

## $q$ -ANALOGUE TRIANGULAR NUMBERS AND DISTANCE GEOMETRY

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ABSTRACT. The so-called “ $q$ -identities” play a major role in classical combinatorics. Most of them can be viewed as arising somehow in the context of hypergeometric series. Here we present a “sum of squares” identity involving  $q$ -analogues of the triangular numbers that, by contrast, arises in the context of distance geometry.

### §1. INTRODUCTION

Our purpose is to establish the following  $q$ -identity, and explain its relation to distance geometry. For integers  $n \geq 1$  define

$$P_n(q) = n^2 q^{2(n-1)} - \sum_{k=1}^{n-1} k q^{2(k-1)}$$

and

$$S_n(q) = \frac{\sqrt{P_n(q)}}{n} + \sum_{m=1}^{n-1} \frac{\sqrt{P_m(q)}}{m(m+1)}.$$

Set  $S_0(q) = 0$ .

**Theorem.**  $\sum_{k=0}^n (S_n(q) - S_k(q))^2 = nq^{2(n-1)}$ .

For  $q < 1$  and  $q$  sufficiently close to 1, this involves only real numbers. The  $n$ th triangular number  $P_n$  is

$$P_n = \sum_{k=1}^n k = n(n+1)/2;$$

two familiar properties of triangular numbers are

$$n^2 = P_n + P_{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{P_n} = 2.$$

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The first immediately shows that  $P_n(1) = P_n$ , so  $P_n(q)$  is a  $q$ -analogue of  $P_n$ , while the sum in the definition of  $S_n(q)$  is reminiscent of the second property. Simple estimates show that all the zeros of  $P_n(q)$  lie strictly inside the unit circle.

The main connection of the identity with distance geometry is given in §3. Also, the proof of the identity in §2 below involves an incidental generalization of the familiar parallelogram law (replace the  $a_i$  by vectors, and interpret  $a_i a_j$  as the inner product  $(a_i, a_j)$ ; then the case  $n = 2$  of the Lemma is the parallelogram law).

## §2. THE PROOF

For  $n \geq 1$  define

$$L_n(a_1, \dots, a_n) = \frac{a_n}{n} + \sum_{i=1}^{n-1} \frac{a_i}{i(i+1)}$$

and set  $L_0 = 0$ .

**Lemma.**

$$\begin{aligned} F(a_1, \dots, a_n) &:= \sum_{k=0}^n [L_n(a_1, \dots, a_n) - L_k(a_1, \dots, a_k)]^2 \\ &= L_n(a_1^2, a_2^2, \dots, a_n^2). \end{aligned}$$

*Proof.* Let

$$\phi(t, j) = \begin{cases} 1/(j(j+1)), & j < t, \\ 1/t, & j = t, \\ 0, & j > t. \end{cases}$$

Then

$$F_{ij} := \frac{\partial^2 F}{\partial a_i \partial a_j} = 2 \sum_{k=0}^n [\phi(n, j) - \phi(k, j)][\phi(n, i) - \phi(k, i)].$$

If  $k > \min(i, j)$ , then one of the two factors above is 0. Hence

$$F_{ij} = 2 \sum_{k=0}^{\min(i, j)} [\phi(n, j) - \phi(k, j)][\phi(n, i) - \phi(k, i)].$$

First, say  $i < j < n$ . Then

$$F_{ij} = \left( 2 \sum_{k=0}^{i-1} \right) + \frac{2}{j(j+1)} \left[ \frac{1}{i(i+1)} - \frac{1}{i} \right]$$

where the sum to  $(i-1)$  is

$$2i \frac{1}{i(i+1)} \frac{1}{j(j+1)}.$$

Secondly, say  $i < j = n$ . Then

$$F_{ij} = \left( 2 \sum_{k=0}^{i-1} \right) + \frac{2}{n} \left[ \frac{1}{i(i+1)} - \frac{1}{i} \right]$$

where the sum to  $(i-1)$  is  $2/(n(i+1))$ . In either case  $F_{ij} = 0$ , so  $F_{ij}$  has no “mixed terms”  $a_i a_j$  with  $i \neq j$ . Next, say  $i = j < n$ . Then

$$F_{ij} = \left( 2 \sum_{k=0}^{i-1} \right) + 2 \left[ \frac{1}{i(i+1)} - \frac{1}{i} \right] \left[ \frac{1}{i(i+1)} - \frac{1}{i} \right]$$

where the sum to  $(i-1)$  is

$$\frac{2i}{[i(i+1)]^2}.$$

Upon dividing by  $2!$  we find that the coefficient of  $a_i^2$  is  $1/(i(i+1))$ . Finally, the coefficient of  $a_n^2$  is  $1/n$  by direct inspection. This proves the Lemma.  $\square$

We can now prove the Theorem. By the Lemma, the sum of the squares is

$$\sum = \frac{P_n(q)}{n} + \sum_{k=1}^{n-1} \frac{P_m(q)}{k(k+1)}.$$

Clearly this polynomial in  $q$  has  $nq^{2(n-1)}$  as its highest order term. Its constant term is

$$\frac{1}{2} - \left( \sum_{k=2}^{n-1} \frac{1}{k(k+1)} + \frac{1}{n} \right) = \frac{1}{2} - \left( 1 - \frac{1}{2} \right) = 0.$$

In general, the coefficient of  $q^{2m}$  is

$$\begin{aligned} & \frac{(m+1)^2}{(m+1)(m+2)} - \left( \sum_{k=m+2}^{n-1} \frac{m+1}{k(k+1)} + \frac{m+1}{n} \right) \\ &= (m+1) \left( \frac{1}{m+2} - \left( \sum_{k=m+2}^{n-1} \frac{1}{k(k+1)} + \frac{1}{n} \right) \right) \end{aligned}$$

and (by partial fractions, for example) this is easily seen to be 0.

*Remark.* If the exponent 2 on the left-hand summand of the Lemma is replaced by 1, the sum becomes simply  $a_n$ . Essentially the same proof works (with rather simpler details). Now, by Hölder’s inequality,

$$f(n, q, \lambda) := \left( \frac{1}{n} \sum |S_n(q) - S_k(q)|^\lambda \right)^{1/\lambda} \leq q^{n-1}$$

for  $1 \leq \lambda \leq 2$ . Here  $f(n, 1, \lambda)$  seems to be decreasing (slowly) in  $n$ . Set

$$g(\lambda) = \liminf_{n \rightarrow \infty} f(n, 1, \lambda), \quad 1 \leq \lambda \leq 2.$$

From the above it is not hard to see that

$$\frac{1}{\sqrt{2}} = g(1) \leq g(\lambda) \leq g(2) = 1, \quad 1 \leq \lambda \leq 2,$$

and from a familiar property of power means we see that  $g(\lambda)$  is non-decreasing. Probably  $\liminf$  may be replaced by  $\lim$ , but the author has not been able to identify  $g(\lambda)$ .

## §3. CONNECTION WITH DISTANCE GEOMETRY

A fundamental question in distance geometry is the following: given a set of points  $S$  and an assignment of a distance to every pair of distinct points in  $S$ , can one find in a given metric space  $M$  a set of points  $S^*$  in one-to-one correspondence with  $S$  and with corresponding distances equal? This, of course, is the problem of isometric embedding. For example, if  $|S| = 4$  and every distance is 1, this can be done if  $M = E^3$  but not if  $M = E^2$ . For a definitive treatise on distance geometry see [B].

We now restrict ourselves to a quite specific question. Given a curve in a metric space  $M$ , does it contain a sequence of distinct points  $x_0, x_1, x_2, x_3, \dots$  (in that order, say) such that the distance from  $x_n$  to the previous points (i.e., to the  $x_i$  with  $0 \leq i \leq n-1$ ) is on average always 1? For the Euclidean line the sequence of initial segments of the harmonic series, namely

$$0, \quad 1, \quad 1 + \frac{1}{2}, \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \dots,$$

is quickly seen to have this property. Next, fix  $q < 1$ . Is there such a sequence for which the average distance from  $x_n$  to the previous  $x_i$  is  $q^{n-1}$  (this is the  $q$ -analogue of the previous question)? Here  $x_0 = 0$  and

$$x_n = q^{n-1} + \sum_{m=2}^n \frac{q^{m-2}}{m}$$

is quickly seen to have this property. Moreover, the  $x_n$  converge to a limit, namely

$$x_\infty = -[q + \log(1 - q)]/q^2.$$

(The monotone dependence of  $x_\infty$  on  $q$  is equivalent to the inequality  $-\log(1 - q) < q + q^2/(2(1 - q))$ .) The problem is more interesting, however, if we replace average distance by *root-mean-square distance*. In this case, for  $q < 1$ , *no such sequence can be infinite*, but for  $q$  sufficiently close to 1 such a sequence may have an *arbitrarily large* number of terms. To see this, simply note that since the  $x_k$  are an increasing sequence that is determined sequentially, they must coincide with the  $S_n(q)$  of the Theorem! Now

$$S_{n+1}(1) - S_n(1) = \frac{-\sqrt{P_n}}{n+1} + \frac{\sqrt{P_{n+1}}}{n+1} > 0,$$

so the  $S_n(q)$  form an increasing sequence when  $q$  is close enough to 1. Moreover, for  $q < 1$  fixed but  $n$  large,

$$\begin{aligned} P_n(q) &\leq n^2 q^{2(n-1)} - \sum_{k=1}^{[3n/4]} k q^{2(k-1)} \\ &\leq n^2 q^{2(n-1)} - q^{[3n/2]} \sum_{k=1}^{[3n/4]} k \\ &\leq n^2 q^{2(n-1)} - \frac{1}{8} n^2 q^{[3n/4]} < 0. \end{aligned}$$

Hence the process always terminates.

*Remarks.* This geometric problem was suggested by the 1964 theorem of Gross [G, C-F-G] in game theory that has found a home in distance geometry. For a

survey and further references see [C-M-Y]. For a survey of  $q$ -analogues, see [A]. An application of isometric embedding to a sum of distances problem is given in [S].

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