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ON 2D PACKINGS OF CUBES IN THE TORUS

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ABSTRACT. The 2D packings of cubes (i.e. squares) in the torus $\mathcal{T}^2 = [0,1)^2$ are considered. We obtain the exact expression $N_2(\lambda) = \lfloor \lambda \lfloor \lambda \rfloor \rfloor$ for the quantity $N_2(\lambda)$, the maximal number of 2D cubes in a packing. (Here $1/\lambda$ is the length of sides of cubes, $\lambda \in \mathbf{R},\ \lambda > 2$.) Corresponding best packings are constructed. Both rank 1 best lattice packings and rank 2 best lattice packings are given.

1. Introduction

This article is concerned with the classification and investigation of *the 2D best packing of cubes* (i.e. squares) in the standard torus $\mathcal{T}^2 = [0,1)^2$.

First of all we consider lattice packings; some of these will turn out to be extremal for the packing problem. According to the classification in [1] or [2], the lattices (in our 2D case) are either *rank 1* lattices or *rank 2* lattices. First, we study lattices of the *rank 1* form,

$$\left(\left\{ \frac{ka_1}{N} \right\}, \left\{ \frac{ka_2}{N} \right\} \right) \in \mathcal{T}^2, \quad k = 0, \cdots, N-1, \tag{1}$$

where $1 \leq a_1 < a_2 < N,\ a_j \in \mathbf{N}\ (j = 1,2)$. We denote by $\{t\}$ the fractional part of the number t. The lattice (1) is of rank 1 , because it consists of integer multiples (modulo 1) of the single vector $(a_1, a_2)/N$.

We also consider *rank 2* lattices. For example, if $n \geq 2$ the lattice

$$\left(\frac{k_1}{n}, \frac{k_2}{n} \right) \in \mathcal{T}^2, \quad k_1, k_2 = 0, \cdots, n-1, \tag{2}$$

is of rank 2 (the maximum for a 2D lattice). We shall be interested in rank 2 lattices of the more general form [1]:

$$\left(\left\{ \frac{ja_1}{nr} + \frac{k_1}{n} \right\}, \left\{ \frac{ja_2}{nr} + \frac{k_2}{n} \right\} \right) \in \mathcal{T}^2, \tag{3}$$
$$k_1, k_2 = 0, \cdots, n-1,\ j = 0, \cdots, r-1.$$

Later we shall use (1)–(3) as a foundation for studying best packing.

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In conformity with the traditional *packing problem*, for $1 \le p \le \infty$ we define spheres

$$B_p\left(x^{(0)};R\right) = B_p^s\left(x^{(0)};R\right) = \left\{x \in l_p^{(s)} \ : \ |x - x^{(0)}| \le R\right\}. \tag{4}$$

Here s is the dimension, $x^{(0)}$ the sphere center and R the sphere radius; and $l_p^{(s)}$ is the space with the metric

$$|x-y| = \left(\sum_{j=1}^{s} |x_j - y_j|^p\right)^{1/p},$$

where $x = (x_1, \cdots, x_s)$, $y = (y_1, \cdots, y_s) \in \mathbf{R}^s$. In particular, when $p = \infty$ and $s = 2$ the spheres become squares.

Definition 1.1. A packing of spheres $B_p^s(\cdot;R)$ in an *s–dimensional manifold M* is a system

$$\left\{B_p^s\left(x^{(k)};R\right)\right\}_{k=1}^{N}, \tag{5}$$

where

$$B_p^s\left(x^{(k)};R\right) \subset M, \quad k = 1, \cdots, N, \tag{6}$$

$$\mathrm{mes}_{\mathbf{R}^s}\left\{B_p^s\left(x^{(k_1)};R\right) \cap B_p^s\left(x^{(k_2)};R\right)\right\} = 0, \quad k_1 \ne k_2,\ k_1, k_2 = 1, \cdots, N. \tag{7}$$

We take in our case $s = 2,\ p = \infty,\ M = \mathcal{T}^2$, and study the quantity $N_2(\lambda)$, which is the maximal number of spheres (i.e. squares) with sides of length $1/\lambda$ in the packing. Such packings are of use for some number theory problems [3], [4], although generally speaking the interesting question is the behavior of $N_s(\lambda)$ for all $s \in \mathbf{N},\ s \ge 2$.

The general problem of determining $N_s(\lambda)_p$ $(1 \le p \le \infty)$ is not solved yet. V. A. Yudin [5] presented a promising approach: a duality between some extremal problems for trigonometric polynomials and the packing problem in Euclidean space. We face the problem of the asymptotic behavior of the *packing fraction*

$$\frac{N_s(\lambda)_p}{\lambda^s},$$

when $s \to \infty$. Even the existence of a limit of the packing fraction is not proved so far. However, for $p = \infty$ packings and the packing fraction were studied by S. B. Stechkin [3], who showed:

For arbitrary fixed $\lambda > 2$ there exists a number $A = A(\lambda)$

$$\textit{such that } \frac{1}{s}\ln N_s(\lambda) = A(1 + o(s^{-1})) \quad (s \to \infty).$$

We denote by $\lfloor t \rfloor$ the integer part of the number t. The main result of this paper is the following explicit expression for $N_2(\lambda)$. The proof is carried out through the construction of explicit lattices for which $N = N_2(\lambda)$, thus best packings are constructed for all $\lambda > 2$.

Theorem 1.1.

$$N_2(\lambda) = \lfloor \lambda \lfloor \lambda \rfloor \rfloor \,. \tag{8}$$

The result is of course trivial if λ is an integer.

2. The upper bound

We first establish the upper bound

$$N_2(\lambda) \le \lfloor \lambda \lfloor \lambda \rfloor \rfloor \,, \tag{9}$$

giving an explicit proof as a matter of convenience; the method used in this proof is of common knowledge and gives the more general bound

$$N_s(\lambda) \le \lfloor \lambda \cdot N_{s-1}(\lambda) \rfloor \le \lfloor \lambda \lfloor \lambda \lfloor \lambda \cdots \rfloor \rfloor \rfloor \tag{10}$$

in the s-dimensional case. Note that there are s pairs of square brackets in (10).

Let $P = P(\lambda)$ be an arbitrary best packing of squares, with $1/\lambda$ the length of sides of the squares (i.e. the radii of the spheres equal $1/(2\lambda)$):

$$P = \left\{ B_\infty^2 \left(x^{(k)}; 1/(2\lambda) \right) \right\}_{k=1}^{N_2(\lambda)} .$$

Denote by $K = K(P)$ the set

$$K = \left\{ x \in \mathcal{T} \ : \ x \notin B_\infty^2 \left(x^{(k)}; 1/(2\lambda) \right) \in P, \quad k = 1, \cdots, N_2(\lambda) \right\} .$$

Consider the function

$$f_P(x) = \begin{cases} \lambda^2, & \text{if } x \in \mathcal{T} \setminus K, \\ 0, & \text{otherwise.} \end{cases}$$

One can see that

$$N_2(\lambda) = \int_0^1 \int_0^1 f_P(x) dx. \tag{11}$$

After the reduction of the double integral (11) to an iterated integral we obtain the inequality

$$N_2(\lambda) \le \left(\int_0^{1/\lambda} \lambda^2 dx \right) \cdot N_1(\lambda). \tag{12}$$

Because $N_1(\lambda) = \lfloor \lambda \rfloor$ and $N_2(\lambda) \in \mathbf{N}$, the estimate (12) implies (9). □

3. The lower bound

Obviously an arbitrary $\lambda \in \mathbf{R}$ ($\lambda \ge 2$) can be expressed in the form

$$\lambda = \lfloor \lambda \rfloor + \frac{m}{\lfloor \lambda \rfloor} + \frac{\varepsilon}{\lfloor \lambda \rfloor}, \tag{13}$$

where $m = 0, \cdots, \lfloor \lambda \rfloor - 1,\ 0 \le \varepsilon < 1$. Consider the rank 1 lattice (1) with

$$a_1 = 1,\ a_2 = \lfloor \lambda \rfloor, \quad N = \lfloor \lambda \rfloor^2 + m = \lfloor \lambda \lfloor \lambda \rfloor \rfloor. \tag{14}$$

Now we have to check whether this lattice can be the set of centers of the packing with the side length of squares equal to $1/\lambda$. So the fulfillment of the condition (7) with $R = \lambda/2$ needs to be tested.

For the case of a lattice it is obvious that one center of the two squares in (7) can be arbitrary and fixed, thus we assume $k_1 = 0$. Furthermore, we may restrict the range of k_2,

$$k_2 = 1, \cdots, \left\lfloor \frac{N}{2} \right\rfloor.$$

First assume that $\lfloor \lambda \rfloor \le k_2 \le \lfloor \frac{N}{2} \rfloor$. Then the first coordinate of $x^{(k_2)}$ satisfies

$$\left\{ \frac{k_2}{N} \right\} = \frac{k_2}{N} \le \frac{1}{2}. \tag{15}$$

Denote by $(x)_{l_\infty^{(2)}}$ the distance (in the space $l_\infty^{(2)}$) between $x \in l_\infty^{(2)}$ and the nearest integer vector $z \in \mathbf{Z}^2$. Then by (15) we see that

$$(x^{(k_2)})_{l_\infty^{(2)}} \ge \frac{k_2}{N} \ge \frac{\lfloor \lambda \rfloor}{N} \ge \frac{1}{\lambda}, \quad \lfloor \lambda \rfloor \le k_2 \le \left\lfloor \frac{N}{2} \right\rfloor. \tag{16}$$

Now consider the case $1 \le k_2 \le \lfloor \lambda \rfloor - 1$. Then the second coordinate of $x^{(k_2)}$ satisfies

$$\left\{ \frac{k_2 \cdot \lfloor \lambda \rfloor}{N} \right\} = \frac{k_2 \cdot \lfloor \lambda \rfloor}{N} \ge \frac{\lfloor \lambda \rfloor}{N} \ge \frac{1}{\lambda} \tag{17}$$

and

$$\left\{ \frac{k_2 \cdot \lfloor \lambda \rfloor}{N} \right\} \le \frac{(\lfloor \lambda \rfloor - 1)\lfloor \lambda \rfloor}{N} \le 1 - \frac{1}{\lambda}. \tag{18}$$

Comparing (17) and (18) gives

$$(x^{(k_2)})_{l_\infty^{(2)}} \ge \frac{1}{\lambda}, \quad k_2 = 1, \cdots, \lfloor \lambda \rfloor - 1. \tag{19}$$

The correctness of (7) follows from (16) and (19). Thus a packing with $N = \lfloor \lambda \lfloor \lambda \rfloor \rfloor$ squares of side $1/\lambda$ is constructed (see (14)), and the lower estimate

$$\lfloor \lambda \lfloor \lambda \rfloor \rfloor \le N_2(\lambda) \tag{20}$$

is now proved. □

Comparing estimates (9) and (20) gives (8).

4. Examples and remarks

Example 4.1. The lattice given by (1), (14) with $\lambda = 5/2$ (in which as a result $\lfloor \lambda \rfloor = 2,\ m = 1,\ \varepsilon = 0$ and $N = N_2(5/2) = 5$) is noteworthy, it being one of the Fibonacci lattices: these are the lattices of the form

$$\left(\frac{k}{F_l}, \left\{ \frac{k \cdot F_{l-2}}{F_l} \right\} \right), \quad k = 0, \cdots, F_l - 1 , \tag{21}$$

where $F_1 = F_2 = 1,\ F_l = F_{l-1} + F_{l-2}$ for $l \ge 3$. Fibonacci lattices (21) are of use as a foundation for cubature formulae, and have been studied extensively [6]–[9]. However, only one Fibonacci lattice, namely that with $l = 5$ and $F_l = 5$, is extremal for the packing problem.

We pay attention to Fibonacci lattices and the corresponding packings because of their interesting geometrical properties. Recently H. Niederreiter and I.H. Sloan proved the following result [8].

Theorem A. *For $l = 2t + 1$ with $t = 1, 2, \ldots$ the Fibonacci lattice (21) has a square unit cell determined by the vectors*

$$\mathbf{a} = \left(F_t/F_{2t+1},\ (-1)^{t-1}F_{t+1}/F_{2t+1}\right) \ \textit{and}$$
$$\mathbf{b} = \left(F_{t+1}/F_{2t+1},\ (-1)^{t}F_t/F_{2t+1}\right).$$

For the case $l = 2t + 1$ we can therefore calculate the maximal length $l_{max}(t)$ of packing squares in the Fibonacci lattice: since the distance (in the maximum norm) of the nearest non-lattice points is F_{t+1}/F_{2t+1}, the maximal length is

$$l_{max}(t) = \frac{F_{t+1}}{F_{2t+1}} .$$

Hence the corresponding extremal value of $\lambda(t)$ is

$$\lambda(t) = \frac{1}{l_{max}(t)} = \frac{F_{2t+1}}{F_{t+1}},$$

and the number of squares in this packing is $N = F_{2t+1}$. On the other hand, the corresponding optimal value of N is

$$N_2(\lambda(t)) = \lfloor \lambda(t)\lfloor \lambda(t)\rfloor\rfloor = \lfloor (F_{2t+1}/F_{t+1})\lfloor F_{2t+1}/F_{t+1}\rfloor\rfloor .$$

For instance

$$\begin{aligned}
&\lambda(1) = 2,\ N_2(\lambda(1)) = 4 > F_{2t+1} = 2 ;\\
&\lambda(2) = 5/2,\ N_2(\lambda(2)) = 5 = F_{2t+1} ;\\
&\lambda(3) = 13/3,\ N_2(\lambda(3)) = 17 > F_{2t+1} = 13 ;\\
&\lambda(4) = 34/5,\ N_2(\lambda(4)) = 40 > F_{2t+1} = 34 ;\\
&\lambda(5) = 89/8,\ N_2(\lambda(5)) = 122 > F_{2t+1} = 89 ;\\
&\lambda(6) = 233/13,\ N_2(\lambda(6)) = 304 > F_{2t+1} = 233 ; \ldots .
\end{aligned}$$

To obtain an asymptotic result we can consider *the quality deficiency* of the Fibonacci rule, defined by

$$q(t) = \frac{N_2(\lambda(t)) - F_{2t+1}}{N_2(\lambda(t))} = 1 - \frac{F_{2t+1}}{\lfloor (F_{2t+1}/F_{t+1})\lfloor F_{2t+1}/F_{t+1}\rfloor\rfloor} .$$

For instance

$$\begin{aligned}
&q(1) = 2/4 = 1/2 ;\\
&q(2) = 0/5 = 0 ;\\
&q(3) = 4/17 ;\\
&q(4) = 6/40 = 3/20 ;\\
&q(5) = 33/122;\\
&q(6) = 71/304 ; \ldots .
\end{aligned}$$

We can use the following property of Fibonacci numbers (see [8, equation (2.2)])

$$F_t^2 + F_{t+1}^2 = F_{2t+1} ,$$

together with

$$\frac{F_t}{F_{t+1}} \to \alpha \quad \text{as} \quad t \to \infty ,$$

where

$$\alpha = \frac{\sqrt{5}-1}{2},$$

to obtain

$$q(t) \to \frac{\alpha^2}{1+\alpha^2} = 0.276393 \approx 28\% \ .$$

Summing up the results, we can say that Fibonacci lattice packing with $l = 2t+1$ is asymptotically worse than the best lattice packing, from the present point of view, by about 28 percent.

5. Alternative best packings

An explicit best packing, namely (14), is given in the proof of Theorem 1.1. In this section we give alternative best lattice packings. One is a family of rank 1 packings generalising (14), the other is a rank 2 packing.

Let the integer m in the representation (13) of λ be coprime with $\lfloor\lambda\rfloor$, and let $\epsilon = 0$. For any m_1 such that

$$m_1 \mid m \tag{22}$$

consider the lattice (1) with

$$a_1 = m_1,\ a_2 = \lfloor\lambda\rfloor, \quad N = N_2(\lambda) = \lfloor\lambda\rfloor^2 + m\ . \tag{23}$$

If $m_1 = 1$ we recover (14). For any value of m_1 that divides m the rank 1 lattice (1), (22), (23) gives a best packing, as we now prove by showing that (7) is satisfied with $R = 1/(2\lambda)$ and sphere centres given by

$$x^{(r)} = \left(\left\{\frac{rm_1}{N}\right\}, \left\{\frac{r\lfloor\lambda\rfloor}{N}\right\}\right), \quad r = 0, \cdots, N-1.$$

Because m and $\lfloor\lambda\rfloor$ are coprime it follows that m_1 and $N = \lfloor\lambda\rfloor^2 + m$ are coprime too. It follows also that N and $\lfloor\lambda\rfloor$ are coprime. From the first of these, there exist integers t and s such that

$$m_1 t + Ns = \lfloor\lambda\rfloor\ . \tag{24}$$

Now we introduce the number

$$M = \lfloor\lambda\rfloor t + m_2\ , \tag{25}$$

where

$$m_2 = \frac{m}{m_1}, \tag{26}$$

and t is as in (24). Next we transform (24) and (25) to the form

$$m_1 t + Ns - \lfloor\lambda\rfloor = 0\ , \tag{27}$$

$$\lfloor\lambda\rfloor t - M + m_2 = 0\ . \tag{28}$$

Multiplying (27) by m_2 and (28) by $\lfloor\lambda\rfloor$ and adding gives

$$\left(m_1 m_2 + \lfloor\lambda\rfloor^2\right) t + Nsm_2 - M\lfloor\lambda\rfloor = 0\ . \tag{29}$$

But (26) is equivalent to

$$m_1 m_2 + \lfloor\lambda\rfloor^2 = N, \tag{30}$$

which with (29) gives

$$M\lfloor\lambda\rfloor = Nv, \quad \text{for some } v \in \mathbf{Z}.$$

Because $\lfloor\lambda\rfloor$ and N are coprime, it follows in turn that

$$M = Nw, \quad \text{for some } w \in \mathbf{Z}.$$

Hence we can write equation (25) in the form

$$\lfloor\lambda\rfloor t = Nw - m_2 \; . \tag{31}$$

Now let t_0 (with $0 \le t_0 \le N-1$) be the residue of t modulo N. Then with the aid of (24) and (31) we obtain the following expression for the point $x^{(t_0)}$ of the lattice (1), (23),

$$x^{(t_0)} = \left(\left\{\frac{t_0 m_1}{N}\right\}, \left\{\frac{t_0\lfloor\lambda\rfloor}{N}\right\}\right) = \left(\left\{\frac{t m_1}{N}\right\}, \left\{\frac{t\lfloor\lambda\rfloor}{N}\right\}\right) = \left(\frac{\lfloor\lambda\rfloor}{N}, 1 - \frac{m_2}{N}\right) \; . \tag{32}$$

We also have

$$x^{(1)} = \left(\frac{m_1}{N}, \frac{\lfloor\lambda\rfloor}{N}\right).$$

On noting that

$$\frac{m_1}{N} < \frac{1}{\lambda} = \frac{\lfloor\lambda\rfloor}{N}, \quad \frac{m_2}{N} < \frac{1}{\lambda} = \frac{\lfloor\lambda\rfloor}{N},$$

we see that the four points

$$x^{(1)}, x^{(N-1)}, x^{(t_0)}, x^{(N-t_0)} \tag{33}$$

are all at a distance (in the space $l_\infty^{(2)}$ and the setting $\mathcal{T}^2$) of $\lfloor\lambda\rfloor/N = 1/\lambda$ from $x^{(0)} = 0$. It only remains to show that no other points of the lattice (1), (23) are closer than $1/\lambda$ to $x^{(0)} = 0$.

To this end we consider the lattice in $\mathbf{R}^2$ defined by

$$\mathcal{L}_1 = \{j_1\mathbf{a} + j_2\mathbf{b} \; : \; j_1, j_2 \in \mathbf{Z}\}, \tag{34}$$

where

$$\mathbf{a} = (m_1/N, \; \lfloor\lambda\rfloor/N) = x^{(1)},$$

$$\mathbf{b} = (\lfloor\lambda\rfloor/N, -m_2/N) = x^{(t_0)} - (0,1).$$

The volume $V(\mathcal{L}_1)$ of the unit cell of the lattice $\mathcal{L}_1$ is

$$V(\mathcal{L}_1) = \left|\det\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2\\ \mathbf{b}_1 & \mathbf{b}_2\end{pmatrix}\right|$$

$$= \left|\det\begin{pmatrix} m_1/N & \lfloor\lambda\rfloor/N \\ \lfloor\lambda\rfloor/N & -m_2/N\end{pmatrix}\right| = \left|\frac{-m_1m_2 - \lfloor\lambda\rfloor^2}{N^2}\right| = \frac{1}{N} \; .$$

The lattice $\mathcal{L}_1$ therefore contains exactly N points in $[0,1)^2$. If we now identify points of $\mathcal{L}_1$ that differ by integer vectors, then it is clear that $\mathcal{L}_1$ is a sublattice of $\mathcal{L}$, since the generators $x^{(1)}$ and $x^{(2)}$ of $\mathcal{L}_1$ are two of the points of $\mathcal{L}$. Since $\mathcal{L}_1$ has the same order N as $\mathcal{L}$, it follows that $\mathcal{L}_1$ and $\mathcal{L}$ coincide.

The representation (34) of the lattice $\mathcal{L}$ allows the distances (in $l_\infty^{(2)}$) of lattice points from $x^{(0)} = 0$ to be computed easily. In particular, the point $j_1\mathbf{a} + j_2\mathbf{b}$ of $\mathcal{L}_1$ (in the setting $\mathbf{R}^2$ and the space $l_\infty^{(2)}$) is easily seen to be at a distance from $x^{(0)}$

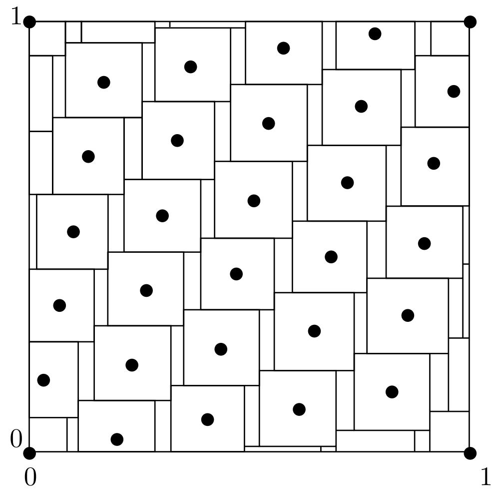

FIGURE 1

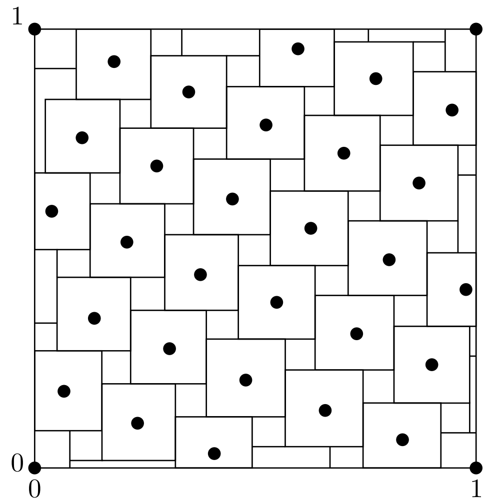

FIGURE 2

of at least $|j_2| \lfloor\lambda\rfloor/N = |j_2|/\lambda$ if j_1 and j_2 are of the same sign or j_1 is zero; and at least $|j_1| \lfloor\lambda\rfloor/N = |j_1|/\lambda$ if j_1 and j_2 are of opposite signs or j_2 is zero. Thus there are no points closer to $x^{(0)} = 0$ than the four points $\mathbf{a}, -\mathbf{a}, \mathbf{b}, -\mathbf{b}$. These four points correspond to the four points in (33) when lattice points differing by an integer vector are identified. □

Consequently, if m is coprime with $\lfloor\lambda\rfloor$, then the rank 1 best packing (14) is not unique. Figures 1 and 2 show the packings (14) and (23) respectively for the case $\lambda = 29/5,\ \lfloor\lambda\rfloor = 5,\ m = 4$ and $N = 29$, with $m_1 = 2$ in the case of Figure 2. For the choice $m_1 = 4$ (not shown) the reader may easily verify that the lattice is just the rotation through a right angle of the lattice in Figure 1.

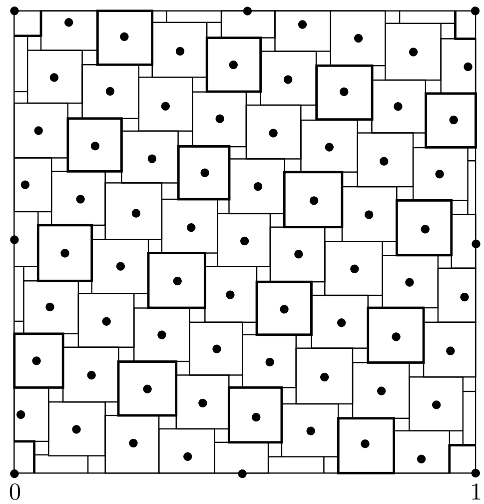

FIGURE 3

Now we construct a rank 2 lattice best packing. Let $\lambda = \lfloor\lambda\rfloor + m/\lfloor\lambda\rfloor$ and $n \in \mathbf{N}$ $(n \geq 2)$ be such that

$$nm < \lfloor\lambda\rfloor. \tag{35}$$

Consider the number λ_* defined by

$$\lambda_* = n\lambda = n\lfloor\lambda\rfloor + \frac{nm}{\lfloor\lambda\rfloor}. \tag{36}$$

Then (35) and (36) give

$$\lfloor\lambda_*\rfloor = n\lfloor\lambda\rfloor,\ \lambda_* = \lfloor\lambda_*\rfloor + m_*/\lfloor\lambda_*\rfloor,\ \text{with } m_* = n^2 m\ , \tag{37}$$

$$N_* = \lfloor\lambda_*\rfloor^2 + m_* = n^2 N\ .$$

We assert that the rank 2 lattice of the form (3), where

$$a_1 = 1,\ a_2 = \lfloor\lambda\rfloor,\ r = N = \lfloor\lambda\rfloor^2 + m\ , \tag{38}$$

is another best packing for squares of side $1/\lambda_*$ and N_* points. This can be seen by the following argument. It is easy to see that (3) with $r = N$ is simply the lattice (1) reduced in scale by the factor n. (In the language of [1], (39) is an n^s *copy* of the lattice rule (1).) It therefore follows by the second part of the proof of Theorem 1.1 and a trivial geometrical rescaling argument that (7) is satisfied by the rank 2 lattice with parameters (38), and with $R = 1/(2\lambda_*) = n^{-1}/(2\lambda)$. On the other hand, the above lattice packing (3), (38) is a *best* packing, because (using (37))

$$N_* = \lfloor\lambda_*\lfloor\lambda_*\rfloor\rfloor = N_2(\lambda_*)\ .$$

Figure 3 shows the rank 2 best packing (3), (38) for the particular case

$$\lambda = 17/4,\ \lambda_* = 17/2,\ n = 2,\ N = 17,\ N_* = 68\ .$$

This lattice is the 2^2 copy of the 17 point lattice indicated by the highlighted squares in Figure 3.

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