

ON A THEOREM OF PRIVALOV AND NORMAL FUNCTIONS

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ABSTRACT. A well known result of Privalov asserts that if f is a function which is analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, then f has a continuous extension to the closed unit disc and its boundary function $f(e^{i\theta})$ is absolutely continuous if and only if f' belongs to the Hardy space H^1 . In this paper we prove that this result is sharp in a very strong sense. Indeed, if, as usual, $M_1(r, f') = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(re^{i\theta})| d\theta$, we prove that for any positive continuous function ϕ defined in $(0, 1)$ with $\phi(r) \rightarrow \infty$, as $r \rightarrow 1$, there exists a function f analytic in Δ which is not a normal function and with the property that $M_1(r, f') \leq \phi(r)$, for all r sufficiently close to 1.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Δ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. For $0 < r < 1$ and g analytic in Δ we set

$$M_p(r, g) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r, g) = \max_{|z|=r} |g(z)|.$$

For $0 < p \leq \infty$ the Hardy space H^p consists of those functions g , analytic in Δ , for which

$$\|g\|_{H^p} = \sup_{0 < r < 1} M_p(r, g) < \infty.$$

A classical result of Privalov [8, Th. 3.11] asserts that a function f analytic in Δ has a continuous extension to the closed unit disc $\overline{\Delta}$ whose boundary values are absolutely continuous on $\partial\Delta$ if and only if $f' \in H^1$. In particular, $f' \in H^1 \Rightarrow f \in H^\infty$. The question of studying the possibility of obtaining results of this kind if the condition $f' \in H^1$ is slightly weakened has been considered by several authors. A result of Bennet and Stoll [3] shows that if the function f is analytic in Δ and f' is the Cauchy-Stieltjes integral of a finite complex Borel measure on $\partial\Delta$, then f belongs to $BMOA$, the space of all H^1 -functions whose boundary values have bounded mean oscillation on $\partial\Delta$. A stronger result was obtained by Baernstein and Brown in [2]. Indeed, Proposition 3 of [2] shows that if the function f is analytic

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in Δ and $f' \in \text{weak-}H^1$, then f belongs to the mean Lipschitz space $\Lambda(2, 1/2)$, and it is well known that $\Lambda(2, 1/2) \subset BMOA$ (see [7] and [6]).

On the other hand, Yamashita proved in [18] that there exists a function f analytic in Δ with $f' \in H^p$ for all $p \in (0, 1)$ but such that f is not even a normal function in the sense of Lehto and Virtanen [9]. We recall that a function f which is meromorphic in Δ is a normal function if and only if

$$\sup_{z \in \Delta} (1 - |z|^2) \frac{|f'(z)|}{1 + |f(z)|^2} < \infty.$$

We refer to [1] and [15] for the theory of normal functions.

In view of these results, it seems natural to study what happens if we substitute the condition $f' \in H^1$ by a condition of the type “ $M_1(r, f')$ grows to ∞ slowly enough”.

Let us start considering some very simple functions. For every $\epsilon > 0$, we set

$$(1) \quad f_\epsilon(z) = \left(\log \frac{1}{1-z} + i\pi \right)^{1+\epsilon}, \quad z \in \Delta.$$

Then, for every $\epsilon > 0$, f_ϵ is holomorphic in Δ and it is easy to see that

$$(2) \quad M_1(r, f'_\epsilon) = O\left(\left(\log \frac{1}{1-r} \right)^{1+\epsilon} \right), \quad \text{as } r \rightarrow 1,$$

while,

$$(3) \quad f'_\epsilon(r) \approx (1 + \epsilon) \frac{1}{1-r} \left(\log \frac{1}{1-r} \right)^\epsilon, \quad \text{as } r \rightarrow 1.$$

Notice that (3) shows that f_ϵ is not a Bloch function (see [1] for the theory of Bloch functions) and, hence, $f \notin BMOA$. Consequently, we see that the condition

$$M_1(r, f') = O\left(\left(\log \frac{1}{1-r} \right)^{1+\epsilon} \right), \quad \text{as } r \rightarrow 1,$$

for some $\epsilon > 0$ does not even imply that f is a Bloch function. However, we can prove a much stronger result showing that no restriction on the growth of $M_1(r, f')$ other than its boundedness is enough to conclude that f is a normal function. More precisely, we can prove the following result.

Theorem 1. *Let ϕ be any positive continuous function defined in $[0, 1)$ with $\phi(r) \rightarrow \infty$, as $r \rightarrow 1$. Then, there exists a function f analytic in Δ which is not a normal function and having the property that*

$$M_1(r, f') \leq \phi(r), \quad \text{for all } r \text{ sufficiently close to } 1.$$

2. PROOF OF THEOREM 1

Clearly, it suffices to prove that there exists a function f which is analytic and non-normal in Δ and a constant $C > 0$ such that

$$(4) \quad M_1(r, f') \leq C\phi(r), \quad \text{for all } r \text{ sufficiently close to } 1.$$

Also, we may assume without loss of generality that ϕ satisfies also the following two conditions:

$$(5) \quad \phi \text{ is increasing and } \phi(r) \geq 1 \text{ for all } r \in [0, 1),$$

and

$$(6) \quad (1 - r)^2 \phi(r) \rightarrow 0, \quad \text{as } r \rightarrow 1.$$

Indeed, let ϕ be as in Theorem 1. There exists a positive constant A such that $A\phi(r) \geq 1$ for all $r \in [0, 1)$. Then, we set

$$\phi_1(r) = \min \left(A\phi(r), \frac{2}{1 - r} \right), \quad 0 < r < 1,$$

and we let ϕ_2 denote the highest increasing minorant of ϕ_1 , that is,

$$\phi_2(r) = \inf_{r \leq s < 1} \phi_1(s), \quad 0 \leq r < 1.$$

Then it is clear that ϕ_2 is a positive and continuous function in $[0, 1)$ with $\phi_2(r) \leq A\phi(r)$, for all $r \in [0, 1)$ and $\phi_2(r) \rightarrow \infty$, as $r \rightarrow 1$. Furthermore, (5) and (6) hold with ϕ_2 in the place of ϕ .

Hence we shall assume that ϕ satisfies (5) and (6) in addition to the conditions of Theorem 1.

Let $\omega : [0, 1] \rightarrow \mathbb{R}$ be defined as follows:

$$(7) \quad \begin{cases} \omega(0) = 0, \\ \omega(\delta) = \delta\phi(1 - \delta)^{1/2}, \quad 0 < \delta \leq 1. \end{cases}$$

Hence,

$$(8) \quad \phi(r) = \left[\frac{\omega(1 - r)}{1 - r} \right]^2, \quad 0 < r < 1.$$

Using (6), it is easy to see that ω is positive and continuous in $[0, 1]$. Moreover,

$$(9) \quad \frac{\omega(\delta)}{\delta} \rightarrow \infty, \quad \text{as } \delta \rightarrow 0,$$

and (5) implies

$$(10) \quad \frac{\omega(\delta)}{\delta} \text{ is decreasing in } (0, 1]$$

and

$$(11) \quad \omega(\delta) \geq \delta, \quad \text{for all } \delta \in [0, 1].$$

Take a fixed number λ with $0 < \lambda < 1$ and let us consider the sequence of numbers $\{\delta_n\}_{n=0}^\infty$, defined inductively as

$$(12) \quad \begin{cases} \delta_0 = 1, \\ \delta_{n+1} = \min \left\{ \delta \in [0, 1) : \max \left[\frac{\omega(\delta)}{\omega(\delta_n)}, \frac{\omega(\delta_n)\delta}{\delta_n\omega(\delta)} \right] = \lambda \right\}, \quad n \geq 0. \end{cases}$$

This sequence was defined by K. I. Oskolkov in [11, 12, 13] and [14] (see also [10]) under the hypothesis of ω being a modulus of continuity, hence, (see Proposition 2.1 of [4]) in these papers ω is assumed to be increasing and subadditive. However, it is clear that the definition of $\{\delta_n\}$ makes sense in our setting. In the following lemma we shall list the main properties of the sequence $\{\delta_n\}$ which will be used in the sequel.

Lemma 1. *Let ω and λ be as above and let $\{\delta_n\}_{n=0}^\infty$ be defined by (12). Then $\{\delta_n\}$ is a decreasing sequence of positive numbers with $\delta_n \rightarrow 0$, as $n \rightarrow \infty$. Moreover, for all $n \geq 0$, we have*

$$(13) \quad \omega(\delta_{n+1}) \leq \lambda\omega(\delta_n),$$

$$(14) \quad \delta_{n+1} \leq \lambda^2\delta_n,$$

$$(15) \quad \omega(\delta_{n+1})\delta_{n+1} \leq \lambda^3\omega(\delta_n)\delta_n.$$

Furthermore, there exists an absolute constant $\beta > 0$ (which depends only on λ) such that

$$(16) \quad \sum_{k=0}^{\infty} \omega(\delta_k) \min\left(1, \frac{\delta_n}{\delta_k}\right) \leq \beta\omega(\delta_n), \quad n \geq 1.$$

We remark that (16) is the substitute in our setting of the inequality (2.12) of [13].

Proof of Lemma 1. First let us notice that (13) and (14) are direct consequences of the definition of the sequence $\{\delta_n\}$, and then (15) and the fact that δ_n tends monotonically to zero follow trivially.

Since $\{\delta_n\}$ is decreasing, we have

$$(17) \quad \sum_{k=0}^{\infty} \omega(\delta_k) \min\left(1, \frac{\delta_n}{\delta_k}\right) = \sum_{k=0}^n \omega(\delta_k) \frac{\delta_n}{\delta_k} + \sum_{k=n+1}^{\infty} \omega(\delta_k).$$

Notice that (12) implies that

$$\frac{\omega(\delta_k)}{\delta_k} \leq \lambda \frac{\omega(\delta_{k+1})}{\delta_{k+1}}, \quad k \geq 0,$$

and, hence,

$$(18) \quad \frac{\omega(\delta_k)}{\delta_k} \leq \lambda^{n-k} \frac{\omega(\delta_n)}{\delta_n}, \quad 0 \leq k \leq n.$$

On the other hand, (13) implies that

$$(19) \quad \omega(\delta_k) \leq \lambda^{k-n} \omega(\delta_n), \quad k \geq n.$$

Then, using (17), (18) and (19), we deduce that

$$\begin{aligned} \sum_{k=0}^{\infty} \omega(\delta_k) \min\left(1, \frac{\delta_n}{\delta_k}\right) &\leq \sum_{k=0}^n \lambda^{n-k} \omega(\delta_n) + \sum_{k=n+1}^{\infty} \lambda^{k-n} \omega(\delta_n) \\ &\leq 2 \left(\sum_{j=0}^{\infty} \lambda^j \right) \omega(\delta_n) \\ &= \frac{2}{1-\lambda} \omega(\delta_n). \end{aligned}$$

This proves (16) with $\beta = \frac{2}{1-\lambda}$ finishing the proof of Lemma 1. \square

Once Lemma 1 has been proved, we continue the proof of Theorem 1. The function f that we are going to construct to prove Theorem 1 will be of the form $f(z) = B(z)F(z)$, where B will be a Blaschke product while the function F will be given by a series of analytic functions in Δ which converges uniformly on every compact subset of Δ . We start with the construction of the Blaschke product B , but first let us remark that from now on we shall be using the convention that C will denote an absolute positive constant which may be different at each occurrence.

Notice that (13) implies that $\omega(\delta_n) \rightarrow 0$, as $n \rightarrow \infty$, and, hence, there exists a positive integer N such that $\omega(\delta_n) < 1$, if $n \geq N$. Define

$$(20) \quad a_n = 1 - \delta_n \omega(\delta_n), \quad n \geq N.$$

Notice that (15) implies that the sequence $\{a_n\}_{n=N}^\infty$ satisfies the Blaschke condition, that is, $\sum_{n=N}^\infty (1 - |a_n|) < \infty$. Let B denote the Blaschke product whose zeros are $\{a_n\}_{n=N}^\infty$, that is,

$$(21) \quad B(z) = \prod_{n=N}^\infty \frac{a_n - z}{1 - \overline{a_n}z}, \quad z \in \Delta.$$

Now, we set

$$(22) \quad r_n = 1 - \delta_n, \quad n \geq N.$$

Protas proved in [16, p. 394] that

$$\int_{-\pi}^\pi |B'(re^{i\theta})| d\theta \leq 8\pi \sum_k \frac{1 - |a_k|}{1 - r + 1 - |a_k|}, \quad 0 < r < 1.$$

Using this inequality, (22) and (20), we deduce that, for every $n \geq N$, we have

$$(23) \quad \begin{aligned} (1 - r_n)M_1(r_{n+1}, B') &\leq C(1 - r_n) \sum_{k=N}^\infty \frac{1 - |a_k|}{1 - r_{n+1} + 1 - |a_k|} \\ &= C \sum_{k=N}^\infty \omega(\delta_k) \left[\frac{\delta_n \delta_k}{\delta_{n+1} + \delta_k \omega(\delta_k)} \right]. \end{aligned}$$

Now, (11) implies that $\delta_k^2 \leq \delta_{n+1} + \delta_k \omega(\delta_k)$ and hence it follows that

$$(24) \quad \frac{\delta_k \delta_n}{\delta_{n+1} + \delta_k \omega(\delta_k)} \leq \frac{\delta_n}{\delta_k}, \quad k, n \in \mathbb{N}.$$

On the other hand, since the sequence $\{\delta_n\}$ is decreasing, we easily see that

$$\frac{\delta_k \delta_n}{\delta_{n+1} + \delta_k \omega(\delta_k)} \leq 1, \quad \text{if } k > n,$$

which, using again the fact that $\{\delta_n\}$ is decreasing and (24), implies

$$(25) \quad \frac{\delta_k \delta_n}{\delta_{n+1} + \delta_k \omega(\delta_k)} \leq \min \left(1, \frac{\delta_n}{\delta_k} \right), \quad k, n \in \mathbb{N}.$$

Then (23), (25) and (16) give

$$(1 - r_n)M_1(r_{n+1}, B') \leq C\omega(\delta_n), \quad n \geq N,$$

or, equivalently,

$$(26) \quad M_1(r_{n+1}, B') \leq C \frac{\omega(\delta_n)}{\delta_n}, \quad n \geq N.$$

Now we turn to construct the above mentioned function F . We set

$$(27) \quad F(z) = \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{1-z+\omega(\delta_j)\delta_j}, \quad z \in \Delta.$$

Clearly, this series converges uniformly on each compact subset of Δ , and therefore it defines a function which is analytic in Δ . Using (22), (25) and (16), we deduce that, for every $n \geq N$, we have

$$(1-r_n)M_{\infty}(r_{n+1}, F) \leq \delta_n \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{\delta_{n+1} + \omega(\delta_j)\delta_j} \leq C\omega(\delta_n),$$

or, equivalently,

$$(28) \quad M_{\infty}(r_{n+1}, F) \leq C \frac{\omega(\delta_n)}{\delta_n}.$$

Now, we have

$$F'(z) = \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1-z+\omega(\delta_j)\delta_j)^2}, \quad z \in \Delta,$$

and therefore we conclude that

$$(29) \quad \begin{aligned} M_1(r, F') &\leq C \sum_{j=N}^{\infty} \omega(\delta_j)\delta_j \int_{-\pi}^{\pi} \frac{d\theta}{|1+\omega(\delta_j)\delta_j - re^{i\theta}|^2} \\ &= C \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1+\omega(\delta_j)\delta_j)^2} \int_{-\pi}^{\pi} \frac{d\theta}{\left|1 - \frac{re^{i\theta}}{1+\omega(\delta_j)\delta_j}\right|^2} \\ &\leq C \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1+\omega(\delta_j)\delta_j)^2} \frac{1}{1 - \frac{r}{1+\omega(\delta_j)\delta_j}} \\ &\leq C \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j}, \end{aligned}$$

which, with (22), implies

$$(30) \quad (1-r_n)M_1(r_{n+1}, F') \leq C \sum_{j=N}^{\infty} \omega(\delta_j) \frac{\delta_n\delta_j}{\delta_{n+1} + \omega(\delta_j)\delta_j}, \quad n \geq N,$$

and then, noticing that the right hand side of (30) and the right hand side of (23) are the same and arguing as in the proof of (26), we obtain

$$(31) \quad M_1(r_{n+1}, F') \leq C \frac{\omega(\delta_n)}{\delta_n}, \quad n \geq N.$$

Finally, notice that for every j

$$\frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j}$$

is a positive increasing function of r in $(0, 1)$ and hence, using Lebesgue's monotone convergence theorem, we deduce that

$$(32) \quad \lim_{r \rightarrow 1^-} F(r) = \sum_{j=N}^{\infty} \lim_{r \rightarrow 1^-} \frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j} = \infty.$$

Once the functions B and F have been constructed, we set

$$(33) \quad f(z) = B(z)F(z), \quad z \in \Delta.$$

Then, since $|B(z)| \leq 1$ for all z , using (26), (28), (31), (11), (8) and (22), we deduce that, for every $n \geq N$,

$$(34) \quad \begin{aligned} M_1(r_{n+1}, f') &\leq M_1(r_{n+1}, B')M_\infty(r_{n+1}, F) + M_1(r_{n+1}, F') \\ &\leq C \left(\frac{\omega(\delta_n)}{\delta_n} \right)^2 + C \frac{\omega(\delta_n)}{\delta_n} \\ &\leq C \left(\frac{\omega(\delta_n)}{\delta_n} \right)^2 \\ &= C\phi(r_n). \end{aligned}$$

Now, since $M_1(r, f')$ and $\phi(r)$ are increasing functions of r , using (34), we deduce that

$$M_1(r, f') \leq M_1(r_{n+1}, f') \leq C\phi(r_n) \leq C\phi(r), \quad r_n \leq r \leq r_{n+1}, \quad n \geq N.$$

Hence

$$(35) \quad M_1(r, f') \leq C\phi(r), \quad r_N \leq r < 1.$$

Now observe that (15) and (20) imply that the sequence $\{a_n\}$ is uniformly separated (see Chapter 9 of [15]). Hence, there exists $\gamma > 0$ such that

$$(36) \quad (1 - |a_n|^2)|B'(a_n)| = \prod_{\substack{j=N \\ j \neq n}}^{\infty} \left| \frac{a_j - a_n}{1 - a_j a_n} \right| \geq \gamma, \quad n \geq N.$$

Since $B(a_n) = 0$, computing the spherical derivative of f at a_n yields

$$\begin{aligned} (1 - |a_n|^2) \frac{|f'(a_n)|}{1 + |f(a_n)|^2} &= (1 - |a_n|^2) \frac{|F'(a_n)B(a_n) + F(a_n)B'(a_n)|}{1 + |B(a_n)F(a_n)|^2} \\ &= (1 - |a_n|^2)|B'(a_n)||F(a_n)| \end{aligned}$$

which, using (36) and (32), implies

$$(1 - |a_n|^2) \frac{|f'(a_n)|}{1 + |f(a_n)|^2} \rightarrow \infty, \quad \text{as } n \rightarrow \infty,$$

and, hence, we see that f is not a normal function. Notice that (35) shows that f satisfies (4) and so this finishes the proof.

3. FINAL REMARKS AND SOME FURTHER RESULTS

(i) If f is a function which is analytic in Δ and the non-tangential limit $f(e^{i\theta})$ exists almost everywhere on $\partial\Delta$, then (see [8, p. 72]), for $p > 0$, $\omega_p(f, \cdot)$ denotes the integral modulus of continuity of order p of the boundary function $f(e^{i\theta})$, that is,

$$\omega_p(f, \delta) = \sup_{0 < h < \delta} \left(\int_{-\pi}^{\pi} |f(e^{i(t+h)}) - f(e^{it})|^p dt \right)^{1/p}, \quad -\pi < \delta < \pi.$$

It is well known that there is a close connection between the behaviour of $\omega_p(f, \delta)$, as $\delta \rightarrow 0$, and the growth of integral means of the derivative $M_p(r, f')$ as $r \rightarrow 1$ (see Chapter 5 of [8] and [5]). I wish to express my gratitude to Alexei Solianik who, in a private communication, showed to the author that for a certain modulus

of continuity $\omega(\delta)$ with $\frac{\omega(\delta)}{\delta} \rightarrow \infty$, as $\delta \rightarrow 0$, there exists a function $f \in H^1$ with $\omega_1(f, \delta) = O(\omega(\delta))$, as $\delta \rightarrow 0$, and $f \notin BMOA$. This result motivated our work and, in fact, using Theorem 2.1 of [5] and Theorem 1, we can state the following improvement of Solianik's result.

Theorem 2. *Let $\rho(t)$ be a positive increasing function in $[0, 1)$ satisfying the following two conditions.*

(a) *Dini's condition: $\rho(t)/t \in L^1((0, 1))$ and there is a constant C such that*

$$\int_0^t \frac{\rho(s)}{s} ds \leq C\rho(t), \quad 0 < t < 1.$$

(b) *The condition b_1 : There exists a constant C such that*

$$\int_t^1 \frac{\rho(s)}{s^2} ds \leq C\frac{\rho(t)}{t}, \quad 0 < t < 1.$$

If $\frac{\rho(\delta)}{\delta} \rightarrow \infty$, as $\delta \rightarrow 0$, then there exists a function $f \in H^1$ which is not a normal function and satisfying

$$\omega_1(f, \delta) = O(\rho(\delta)), \quad \text{as } \delta \rightarrow 0.$$

(ii) It is well known that if f is a function which is analytic in Δ and has finite Dirichlet integral, that is, if

$$\iint_{|z|<1} |f'(z)|^2 dx dy < \infty,$$

then $f \in \Lambda(2, 1/2) \subset BMOA$. On the other hand, Yamashita proved in [17] that given $0 < p < 2$ there exists a function f analytic in Δ with $\iint_{\Delta} |f'(z)|^p dx dy < \infty$ but such that f is not a normal function.

These results lead us to consider the question of whether or not some restriction on the growth of

$$\iint_{|z|<r} |f'(z)|^2 dx dy, \quad \text{as } r \rightarrow 1,$$

other than its boundedness, is enough to conclude that f is a normal function. Theorem 3 asserts that the answer to this question is negative.

Theorem 3. *Let ϕ be any positive continuous function defined in $[0, 1)$ with $\phi(r) \rightarrow \infty$, as $r \rightarrow 1$. Then, there exists a function f analytic in Δ which is not a normal function and having the property that*

$$\left(\iint_{|z|<r} |f'(z)|^2 dx dy \right)^{1/2} \leq \phi(r), \quad \text{for all } r \text{ sufficiently close to } 1.$$

Proof of Theorem 3. Just as in the proof of Theorem 1, we may assume without loss of generality that the function ϕ also satisfies (5) and (6), and it suffices to prove that there exists a non-normal analytic function f in Δ satisfying

$$(37) \quad \left(\iint_{|z|<r} |f'(z)|^2 dx dy \right)^{1/2} \leq C\phi(r), \quad \text{for all } r \text{ sufficiently close to } 1.$$

Let f be the function defined in the proof of Theorem 1. Since B is a Blaschke product, we easily see that, for $0 < r < 1$,

$$(38) \quad \left(\iint_{|z|<r} |f'(z)|^2 dx dy \right)^{1/2} \leq \left(\iint_{|z|<r} |F'(z)|^2 dx dy \right)^{1/2} + M_\infty(r, F) \left(\iint_{|z|<r} |B'(z)|^2 dx dy \right)^{1/2}.$$

Using arguments similar to those used in the proof of Theorem 1, we can prove that

$$(39) \quad \left(\iint_{|z|<r} |F'(z)|^2 dx dy \right)^{1/2} \leq C \sum_{j=N}^\infty \frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j}, \quad 0 < r < 1,$$

and

$$(40) \quad \left(\iint_{|z|<r} |B'(z)|^2 dx dy \right)^{1/2} \leq C \sum_{j=N}^\infty \frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j}, \quad 0 < r < 1.$$

Notice that the right hand side of (39) coincides with the last term of (29) and then, just as in the proof of (31), we obtain

$$(41) \quad \left(\iint_{|z|<r_{n+1}} |F'(z)|^2 dx dy \right)^{1/2} \leq C \frac{\omega(\delta_n)}{\delta_n}, \quad n \geq N.$$

On the other hand, (40) and (22) show that

$$(42) \quad (1-r_n) \left(\iint_{|z|<r_{n+1}} |B'(z)|^2 dx dy \right)^{1/2} \leq C \sum_{j=N}^\infty \omega(\delta_j) \left[\frac{\delta_n \delta_j}{\delta_{n+1} + \omega(\delta_j)\delta_j} \right], \quad n \geq N.$$

Notice that the right hand side of (42) and the right hand side of (23) coincide and then, arguing as in the proof of (26), we obtain

$$(43) \quad \left(\iint_{|z|<r_{n+1}} |B'(z)|^2 dx dy \right)^{1/2} \leq C \frac{\omega(\delta_n)}{\delta_n}, \quad n \geq N.$$

Then, using (38), (41), (28) and (43) and having in mind (11), the definitions of δ_n and r_n and (8), we obtain

$$(44) \quad \left(\iint_{|z|<r_{n+1}} |f'(z)|^2 dx dy \right)^{1/2} \leq C \phi(r_n), \quad n \geq N.$$

Finally, since $\left(\iint_{|z|<r} |f'(z)|^2 dx dy \right)^{1/2}$ and $\phi(r)$ are increasing functions of r , arguing as in the proof of (35), we see that (44) implies

$$\left(\iint_{|z|<r} |f'(z)|^2 dx dy \right)^{1/2} \leq C \phi(r), \quad r_N \leq r < 1.$$

This proves (37) and, since we already know that f is not a normal function, finishes the proof. \square

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