notin \sigma(EK|_{EH})$ for every $E \in G$, then by the choice of ϵ, $\triangle(P)(\lambda - K)\triangle(P)$ is invertible in $alg\mathcal{L}\triangle(P)$ for every $P \in \mathcal{F}_0$. As in the proof of Theorem 3.2, we have $\lambda \notin \sigma_{\mathcal{L}}(K)$. Thus

$$\sigma_{\mathcal{L}}(K) \subseteq \bigcup_{E \in G} \sigma(EKE).$$

In [4], it is proved that if $\mathcal{L}$ is a nest which has no infinite-dimensional atom, then $T \in alg\mathcal{L}$ is invertible in $\mathcal{L}$ if and only if T is invertible in $alg\mathcal{L} + K(H)$. The analogue holds for a CSL algebra. □

Theorem 3.5. *Let $\mathcal{L}$ be a CSL on H which has no infinite-dimensional atom, and let $alg\mathcal{L} + K(H)$ be closed. Then $T \in alg\mathcal{L}$ is invertible if and only if it is invertible in $alg\mathcal{L} + K(H)$.*

Proof. We need only to prove that if T is invertible in $alg\mathcal{L} + K(H)$, then T is invertible in $alg\mathcal{L}$ for every T in $alg\mathcal{L}$.

Let $T^{-1} = A + K$ for some $A \in alg\mathcal{L}$ and $K \in K(H)$. Then $TA = I - TK$ implies that $TK = I - TA \in alg\mathcal{L}$ and TK is compact. If $(I - TK)PH = PH$ for some P in $\mathcal{L}$, then $TAPH = PH$ and hence $TPH = PH$ for $T^{-1}PH = APH \subseteq PH$. So we suppose that $(I - TK)PH \subset PH$ for some P in $\mathcal{L}$, that is, $1 \in \sigma_{\mathcal{L}(TK)}$.

Since TK is compact and in $alg\mathcal{L}$, by Theorem 3.4, there are only finitely many atoms $E_1, E_2, \cdots, E_n$ of $\mathcal{L}P$ such that $E_i(I - TK)|_{E_iH}$ is not invertible (note that TK is compact), We first prove that, for any interval E of $\mathcal{L}P$ with $EE_i = 0$ for each i, $ETEH = EH$. As in the proof of Theorem 3.4, since TK is compact there are finitely many intervals $F_1, F_2, \cdots, F_m$ of $\mathcal{L}E$ such that $E = F_1 + F_2 + \cdots + F_m$ and $F_i(I - TK)|_{F_iH}$ is invertible in $alg\mathcal{L}F_i$. Therefore we can prove that $ETA|_{EH}$ is invertible in $alg\mathcal{L}E$ as in Theorem 3.2. So $ET|_{EH}$ is invertible in $alg\mathcal{L}E$, and then $ETEH = EH$.

Let P_i be the smallest projection in $\mathcal{L}$ such that $P_iE_i = E_i$, and let $Q_i \in \mathcal{L}P$ be such that $E_i = P_i - Q_i$ for each i. If $P_jE_i = E_i$ for some $i \neq j$, then by the choice of P_j, Q_j, we have $Q_jE_i = E_i$ and then $P_j > Q_j > P_i > Q_i$. Without loss of generality, we assume that $Q_1 < P_1 < \cdots < Q_n < P_n$. So $T|_{Q_1H}$ is invertible in $alg\mathcal{L}Q_1$ by the previous paragraph. Since E_1 is of finite dimension, $TP_1H = P_1H$ by Lemma 2.2 . Since $(Q_2 - P_1)E_i = 0$ for each i, it follows that $(Q_2 - P_1)T|_{(Q_2 - P_1)H}$ is invertible in $alg\mathcal{L}(Q_2 - P_1)$ by the above paragraph. Hence, it is not hard to prove that $T|_{Q_2H}$ is invertible in $alg\mathcal{L}Q_2$ and $TQ_2H = Q_2H$. By induction, $ETE|_{EH}$ is invertible in $alg\mathcal{L}E$ for every interval E of $\mathcal{L}$; in particular, $T|_{PH}$ is invertible in $alg\mathcal{L}P$. Therefore, by the arbitrariness of P, T is invertible in $alg\mathcal{L}$. The proof is complete. □

Remark. When $\mathcal{L}$ has no atoms and $alg\mathcal{L} + K(H)$ is closed, for example, $\mathcal{L}(2^\infty, \leq, M_p)$ (see [6]), the result of Theorem 3.5 is valid clearly. When $\mathcal{L}$ has infinite dimensional atoms, the conclusion does not hold generally.

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