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# INFINITE LOOP SPACES AND NEISENDORFER LOCALIZATION

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Abstract. There is a localization functor $L$ with the property that $L(X)$ is the $p$-completion of $X$ whenever $X$ is a finite dimensional complex. This same functor is shown to have the property that $L(E)$ is contractible whenever $E$ is a connected infinite loop space with a torsion fundamental group. One consequence of this is that many finite dimensional complexes $X$ are uniquely determined, up to $p$-completion, by the homotopy fiber of any map from $X$ into the classifying space $BE$.

## 1. Introduction

Fix a rational prime $p$ and let $\mathcal{L}_p$ denote the homotopy functor defined by localizing with respect to the constant map $\varphi : B\mathbb{Z}/p \to *$, in the sense of Dror Farjoun [4], and then completing at the prime $p$ in the sense of Bousfield-Kan [2]. In symbols, $\mathcal{L}_p(X) = (L_\varphi(X))_p$. This is the localization functor to which the title refers. Now if $X$ is a finite nilpotent complex, it follows from Miller's solution to the Sullivan conjecture that $\mathcal{L}_p(X) \simeq X_p$. At first glance, this would suggest that the functor $\mathcal{L}_p$ is unlikely to yield any new information. However, in [10], Neisendorfer showed that this functor has a remarkable property when applied to the $n$-connected cover, $X\langle n\rangle$, of certain spaces $X$. His main result was the following.

**Theorem 1.** *Let $X$ be a 1-connected finite dimensional complex with $\pi_2(X)$ a torsion group. Then $\mathcal{L}_p(X\langle n\rangle) \simeq X_p$ for any positive integer $n$.*

Thus, up to $p$-completion, no information is lost when one passes to the $n$-connected cover of such a complex! Of course, this is false for more general spaces, where the first $n$ homotopy groups and the corresponding $k$-invariants are irretrievably lost in such a process. Thus Theorem 1 reveals a subtle homotopy property of certain finite dimensional complexes. It is worth noting that Theorem 1 also applies to those iterated loop spaces of $X$ which have the required connectivity.

The main result of this paper is another application of the functor $\mathcal{L}_p$. It shows that the circle and its $n$-fold products are essentially the only infinite loop spaces of finite type which are not destroyed by Neisendorfer localization.

**Theorem 2.** *Let $E$ be a connected infinite loop space whose fundamental group is a torsion group. Then $\mathcal{L}_p(E) \simeq *$ for each prime $p$.*

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This result is about as close as one can get to saying that, in general, the classifying space $B\mathbb{Z}/p$ is the prime ingredient or the major building block in the $p$-primary part of infinite loop spaces of the sort just described. However, under certain special conditions (see Theorem 2.1) one can say more; in that case there is a cellular inequality $B\mathbb{Z}/p \ll E$ in the sense of Dror Farjoun, [5]. Roughly this means that $E$ can be constructed from $B\mathbb{Z}/p$ and its suspensions in much the same way that a $CW$-complex is built out of the sphere $S^0$ and its suspensions. More accurately, it means that $E$ is a member of the smallest class of spaces which contains $B\mathbb{Z}/p$ and is closed under weak equivalences and pointed homotopy colimits, [5].

The following theorem, from [8], is similar to Theorem 2 although the precise connection between the two results is not clear to me. This result is another offspring of the Sullivan conjecture.

**Theorem 3.** *Let $E$ be a connected infinite loop space whose fundamental group is a torsion group, and assume that $Y$ is a nilpotent finite $CW$-complex. Then the function space of based maps from $E$ to the $p$-completion of $Y$ is weakly contractible.*

At any rate, we see that as far as maps into finite complexes are concerned, most infinite loop spaces behave just like $B\mathbb{Z}/p$. It was this fact that led me to wonder if Theorem 2 might be true.

There is a crucial step in the proof of Neisendorfer's theorem where one shows that $\mathcal{L}_p(Y) \simeq *$ when $Y$ is a connected loop space with a torsion fundamental group and only finitely many nonzero higher homotopy groups. Theorem 2 plays the same role in the next result, which deals with the infinite suspension. As usual, let $QX = \operatorname{colim}_n \Omega^n\Sigma^n X$ for a pointed space $X$ and let $\iota : X \to QX$ be the usual inclusion. More precisely, $\iota$ is the map whose adjoint is the identity on $\Sigma^\infty X$. Let $Fib(\iota_X)$ denote the homotopy fiber of the map $\iota$. The final result deals with the homotopy fibration

$$Fib(\iota_X) \xrightarrow{\ i\ } X \xrightarrow{\ \iota\ } QX.$$

**Corollary 4.1.** *Let $X$ be a 1-connected finite dimensional complex with $\pi_2(X)$ a torsion group. There is a homotopy equivalence $\mathcal{L}_p(Fib(\iota_X)) \simeq X_p$ induced by the map $i$.*

In other words, up to $p$-completion, such a space can be recovered from the fiber of its infinite suspension! Notice that the map $X \to QX$ is an equivalence through some finite range of dimensions (the stable range—which, of course, depends on the connectivity of $X$) and one might think that this stable information is lost when one passes to the fiber. However, just as in Neisendorfer's theorem, the fiber somehow retains this data.

## 2. Proofs

Each space in this paper is assumed to have the homotopy type of a $CW$-complex. All spaces and maps are assumed to be pointed. To simplify notation, let $B = B\mathbb{Z}/p$. We will sometimes assume certain spaces are $p$-local without burdening the notation with this assumption. Let $\mathcal{C}(B)$ denote the smallest class of connected spaces which contains $B$ and is closed under both weak equivalences and pointed homotopy colimits. It is not difficult to see that the one-point space $*$, is in this class, as are the iterated suspensions $\Sigma^n B$ for $n = 1, 2, ...$ Following Dror Farjoun, [7], write $X \gg B$ iff $X$ belongs to $\mathcal{C}(B)$. Note also that $\mathcal{L}_p(X) \simeq *$ whenever

$X \gg B$ (*ibid.*, Prop. A7). In the same paper it is noted that for connected spaces $X$ and $Y$, one has

$$\Sigma X \ll Y \quad \text{iff} \quad X \ll \Omega Y.$$

Thus the inequality $\Sigma^n B \ll \Sigma^n B$ implies that $B \ll \Omega^n \Sigma^n B$ for each $n$. Taking colimits, it follows that $QB$ is a member of $\mathcal{C}(B)$. Let $QS^0\langle 0\rangle$ denote the base point component of $QS^0$. This component is a retract of $QB$ by the Kahn-Priddy Theorem, [1]. It is not difficult to see that if $Y$ is a retract of a connected $H$-space $X$, then $Y \gg X$, and hence

$$B \quad \ll \quad QS^0\langle 0\rangle.$$

Since $QS^n\langle n\rangle \;\simeq\; \Omega QS^{n+1}\langle n+1\rangle$, it follows that

$$\Sigma^n B \;\ll\; QS^n\langle n\rangle$$

for each $n$. Thus $\mathcal{L}_p(QS^n\langle n\rangle) \simeq *$ for each $n \geq 0$. The next step is to establish the same result for $QS^n$ for each $n \geq 2$. To do this we will use the fibration

$$QS^n\langle n\rangle \longrightarrow QS^n \longrightarrow K(\mathbb{Z}, n)$$

and the following important result due to Dror Farjoun, [5].

**Theorem 5.** *Given a map $f : X \to Y$ and a fibration $F \longrightarrow C \xrightarrow{\ \pi\ } D$, if $L_f(F) \simeq *$, then $L_f(\pi)$ is a homotopy equivalence.*

There is a technical point that should be mentioned before using this result. Recall that $\mathcal{L}_p$ was defined as the composite of two functors: $L_\varphi$, followed by $p$-completion. Thus it is not obvious that it can be used with Theorem 5. However, Neisendorfer shows in [10], Lemma 1.4, that when applied to simply connected spaces, $\mathcal{L}_p = L_f$ where $f : B \vee M \to *$. Here $B$ is again $B\mathbb{Z}/p$ and $M$ is the Moore space with one nontrivial reduced homology group isomorphic to $\mathbb{Z}[1/p]$ in dimension 1. Thus Theorem 5 can be used with $\mathcal{L}_p$.

Now $\mathcal{L}_p(K(\pi, n)) \simeq *$ for each $n \geq 2$ and for any abelian group $\pi$, according to Casacuberta ([3], §7). Therefore, using Theorem 5 on the fibration over $K(\mathbb{Z}, n)$ mentioned earlier, it follows that the Neisendorfer localization of $QS^n$ is contractible for each $n \geq 2$.

Let $X$ be a 1-connected finite $CW$-complex. Consider the cofibrations which inductively construct its skeleta,

$$\vee S^n \;\to\; X_{n-1} \;\longrightarrow\; X_n,$$

starting with $X_1 = *$. Each of these cofiber sequences involves only a finite number of spheres. The functor $Q(\ \ )$ converts these cofibrations into fibrations,

$$\times Q(S^n) \;\longrightarrow\; Q(X_{n-1}) \longrightarrow Q(X_n).$$

The functor $\mathcal{L}_p$ commutes with finite products, [6], and so Theorem 5 together with the result for $QS^n$ implies that $\mathcal{L}_p(QX) = \mathcal{L}_p(QX_n) = \cdots = \mathcal{L}_p(QX_1) = *$.

Now let $E$ denote a 1-connected infinite loop space. Choose a $CW$-decomposition of $E$ in which every finite subcomplex is 1-connected. It is clear that

$$QE \;\simeq\; \operatorname{colim} Q(K).$$

where $K$ runs through the finite subcomplexes of $E$. Apply the functor $\mathcal{L}_p$ to both sides of this equation and note that, in general,

$$L_f(\operatorname{colim}_\alpha X_\alpha) \quad\simeq\quad L_f(\operatorname{colim}_\alpha L_f(X_\alpha))$$

according to Dror Farjoun, [7].[1] This implies that $\mathcal{L}_p(QE) \simeq *$. Since $E$ is an infinite loop space, it is a retract of $QE$. It follows at once that $\mathcal{L}_p(E)$ must also be contractible.

Finally consider the case where $E$ is not simply connected. Its fundamental group $\pi$ must be an abelian torsion group. Apply Theorem 5 to the fibration

$$E\langle 1\rangle \longrightarrow E \longrightarrow K(\pi, 1).$$

Since $\mathcal{L}_p$ annihilates both the fiber and base space, the conclusion of Theorem 2 follows.

The following result gives some conditions under which $B \ll E$. I am not certain that this result is best possible.

**Theorem 2.1.** *Assume that $E$ is an infinite loop space which is $p$-local, rationally trivial, and $n$-connected where $n \geq 1$. Then* $\Sigma^{n-1}B\mathbb{Z}/p \ll E$.

*Proof.* Since $E$ is $n$-connected and rationally trivial it can be obtained as the homotopy colimit of a direct system of finite complexes with the same two properties. These finite complexes, in turn, can be constructed using mod-$p^r$ Moore spaces, for various $r$, instead of spheres. Express this particular colimit as

$$E \simeq \operatorname{colim}_\alpha L_\alpha.$$

Write $A < X$ to signify that the localization of $X$ with respect to the map $A \to *$ is contractible. The implication $\Sigma A < X \Longrightarrow A \ll X$ is due to Dror Farjoun [5] and plays a crucial role in what follows.

**Lemma 2.1.1.** *If $0 \leq k < n$, then* $\Sigma^k B\mathbb{Z}/p < Q(S^n \cup_{p^r} e^{n+1})$.

Assume for the moment that this inequality is true. Let $L$ denote one of the finite complexes in the expression of $E$ as a colimit. Then $L$ can be obtained in a finite sequence of cofibrations

$$M_q \longrightarrow L_{q-1} \longrightarrow L_q$$

where each $M_q$ is a finite wedge of Moore spaces, each of which is at least $n$-connected. Apply $Q(\quad)$ to this cofiber sequence to obtain a fibration. Using Lemma 2.1.1 it follows that $\Sigma^n B < Q(M_q)$. Apply Theorem 5 to this fibration, where $f : \Sigma^n B \to *$. It follows from a finite induction that $\Sigma^n B < QL$. Since

$$QE \simeq \operatorname{colim}_\alpha QL_\alpha,$$

it follows that $\Sigma^n B < E$ (and hence, that $\Sigma^{n-1}B \ll E$ ) by the same arguments that were used in the proof of Theorem 2.

To verify the lemma, take the cofiber sequence

$$S^n \xrightarrow{p^r} S^n \longrightarrow S^n \cup_{p^r} e^{n+1},$$

apply the functor $Q$ to it, and then take $n$-connected covers. It was shown earlier that $\Sigma^n B < Q(S^n)\langle n\rangle$. Thus Theorem 5, applied to the fibration just obtained, implies that $\Sigma^n B < Q(S^n \cup_{p^r} e^{n+1})\langle n\rangle$. Theorem 5 can also be used to show first

[1] This result seems best possible, since $L_f$ does not commute with direct limits in general. A good example is the case of $QX = \operatorname{colim} \Omega^n\Sigma^n X$ when $X$ is a mod $p$ Moore space. It has been shown that $\mathcal{L}_p(QX) = *$, and yet $\mathcal{L}_p$ acts like the identity on each term in the colimit.

that $B < K(\mathbb{Z}/p^r, 1)$ and then that $\Sigma^{n-1}B < K(\mathbb{Z}/p^r, n)$. The conclusion of the lemma then follows from a final application of Theorem 5 to the fibration

$$Q(S^n \cup_{p^r} e^{n+1})\langle n\rangle \longrightarrow Q(S^n \cup_{p^r} e^{n+1}) \longrightarrow K(\mathbb{Z}/p^r, n).$$

Corollary 4.1 is the most interesting special case of the following result.

**Corollary 4.0.** *Let $X$ be a 1-connected finite dimensional complex and let $E$ be a 1-connected infinite loop space with $\pi_2 E$ torsion. Then $\mathcal{L}_p(Fib(\rho)) \simeq X_p$ for any map $\rho : X \to E$.*

*Proof.* Convert $\rho$ into a fibration, move one step back, and apply Theorems 2 and 5 to the fiber sequence

$$\Omega E \longrightarrow Fib(\rho) \longrightarrow X. \quad \square$$

## Final remarks

The results in this paper give further testament to the power of recent advances made in unstable localization by Bousfield, Dror Farjoun, and others; e.g., see [3]. The preprints of Dror Farjoun, listed below, are available on the Hopf Topology archive. The material they contain will soon appear in a Springer monograph by him.

## References


1. J. F. Adams, *The Kahn-Priddy theorem*, Proc. Camb. Phil. Soc. (1973) **73** 45–55. MR **46:**9976
2. A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Math. **304**, Springer, 1972. MR **51:**1825
3. C. Casacuberta, *Recent advances in unstable localization*, CRM Lecture Notes **6**, (1994), 1–22. MR **95e:**55014
4. E. Dror Farjoun, *Homotopy localization and $v_1$-periodic spaces* Springer Lecture Notes in Math. **1509**, (1991), 104–113. MR **93k:**55013
5. E. Dror Farjoun, *Localizations, fibrations and conic structures*, preprint 1992.
6. E. Dror Farjoun, *Cellular spaces*, preprint 1993.
7. E. Dror Farjoun, *Cellular inequalities*, Contemporary Math. **181** (1995), 159–181. CMP 95:09
8. C. A. McGibbon, *A note on Miller's theorem about maps out of certain classifying spaces*, Proc. Amer. Math. Soc., to appear.
9. H. Miller, *The Sullivan fixed point conjecture on maps from classifying spaces*, Annals of Math. **120** (1984) 39–87; **121** (1985), 605–609. MR **85i:**55012; MR **87k:**55020
10. J. A. Neisendorfer, *Localization and connected covers of finite complexes*, Contemporary Math. **181** (1995), 385–390. MR **96a:**55019


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