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AFFINE AND HOMEOMORPHIC EMBEDDINGS INTO ℓ^2

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ABSTRACT. It is shown that

(1) a locally compact convex subset C of a topological vector space that admits a sequence of continuous affine functionals separating points of C affinely embeds into a Hilbert space;
(2) an infinite-dimensional locally compact convex subset of a metric linear space has a central point;
(3) every σ-compact locally convex metric linear space topologically embeds onto a pre-Hilbert space.

Let C be a convex subset of a separable metric linear space E. The question arises of whether C can be embedded onto a convex subset of ℓ^2, or onto a linear subspace of ℓ^2 provided C is linear.[1] A positive answer to this question would reduce the topological identification of such arbitrary convex sets C to those contained in ℓ^2. Let us mention that this identification problem has a satisfactory solution for complete sets C which are AR's. It is shown in [DT1] and [DT2] that an infinite-dimensional set C which is an AR is either a (contractible) Hilbert cube manifold or a copy of ℓ^2. The case of incomplete C, in particular, the σ-compact case, is far from being settled (even for C which are AR's). By Dugundji's theorem (see [BP2, p. 61]), every convex subset C of a locally convex E is an AR. It is unknown whether this is true for an arbitrary metric linear space E.[1] Therefore the AR-property of C could possibly be an obstacle for embedding C onto a convex subset of ℓ^2.

The first part of the paper is devoted to affine embeddings of locally compact convex sets C into ℓ^2. Due to an elementary observation of Klee (see [BP2, p. 98]) every compact convex subset of a locally convex E affinely embeds into ℓ^2. Following this, the notion of a Keller set was introduced which proved to be important in solving some identification problems, see [BP2]. An infinite-dimensional compact convex set C is a Keller set if it affinely embeds into ℓ^2. It is known that not all infinite-dimensional compact convex sets C are Keller sets. The examples of compact convex subsets C without extreme points given by Roberts (see [R1], [R2], and [KPR]) obviously cannot be affinely embedded into ℓ^2. Refining the construction of Roberts one can find such C with the AR-property ([DM2] and

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[1]By the time this paper was sent to the printer both these questions were answered in the negative; see [Mar] and [Ca].

[NT]). We show that every locally compact (closed) convex subset C of E whose dual E^* separates points of C admits an affine (closed) embedding into ℓ^2; this answers [BP2, Problems 1 and 2, p. 99].

The importance of a Keller space C comes from the fact that C has a so-called central point (see [BP2, p. 157]). This fact was recovered in [vBDHvM] for an arbitrary infinite-dimensional compact convex set C. We show that this remains true for locally compact convex C, solving a question raised in [vBDHvM]. Similarly as in [vBDHvM], we apply this fact to show that the AR-property of a locally compact convex C is equivalent to the homeomorphism extension property between so-called Z_∞-sets in C.

In the second part of the paper we show that every σ-compact linear subspace E that is locally convex embeds onto a dense linear subspace of ℓ^2, which answers [DM1, Problem 587]. The proof is obtained by constructing a pre-Hilbert space H so that the classes of compacta embeddable in E and H coincide. Then we apply the uniqueness theorem on absorbing sets to get a homeomorphism of the linear completions $\widehat{E}$ and $\widehat{H}$ of E and H, respectively, that sends E onto H. Earlier, using the same approach, it was shown [Do3] that such E embeds onto a linear subspace of the countable product of lines. We leave as an open problem the question of whether an arbitrary convex σ-compact set C of such E embeds onto a convex subset of ℓ^2.

1. Affine embeddings of convex local compacta

Let us start with the following fact which belongs to mathematical folklore.

1.0. Lemma. *If K is a locally compact, σ-compact space which admits a sequence $\{g_n\}_{n=1}^\infty$ of continuous real functions separating points of K, then K is metrizable.*

Proof. By our assumpion, K locally embeds in the countable product of lines, and therefore is locally metrizable. Since K is σ-compact, it is Lindelöf, and consequently it is paracompact. Applying [En, Ex. 5.4.A, p. 415], K is metrizable. □

Assume that a locally compact convex set C of a topological vector space E contains the origin $0 \in E$. Then there exists a closed symmetric neighborhood U of 0 in E such that

$(*)$ $U \cap C$ is compact.

We also have for every $\lambda \geq 1$

$(**)$ $(\lambda U) \cap C \subset \lambda(U \cap C)$.

1.1. Theorem. *Let C be a locally compact (closed) convex subset of a topological vector space E. If there exists a sequence of continuous affine functionals $\{g_n\}_{n=1}^\infty$ on C separating points of C, then C admits an affine (closed) embedding into ℓ^2. Here ℓ^2 can be replaced by an arbitrary infinite-dimensional, complete metric linear space.*

Proof. We can assume that $0 \in C$. Pick a closed symmetric neighborhood U of 0 in E such that $(*)$ is satisfied. Let $p_n = \max\{|g_n(x)| : x \in U \cap C\}$. Without loss of generality we may assume that $p_n \leq 2^{-n}$ (if not, replace the functionals g_n by suitable positive multiples). Define $T(x) = (g_n(x))$ for $x \in C$. The estimate for p_n's and $(**)$ together with the assumptions of the theorem imply that T is an injective affine map into ℓ^2 and that, for every $\lambda \geq 1$, the restricted map $T|(\lambda U) \cap C$ is continuous. Hence $T : C \to \ell^2$ is continuous.

Let Y be an infinite-dimensional, complete metric linear space and $|\cdot|$ be a monotone F-norm on Y, i.e., for every number $|\lambda| \leq 1$ and every $y \in Y$ we have $|\lambda y| \leq |y|$. By S. Mazur's lemma (see the proof of [BP2, Prop. 2.2, p. 268]), there are (linearly independent) vectors $y_n \in Y$ such that for every sequence of reals $\{t_n\}_{n=1}^{\infty}$ with $|t_n| \leq 2^{-n}$, $n \geq 1$, the series $\sum_{n=1}^{\infty} |t_n y_n|$ converges and the map $(t_n) \to \sum_{n=1}^{\infty} t_n y_n$ is injective. Let $T(x) = \sum_{n=1}^{\infty} g_n(x) y_n$ for $x \in C$. It is easy to check that T is a well-defined, injective, affine, continuous transformation of C into Y.

Using $(**)$ and $(*)$, C is σ-compact. By 1.0, C is metrizable. Therefore the following lemma completes the proof of the theorem.

1.2. Lemma. *If C is a (closed) locally compact metrizable, convex subset of a topological vector space E and $T : C \to Y$ is an injective continuous affine map into a metric linear space, then T is a (closed) embedding.*

Proof. Let $|\cdot|$ be a monotone F-norm on the space Y. First we shall show that

$$S = T^{-1} : T(C) \to E$$

is continuous at each point $y_0 = T(x_0) \in T(C)$. Replacing T by the map $x \to T(x + x_0) - Tx_0$, it is no loss of generality to assume that $x_0 = 0_E$, $y_0 = 0_Y$. (In the sequel we shall use one symbol 0 for denoting the points 0_E, 0_Y and the number zero.) Let U be a neighborhood of 0 in E satisfying $(*)$. We claim that there exists an $\epsilon > 0$ such that

$(***)$ whenever $|T(x)| < \epsilon$, $x \in C$, then $x \in \operatorname{int} U \cap C$.

If the claim were not true, then there would exist a sequence (x_n) of elements of C such that $x_n \in \partial U \cap C$ and such that $Tx_n \to 0$ (use monotonicity of the F-norm $|\cdot|$). By $(*)$, there would exist a subsequence (z_n) of (x_n) with $Tz_n \to 0$ such that $z_n \to z \in \partial U \cap C$, contradicting the continuity of T.

Since every neighborhood of 0 in E contains a neighborhood satisfying $(*)$, the condition $(***)$ implies the continuity of S at 0.

Assume that C is closed in E and that $y_n = Tx_n \to y \in Y$, $x_n \in C$. Since the set $\{Tx_n\}_{n=1}^{\infty} \cup \{y\}$ is compact, there exists $0 < a \leq 1$ such that $|aTx_n| < \epsilon$ for all n, where ϵ is that of $(***)$. Using the facts that $0 \in C$ and $a \leq 1$, all the points ax_n are in C. By $(***)$ and the fact that T is affine, $ax_n \in \operatorname{int} U \cap C$. By $(*)$, there is a subsequence (az_n) of the sequence (ax_n) such that $az_n \to z \in U \cap C$. Therefore $z_n \to a^{-1}z$ which is in C (since C is closed) and $y = \lim Tx_n = \lim Tz_n = T(a^{-1}z) \in T(C)$. This completes the proof. □

In connection with the last part of our proof let us notice the following elementary fact.

1.3. Remark. If a locally compact convex set C is closed in a metric linear space, then C is also closed in the (linear) completion $\widehat{E}$ of E. Apply 1.2 to the inclusion map of C into $\widehat{E}$.

Let us note the following generalization of [B, Cor. 1].

1.4. Corollary. *Every metrizable locally compact convex (closed) subset C of a topological vector space E whose dual E^* separates points of E admits an affine (closed) embedding into ℓ^2.*

Proof. We can assume that $0 \in C$. Let U be a closed neighborhood of 0 in E satisfying $(*)$. Fix a metric d on C. Since the set $A_\epsilon = \{k_1 - k_2 : k_1, k_2 \in U \cap C,$ and $d(k_1, k_2) \geq \epsilon\}$, $\epsilon > 0$, is compact, there are finitely many functionals that separate points of A_ϵ from 0. It follows that there exists a sequence of functionals $\{x_n^*\}_{n=1}^\infty \subset E^*$ that separate 0 from points of $(U \cap C) - (U \cap C) \setminus \{0\}$. Consequently, $\{x_n^*\}_{n=1}^\infty$ separates points of $U \cap C$. By $(**)$, $\{x_n^*\}_{n=1}^\infty$ separates points of C and 1.1 is applicable. □

2. Central points and application to the AR-property

A closed subset A of a metric space X is called a Z_∞-set [Tor], if every map of an n-dimensional cube I^n, $n \geq 1$, into X can be approximated by maps whose images miss A. If the space X is an ANR, then the set A is simply called a Z-set. The distinction of the terminology comes from the fact that for ANR spaces X the above property guarantees that every map of X into X can be approximated by maps whose images miss X (the property commonly understood to describe the fact that A is a Z-set in X). If A is not necessarily closed and satisfies the mapping condition for a Z_∞-set, then A is called locally homotopy negligible in X. We will use the above notions for the case where X is a convex subset of a metric linear space and hence, in general, it is merely contractible and locally contractible (and perhaps, is not an AR).

Throughout this section C will denote an infinite-dimensional locally compact convex subset of a complete metric linear space E endowed with an F-norm $|\cdot|$. We say that $x_0 \in C$ is a central point for C if the set $x_0 + [0,1) \cdot (C - x_0)$ is a countable union of Z_∞-sets in C. Our main result is

2.1. Theorem. *There exists a central point for C.*

The proof employs a few auxiliary lemmas.

2.2. Lemma. *The set $x_0 + [0,1) \cdot C$ is a countable union of Z_∞-sets iff each $x_0 + [0, 1 - \frac{1}{n}) \cdot C$ is locally homotopy negligible in C.*

Proof. Since each $x_0 + [0, 1 - \frac{1}{n}) \cdot C$ is a σ-compact subset of C and since a compact subset of a locally homotopy negligible set is a Z_∞-set, the fact that each $x_0 + [0, 1 - \frac{1}{n}) \cdot C$ is locally homotopy negligible implies that $x_0 + [0,1) \cdot C$ is a countable union of Z_∞-sets in a complete metrizable space C. By [Tor, Cor. 2.7] it follows that $x_0 + [0,1) \cdot C$ is locally homotopy negligible. □

2.3. Lemma. *For every compact convex subset $C_0 \subseteq C$ the set $C - C_0$ is convex and locally compact.*

Proof. Clearly $C - C_0$ is convex and we need to verify that $C - C_0$ is locally compact. By $(**)$ it is enough to check the local compactness at $0 \in C - C_0$. Using the local compactness of C, for every $x \in C$ there exists $\delta_x > 0$ such that

$$C_x = \{y \in C : |x - y| \leq 2\delta_x\} \text{ is compact.} \tag{1}$$

By the compactness of C_0, there are $x_1, x_2, \dots, x_k \in C_0$ such that

$$C_0 \subset \bigcup_{i=1}^{k} B_{x_i}, \tag{2}$$

where $B_x = \{y \in C : |x - y| < \delta_x\}$, $x \in C_0$. Set $\delta = \min\{\delta_{x_1}, \delta_{x_2}, \dots, \delta_{x_k}\}$. We claim that

$$\{y - x : y \in C, x \in C_0 \text{ and } |y - x| \le \delta\} \tag{3}$$

is compact. Given an arbitrary $x \in C_0$, there exists i, $1 \le i \le k$, with $x \in B_{x_i}$, i.e., $|x - x_i| < \delta_{x_i}$. Now, pick any $y \in C$ with $|y - x| \le \delta$. We have

$$|y - x_i| \le |y - x| + |x - x_i| \le \delta + \delta_{x_i} \le 2\delta_{x_i}\,;$$

hence $y \in C_{x_i}$. Consequently,

$$\{y - x : y \in C,\ x \in C_0 \text{ and } |y - x| < \delta\} \subset (\bigcup_{i=1}^{k} C_{x_i}) - C_0\,.$$

By (1), the lemma follows. □

Before we provide a proof of 2.1 let us modify [vBDHvM, Lm. 4.1]. Every point $x_0 \in C$ satisfying the assertion of 2.4 turns out to be a central point of C.

2.4. Lemma. *There exists $x_0 \in C$ such that for every compact convex $C_0 \subseteq C$ we have*

$$\inf_{0 \ne z \in C} \operatorname{diam}_{|\cdot|}\big([0,\infty) \cdot (z - x_0) \cap (C - C_0)\big) = 0\,.$$

Proof (cf. [vBDHvM, proof of Lm. 4.1]). Assume $0 \in C$. Take $\delta > 0$ so that $U = \{x \in C : |x| \le \delta\}$ is compact. Fix a dense subset $\{x_n\}_{n=1}^{\infty}$ in C such that $x_n \ne x_m$ for $n \ne m$. Pick a sequence of positive reals $\{\lambda_n\}_{n=1}^{\infty}$ satisfying

$$\sum_{n=1}^{\infty} \lambda_n \le 1 \quad \text{and} \quad \sum_{n=1}^{\infty} |\lambda_n x_n| \le \delta\,.$$

It follows that the series $\sum_{n=1}^{\infty} \lambda_n x_n$ converges to some $x_0 \in U$. We claim that x_0 satisfies the assertion of the lemma. Otherwise, there would exist a compact convex subset $C_0 \subset C$ such that

$$\operatorname{diam}_{|\cdot|}\big([0,\infty) \cdot (z - x_0) \cap (C - C_0)\big) > \epsilon$$

for some $\epsilon > 0$ and for all $z \in C \setminus \{x_0\}$. From the definition of x_0 it follows that if $k \ne l$ (and therefore $x_k \ne x_l$) and if $0 < t < \min(\lambda_k, \lambda_l)$, then

$$x_0 + t(x_k - x_l) = \sum_{n=1}^{\infty} \lambda_n x_n + t x_k - t x_l = \sum_{n=1}^{\infty} \mu_n x_n,$$

where all μ_k are positive and $\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \mu_n \le 1$, whence $x_0 + t(x_k - x_l) \in C$. Therefore

$$\operatorname{diam}_{|\cdot|}\big([0,\infty) \cdot (x_k - x_l) \cap (C - C_0)\big) > \epsilon$$

for $k \ne l$. Hence

$$B = \Big(\bigcup_{k,l} [0,\infty) \cdot (x_k - x_l)\Big) \cap \{x \in E : |x| < \frac{\epsilon}{2}\} \subset C - C_0\,.$$

Since $\{x_n\}_{n=1}^{\infty}$ is dense in C, we conclude that

$$\operatorname{span}(C) = [0,\infty) \cdot (C - C) \subseteq \overline{\bigcup_{k,l} [0,\infty) \cdot (x_k - x_l)}.$$

This would imply

$$\{x \in \operatorname{span}(C) : |x| < \frac{\epsilon}{2}\} \subset \overline{B} \subset \overline{C - C_0}$$

(the closure taken in E). Since $C - C_0$ is locally compact (Lemma 2.3), so is $\overline{C - C_0}$. By the infinite-dimensionality of $\operatorname{span}(C)$, no neighborhood of it can be locally compact, a contradiction. □

2.5. Remark. If C is finite-dimensional, it has nonempty interior with respect to its affine hull and therefore the assertion of Lemma 2.4 is then false.

Proof of 2.1 (see [BP1, Lm. 2.7]). We will show that any point x_0 satisfying the assertion of 2.4 is a central point. We may assume that $x_0 = 0$. By 2.2, it is enough to show that $[0, 1 - \frac{1}{k}) \cdot C$ is locally homotopy negligible in C. Fix a map $f : I^n \to C$ and $\epsilon > 0$. Find $f_1 : I^n \to C \cap L = C'$, where L is a finite-dimensional linear subspace of E, such that $d(f, f_1) < \frac{\epsilon}{2}$ and $C_0 = \operatorname{conv}\big(f_1(I^n)\big) \subset \operatorname{int}_L C'$. By Remark 2.5,

$$\inf_{0 \neq z \in C'} \operatorname{diam}_{|\cdot|}\big([0,\infty) \cdot z \cap (C - C_0)\big) \geq \inf_{0 \neq z \in C'} \operatorname{diam}_{|\cdot|}\big([0,\infty) \cdot z \cap (C' - C_0)\big) > 0\,,$$

and by 2.4, there exists $q \in C \backslash L$ such that

$$\operatorname{diam}_{|\cdot|}\big(([0,\infty) \cdot q) \cap (C - C_0)\big) < \frac{\epsilon}{2}\,.$$

Write $X = \operatorname{span}(L, q)$, $K = (1 - \frac{1}{n+1}) \cdot (C \cap X)$ and $A = f_1(I^n)$. Define $\big\{f_2(x)\big\} = (x + [0,\infty) \cdot q) \cap \partial_X(K)$ for $x \in A$. Since $A \subset \operatorname{int}_L C'$, we conclude that $f_2(x)$ consists precisely of one point. We have $f_2(x) \in C$ and $f_2(x) \in C \backslash [0, 1 - \frac{1}{n}) \cdot C$. Moreover, $f_2 : A \to C$ is continuous, and

$$d(f_2, \mathrm{id}) \leq \operatorname{diam}_{|\cdot|}\big([0,\infty) \cdot q \cap (C - C_0)\big) < \frac{\epsilon}{2}\,.$$

Writing $g = f_2 \circ f_1$ we see that $d(g, f) < \epsilon$ and $g(I^n) \subset C \backslash [0, 1 - \frac{1}{n})C$. □

2.6. Remark. The above argument shows that every $x_0 \in C$ satisfying the assertion of 2.4 is also a central point of $\overline{C}$.

2.7. Corollary. *If C has the homeomorphism extension property for Z_∞-sets, then C is an* AR.

Proof. By a result of [Do2], C is an AR iff for every compact subset $A \subset C$ the identity map id_A can be approximated by maps with finite-dimensional ranges. Take a central point x_0 for C (Theorem 2.1). We can assume that $x_0 = 0$. Clearly the sequence of maps $x \to (1 - \frac{1}{n}) \cdot x$, $x \in A$, converges to id_A and has ranges which are Z_∞-sets. Consequently, we may assume that A itself is a Z_∞-set. By [CDM, Prop. 3.5], there is a copy of the Hilbert cube Q contained in $\frac{1}{2} \cdot C$. Since Q is a Z_∞-set in C, by our assumption, there is a homeomorphism h of C with $h(A) \subset Q$. The sequence $(h^{-1}\pi_n h)$, where π_n are standard projections $Q = \prod_{k=1}^{\infty} I_k$, $I_k = I$, onto $\prod_{k=1}^{n} I_k$, approximates id_A and $\dim\big(h^{-1}\pi_n h(A)\big) < \infty$, $n = 1, 2, \ldots$. □

Applying [DT1, Th. 2] we obtain

2.8. Corollary. *C has the homeomorphism extension property for Z_∞-sets iff C is a Hilbert cube manifold.*

In [B, Rm. 3] it was shown that if E is locally convex, then C is homeomorphic to $Q\backslash K$, where K is a Z-set in Q. It suggests the following

2.9. Question. *Assume that C is an* AR. *Is then C homeomorphic to $Q\backslash K$ for some Z-set K in the Hilbert cube Q?*

Let us also ask

2.10. Question. *Assume C is an* AR. *Is then $\overline{C}$ an* AR*?*

Note that if $\overline{C}$ is homogeneous, then the answer to 2.10 is "yes".

2.11. Remark. Assume that C is closed in E and that C has the homeomorphism extension property for its Z_∞-sets. Write $cc(C) = \{x \in E : \exists(y \in C)\ (y + [0,\infty) \cdot x \subset C)\}$ for the characteristic cone for C. We have:

(i) If $cc(C) = \{0\}$, then C is homeomorphic to Q.
(ii) If $cc(C)$ is a linear subspace of E, then C is homeomorphic to $Q \times R^n$, where $n = \dim\big(cc(C)\big)$, $n \geq 1$.
(iii) If $cc(C)$ is not linear, then C is homeomorphic to $Q \times [0,\infty)$.

Apply a result of [Do1].

In connection with 2.3 let us note that if C is a cone over a Keller set, then $C - C$ is not locally compact (it contains a closed infinite-dimensional linear subspace). Therefore the assumption that C_0 is compact is essential in 2.3. We have the following corresponding fact for topological groups.

2.12. Remark. Let G be a topological group. Assume $L \subset G$ is locally compact and $K \subset G$ is compact. If L is closed in G, then $L + K$ is locally compact. To justify this, pick $g = l + k$, $l \in L$ and $k \in K$. Since $(g - K) \cap L$ is a compact subset of L, it is contained in a compact neighborhood U in L. We claim that $U + K$ contains a neighborhood g (then, necessarily, relatively compact) in $L + K$. Otherwise, g will be an accumulation point of elements $l' + k'$, $l' \in L\backslash U$ and $k' \in K$. Using the compactness of K we easily get a contradiction.

If L happens not to be closed, then $L + K$ might not be locally compact. (Take $G = R^2$ with the addition, $L = \Big\{(0,t) : |t| < 1\Big\}$, and $K = \Big\{(\frac{1}{n}, -1) : n = 1, 2, \dots\} \cup \{(0,-1)\Big\}$. Clearly $(0,0) \in L + K$ has no compact neighborhood in $K + L$.)

The assertion of 2.12 and the above example are due to J. Grabowski.

3. Embedding of σ-compact linear spaces onto pre-Hilbert spaces

Let $\mathcal{T}$ be a topological class of compacta. A subset Y of a copy X of ℓ^2 is called $\mathcal{T}$-absorbing provided $Y = \bigcup_{n=1}^{\infty} K_n$, $K_n \in \mathcal{T}$, and the following condition is satisfied:

(abs) for every $K \in \mathcal{T}$ and a closed set $A \subset K$, every map $f : K \to X$ that restricts on A to an embedding into Y can be arbitrarily closely approximated by embeddings into Y that agree with f on A.

We will make use of the following version of the uniqueness theorem for absorbing sets [BP2, p. 123].

3.1. Theorem. *Let Y_1 and Y_2 be σ-compact subsets of the copies X_1 and X_2 of ℓ^2, respectively. Assume Y_i is a $\mathcal{K}(Y_i)$-absorbing set, where $\mathcal{K}(Y_i)$ is the class of compacta embeddable in $Y_i, i = 1, 2$. If Y_i can be represented as $\bigcup_{n=1}^{\infty} K_n^i$, where K_n^i are compacta such that $K_n^1 \in \mathcal{K}(Y_2)$ and $K_n^2 \in \mathcal{K}(Y_1)$ for all $n = 1, 2, \ldots$ and $i = 1, 2$, then there exists a homeomorphism h of X_1 onto X_2 with $h(Y_1) = Y_2$.*

The proof of 3.1 can be obtained by using a standard back and forth argument [BP2, p. 123]. Note that if Y is $\mathcal{T}$-absorbing, then the condition (abs) remains true for an arbitrary compactum which is a union of two elements of $\mathcal{T}$, see [DM1, p. 412] (consequently, there is no need to require that $\mathcal{T}$ is additive).

We apply 3.1 with (X, Y) a pair of locally convex separable metric linear spaces so that $Y = E$ is infinite-dimensional and σ-compact and $X = \widehat{E}$ is its linear completion. By the Kadec-Anderson theorem [BP2, p. 189], $\widehat{E}$ is a copy of ℓ^2. The following fact was proved in [Do3].

3.2. Proposition. *The space E is a $\mathcal{K}(E)$-absorbing subset of $\widehat{E}$.*

3.3. Lemma. *Expressing $E = \bigcup_{n=1}^{\infty} K_n$, where K_n are compacta, there exists a linear (not necessarily continuous) injective transformation T of E into ℓ^2 such that $T \mid K_n$ is continuous, $n = 1, 2, \ldots$.*

Proof. Assume $K_1 \subseteq K_2 \subseteq \cdots$. Pick a sequence of continuous linear functionals $\{x_n^*\}_{n=1}^{\infty}$ which separates points of E. Let

$$p_n = \max_{x \in K_n} \Big(|x_1^*(x)|, |x_2^*(x)|, \ldots , |x_n^*(x)| \Big) .$$

Write $T(x) = \left(\frac{x_n^*(x)}{p_n 2^n} \right)$ and observe that for every $x \in K_n$ and every $i \geq n$ we have $\left| \frac{x_i^*(x)}{p_i 2^i} \right| \leq \frac{1}{2^i}$. It follows that $T(K_n)$ is contained in a compact subset of ℓ^2. Since the topology on compacta in ℓ^2 coincides with the coordinatewise convergence topology, $T \mid K_n$ is continuous. Clearly, T is a linear transformation of E into ℓ^2. □

3.4. Theorem. *Every σ-compact locally convex metric linear space E is homeomorphic to a pre-Hilbert space H. Moreover, there exists a homeomorphism of the linear completions $\widehat{E}$ of $\widehat{H}$ of E and H, respectively, which sends E onto H.*

Proof. Represent $E = \bigcup_{n=1}^{\infty} K_n$, where K_n are compacta. Pick a transformation T of 3.3. Apply 3.1 with $Y_1 = E$ and $Y_2 = T(E)$. By 3.2, each Y_i is a $\mathcal{K}(Y_i)$-absorbing set. Since $T \mid K_n$ is continuous the other assumption of 3.1 is also satisfied. Now, the theorem follows from 3.1. □

3.5. Remark. The Hilbert space $\widehat{H}$ in Theorem 3.4 can be replaced by an arbitrary locally convex complete separable metric linear space F. Let $\{(x_n, x_n^*)\}_{n=1}^{\infty}$ be a biorthogonal sequence (i.e., $x_n^*(x_n) = 1$ and $x_n^*(x_k) = 0$ for $n \neq k$) such that the x_n's are linearly dense in F and the x_n^*'s separate points of F (see, e.g., [Kl, Cor. 2.3]). Replacing, if necessary, the x_n's by suitable scalar multiples the formula $S((t_n)) = \sum_{n=1}^{\infty} t_n x_n$ defines a continuous linear injection of ℓ^2 onto a dense subspace of F. Now, the argument of 3.4 applies, and there exists a homeomorphism of $\widehat{H}$ onto F which carries $T(E)$ onto $S(T(E))$.

We do not know whether the assertion of 3.4 can be extended over all convex σ-compacta C of locally convex metric linear spaces E. The case where the closure $\overline{C}$ of C in $\widehat{E}$ is locally compact is obviously settled by Theorem 1.1. Assume that $\overline{C}$ is nonlocally compact. By a result of [DT2], $\overline{C}$ is then homeomorphic to ℓ^2. Note that 3.3 easily extends for the convex case. However, we are not able to show that C is a $\mathcal{K}(C)$-absorbing set in $\overline{C}$ and consequently we cannot apply 3.1 (it is even unclear how to check the condition (abs) for such C contained in ℓ^2). In general, we do not know whether C must be homogeneous. The following example shows that not all convex sets with nonlocally compact closures are homogeneous.

3.6. Example. Let $C = B \cup A$, where B is the open unit ball in ℓ^2 and A is a copy of the rationals embedded into its sphere. Clearly, C is convex and each point $p \in A \subset C$ has no complete neighborhood while $0 \in C$ has a basis of complete neighborhoods in C.

Let us finally recall that, according to [CDM], C is a $\mathcal{K}(C)$-absorbing set in $\overline{C}$ in the following instances:

(1) if C is a countable union of finite-dimensional compacta,
(2) if C contains a Keller set.

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