A METRIC CONDITION WHICH IMPLIED DIMENSION \( \leq 1 \)

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ABSTRACT. A class of 1-dimensional spaces is distinguished by special type embeddings in compacta, or a corresponding metric property. In this setting, a simple proof of the Oversteegen-Tymchatyn theorem that the spaces of homeomorphisms of the Sierpiński’s Carpet and the Menger Universal Curve have dimension \( \leq 1 \) is given.

1. INTRODUCTION

We consider only separable metrizable spaces, and by a compactum we mean a compact metrizable space.

**Definition 1.1.** A subset \( X \) of a compactum \( K \) is \( L \)-embedded in \( K \) if for any open cover \( U \) of \( K \) there is a neighbourhood \( U \) of \( X \) in \( K \) such that every continuum in \( U \) is contained in some element of \( U \).

**Theorem 1.2.** \( L \)-embedded subsets of compacta are at most 1-dimensional.

We prove this theorem in sec. 3, where we consider a metric condition \((L)\), related to \( L \)-embeddings. Let us notice that a non-trivial connected set can be \( L \)-embedded in a compactum, cf. sec. 4.

There is a link between \( L \)-embeddings and a notion of almost 0-dimensionality introduced in [4].

**Definition 1.3 (Oversteegen - Tymchatyn [4]).** A space \( X \) is almost 0-dimensional provided there is a countable basis \( B \) in \( X \) such that for each pair \( G, H \) of elements of \( B \) with disjoint closures there is an open-and-closed set \( W \) with \( G \subset W \subset X \setminus H \).

**Theorem 1.4 (Oversteegen - Tymchatyn [4]).** Almost 0-dimensional spaces are at most 1-dimensional.

The spaces of homeomorphisms of the Sierpiński’s Carpet, the Menger Universal Curve, or more generally, the Menger compacta \( M_{kn} \) with \( k > n \), are almost 0-dimensional; see Oversteegen and Tymchatyn [4], Th.5, cf. also [1], Th.1.3. Theorem 1.4 provided the first proof that the homeomorphism spaces are 1-dimensional (the inequality \( \geq \) was established earlier by Brechner [1], Corollary 3.1.1 and 3.2.1, cf. [1], Question 1 on page 533).

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However, the proof of Theorem 1.4 given in [4], based on a notion of R-trees, is rather complicated, and one of our objectives is to give a simple proof of this very interesting result. To this end, one can show that any almost 0-dimensional space can be L-embedded in some compactum, and then apply Theorem 1.2. We decided to include also, in sec. 2, an even more direct proof of the Oversteegen-Tymchatyn theorem, where L-embedding is used only implicitly, bypassing some difficulties in Theorem 1.2.

2. A proof of the Oversteegen-Tymchatyn theorem

Let $X$ be an almost 0-dimensional space, and let $(A_0, B_0), (A_1, B_1)$ be two pairs of disjoint closed sets in $X$. We have to find partitions $L_i$ in $X$ between $A_i$ and $B_i$ with $L_0 \cap L_1 = \emptyset$; cf. [2], 1.7.9.

Let $\mathcal{B}$ be a countable basis described in Definition 1.3. For each pair $G, H$ in $\mathcal{B}$ with disjoint closures fix a continuous map from $X$ to $\{0, 1\}$ taking $G$ to 0 and $H$ to 1, and arrange the mappings into a sequence $f_1, f_2, \ldots$. Let $g_i : X \to [0, 1]$ be continuous maps with $A_i = g_i^{-1}(0), B_i = g_i^{-1}(1), i = 0, 1$, and finally, let $\rho$ be a totally bounded metric for $X$.

Let us consider a totally bounded metric $d$ on $X$, defined by

$$d(x, y) = \rho(x, y) + \sum_{i=0}^{1} |g_i(x) - g_i(y)| + \sum_{i=1}^{\infty} 2^{-i} |f_i(x) - f_i(y)|.$$

Let $K$ be the compact completion of $X$ with respect to $d$. (At this point, one could check, ignoring $g_i$, that $X$ is L-embedded in $K$. We choose a more direct argument, although L-embedding will be hidden in our reasoning.)

For each $G \in \mathcal{B}$ choose $G^*$ open in $K$ with $G^* \cap X = G$, and let

$$U = \bigcup \{G^* : G \in \mathcal{B}, \ diam G^* < 1/3\},$$

where the diameter $diam$ refers to the extension of the metric $d$ over $K$.

If a continuum $C$ in $K$ intersects both sets $G^*$ and $H^*$, their closures must meet, for otherwise the extension over $K$ of a function $f_i$ separating $G$ and $H$ would split $C$ into two separate pieces. It follows that all continua in $U$ have diameter < 1, and therefore no continuum in $U$ joins the closures $clA_i$ and $clB_i$ of $A_i$ and $B_i$ in $K$, $i = 0, 1$.

Let $\varphi : K \to [0, 1]$ be a continuous map with $U = \varphi^{-1}(0, 1]$ and let us consider the compact rings

$$K_n = \varphi^{-1}[1/(n + 1), 1/n], \ n = 1, 2, \ldots.$$ 

Then no component of $K_n$ joins $clA_i$ and $clB_i$, and since both collections \{$K_{2n-i} : n = 1, 2, \ldots\}$, where $i = 0, 1$, are discrete in the space $U$, there are partitions $S_i$ in $U$ between $clA_i \cap U$ and $clB_i \cap U$ with $S_i \cap \bigcup \{K_{2n-1} : n = 1, 2, \ldots\} = \emptyset$. Then $S_0 \cap S_1 = \emptyset$, and $L_i = S_i \cap X$ are the partitions we were looking for.

3. A metric condition which implies dimension $\leq 1$

In a metric space $X$, endowed with a metric $\rho$, we call a pair of sets $A, B$ distant, provided

$$\inf \{\rho(a, b) : a \in A, b \in B\} > 0.$$
**Definition 3.1.** Given a pair \( A, B \) of disjoint sets in a metric space \( X \) we call a pair of open sets \( G \supset A, H \supset B \) an \( L_\epsilon \)-enlargement for the pair \( A, B \) if \( G \cap H = \emptyset \) and \( X \setminus (G \cup H) \) is a union of a discrete collection \( \mathcal{F} \) of closed sets of diameter \( \leq \epsilon \) such that for each \( F \in \mathcal{F} \) the sets \( \text{cl}G \cap F \) and \( \text{cl}H \cap F \) are distant.

A metric space \( X \) has property \((L)\) if for each pair of distant sets \( A, B \) in \( X \) and every \( \epsilon > 0 \), there is an \( L_\epsilon \)-enlargement for the pair \( A, B \).

**Theorem 1.2** follows immediately from the following

**Proposition 3.2.** Each \( L \)-embedded subspace of a compactum \( K \) has property \((L)\) with respect to any metric inherited from \( K \), and each separable metric space with property \((L)\) is at most 1-dimensional.

**Proof.** (A) Let \( X \) be \( L \)-embedded in a compactum \( K \), let \( \rho \) be a metric on \( K \), and let \( A, B \) be a pair of subsets of \( X \) with

\[
\inf\{\rho(a, b) : a \in A, b \in B\} = \delta > 0.
\]

Let \( \epsilon > 0 \). We have to find an \( L_\epsilon \)-enlargement for the pair \( A, B \). To this end, let us choose an open neighbourhood \( U \) of \( X \) in \( K \) such that all continua in \( U \) have diameter \( \leq (1/2) \min(\epsilon, \delta) \). Let \( \varphi : X \rightarrow [0, 1] \) be a continuous map with \( U = \varphi^{-1}(0, 1] \), and let \( K_n = \varphi^{-1}[1/(n+1), 1/n] \). In each compactum \( K_n \), no component joins the closures \( \text{cl}A \) and \( \text{cl}B \) of \( A \) and \( B \) in \( K \), and hence each \( K_n \) can be split into two disjoint closed sets containing \( \text{cl}A \cap K_n \) and \( \text{cl}B \cap K_n \), respectively. The collection of “even rings” \( K_{2n} \), \( n = 1, 2, \ldots \), being discrete in \( U \), one can find open sets \( V, W \) in \( U \) with \( \text{cl}A \cap U \subset V \), \( \text{cl}B \cap U \subset W \), \( clV \cap clW \cap U = \emptyset \) and \( \bigcup\{K_{2n} : n = 1, 2, \ldots \} \subset V \cup W \). Then \( U \setminus (V \cup W) \) is contained in the union of the collection of “odd rings” \( K_{2n-1} \), \( n = 1, 2, \ldots \), which is discrete in the space \( U \). Each \( K_{2n-1} \) is a compactum whose components have diameter \( < \epsilon \), and therefore \( K_{2n-1} \) can be split into finitely many disjoint compacta of diameter \( \leq \epsilon \). The compactness guarantees that the traces of \( \text{cl}V \) and \( \text{cl}W \) on any of these pieces are distant. Therefore, the pair \( G = V \cap X \) and \( H = W \cap X \) provides an \( L_\epsilon \)-enlargement for \( A, B \).

(B) Let \( X \) be a metric space with property \((L)\). Let \( F \) be a closed set in \( X \) and \( p \in X \setminus F \). We have to find open sets \( V, W \) in \( X \) with \( p \in V, F \subset W \), and \( \dim(X \setminus (V \cup W)) \leq 0 \).

We shall define two increasing sequences of open sets

\[
p \in G_1 \subset G_2 \subset \ldots , \quad F \subset H_1 \subset H_2 \subset \ldots
\]

such that \( \text{cl}G_n \cap \text{cl}H_n = \emptyset \) and each pair \( G_{n+1}, H_{n+1} \) is an \( L_{1/n} \)-enlargement for the pair of the closures \( \text{cl}G_n, \text{cl}H_n \).

We begin with a distant pair \( G_1, H_1 \) of open sets containing \( p \) and \( F \), respectively. Suppose \( G_n, H_n \) have been defined. For \( n = 1 \), we get \( G_2, H_2 \) directly from property \((L)\).

Assume \( n \geq 2 \), and let \( X \setminus (G_n \cup H_n) \) be the union of a discrete family \( \mathcal{F} \) of closed sets such that for each \( F \in \mathcal{F} \) the sets

\[
A(F) = \text{cl}G_n \cap F , \quad B(F) = \text{cl}H_n \cap F
\]

are distant. Let \( G(F) \supset A(F) \) and \( H(F) \supset B(F) \) be an \( L_{1/n} \)-enlargement for the pair \( A(F), B(F) \). For each \( F \in \mathcal{F} \), let \( U(F) \supset F \) be an open set such that the
collection of closures \( \{ \text{cl } U(F) : F \in \mathcal{F} \} \) is discrete. Then we can define

\[
G_{n+1} = G_n \cup \bigcup \{ G(F) \cap U(F) : F \in \mathcal{F} \},
\]

\[
H_{n+1} = H_n \cup \bigcup \{ H(F) \cap U(F) : F \in \mathcal{F} \}.
\]

Finally, we put \( V = \bigcup_n G_n \), \( W = \bigcup_n H_n \). For each \( \epsilon > 0 \), the complement \( X \setminus (V \cup W) \) is a union of a discrete collection of closed sets of diameter \( \leq \epsilon \), and therefore it is at most 0-dimensional. \( \square \)

4. Examples

4.1. Let \( X \) be a subspace of a compactum \( K \) such that all but countably many points of \( X \) have a basis of closed-and-open sets in \( K \). Then, as one can easily check, \( X \) is \( L \)-embedded in \( K \). In particular, a one-dimensional set defined by Kuratowski \([3]\) is \( L \)-embedded in its closure in the plane.

We have already noticed (cf. the remark in brackets, following the definition of \( d \) in sec. 2) that each almost 0-dimensional space can be \( L \)-embedded in some compactum. The construction of Kuratowski we have just mentioned provides \( L \)-embedded sets which are not totally disconnected, and hence not almost 0-dimensional. But, as we shall see in the next example, non-trivial \( L \)-embedded sets may be even connected.

4.2. Let \( S \) be the space of the points in the separable Hilbert space \( l^2 \) with all coordinates rational. Then \( S \) is almost 0-dimensional \([4]\), sec.1. Roberts \([5]\) proved that one can add to \( S \) a single point \( p \), obtaining a connected space \( X = \{ p \} \cup S \). We shall show that there is a compactification \( K \) of \( X \) in \( l^2 \).

Let \( \rho \) be a totally bounded metric for \( X \). Let \( \mathcal{B} \) be a basis in \( S \) described in Definition 1.3 and let \( f_i : S \to \{ -1, 1 \} \) be continuous maps such that for each pair \( G, H \) of elements of \( \mathcal{B} \) with disjoint closures in \( S \), there is some \( f_i \) which takes \( G \) to \(-1\) and \( H \) to \(-1\). Let us consider continuous maps \( u_i \) on \( X \) defined by \( u_i(p) = 0 \) and \( u_i(x) = \rho(x, p) \cdot f_i(x) \) for \( x \in S \). Then

\[
d(x, y) = \rho(x, y) + \sum_{i=1}^{\infty} 2^{-i} | u_i(x) - u_i(y) |
\]

is a totally bounded metric on \( X \). We shall verify that \( X \) is \( L \)-embedded in the compact completion \( K \) of \( X \) with respect to \( d \).

The metric \( \rho \) and the functions \( u_i \) extend continuously over \( K \), and we shall keep the same symbols for the extensions. Let

\[
Z = \{ x \in K : \rho(x, p) = 0 \},
\]

\[
G_i = \{ x \in K : u_i(x) < 0 \}, \quad H_i = \{ x \in K : u_i(x) > 0 \}.
\]

One readily checks that

\[
K \setminus Z \subset G_i \cup H_i \quad \text{for } i = 1, 2, \ldots .
\]

Let \( \mathcal{U} \) be an open cover of \( K \) and let \( \delta > 0 \) be such that each set of diameter \( \leq \delta \) in \( K \) is contained in some element of \( \mathcal{U} \). For every \( G \in \mathcal{B} \) choose \( G^* \subset K \setminus Z \), open in \( K \), with \( G^* \cap X = G \), and let \( W \) be the union of the sets \( G^* \) of diameter \( \leq \delta/16 \). The neighbourhood \( U \) of \( X \) in \( K \) required by Definition 1.1, will be the union of \( W \) and the open \( \delta/4 \)-ball \( B(p, \delta/4) \) about \( p \) in \( K \). Let us check that each continuum \( C \)
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in $U$ has diameter $\leq \delta$. Aiming at a contradiction, suppose that diam$C > \delta$. Then there is a continuum $T$ in $C \cap W$ of diameter $\geq \delta/4$. Indeed, consider $q \in C$ with $d(p, q) > \delta/2$. If $C$ is disjoint from $B(p, \delta/4)$, we can take $T = C$. Otherwise, let $T$ be any continuum in $C$ joining $q$ with the boundary of $B(p, \delta/4)$. Let us consider $a, b \in T$ with $d(a, b) \geq \delta/4$. Since $T \subset W$, there are $G, H \in B$ of diameter $\leq \delta/16$, with $a \in G^*$ and $b \in H^*$. But then the closures of $G$ and $H$ are disjoint and, for some $i$, $u_i$ is negative on $G$ and positive on $H$. By $(\star)$, $T$ being disjoint from $Z$, the function $u_i$ changes its sign, never vanishing on $T$, which contradicts connectivity of $T$.

REFERENCES

3. K.Kuratowski, Une application des images de functions á la construction de certains ensembles singuliers, Mathematica 6 (1932), 120-123.

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