A NOTE ON A HOMOLOGY SPHERE

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Abstract. Here we give a solution to a problem of Y. Matsumoto which was posed in “Kirby’s problem list”

In this note we solve a problem posed by Y. Matsumoto in Kirby’s problem list (Problem 4.28 (A) of [K]). This problem should have been solved ten years ago after Donaldson’s Theorem-C in [D] which imposed a restriction to intersection forms of certain 4 manifolds because the solution does not use any 4-manifold techniques developed since then. The problem is whether the 4-manifold $M$ obtained by attaching a pair of two handles to a 4-ball $B^4$ along the two linked left handed trefoil knots, as in Figure 1, contains a smoothly imbedded wedge of 2-spheres representing generators of $H_2(M)$? An affirmative answer implies that $\partial M$ bounds a contractible manifold $W$. We show that this is not the case. In fact $\partial M$ does not bound a 4-homology ball $W$ with $\pi_1(\partial M) \to \pi_1(W)$ onto.

![Figure 1](image_url)

The 4-manifolds represented by Figures 1, 2, 3, 4 have the same boundary: By blowing down the $(+1)$-framed 2-handle in Figure 4 we get Figure 3; Figure 3 is obtained from Figure 2 by sliding the $(-1)$-framed handle over one of the 0-framed trefoil knots, as indicated by the dotted arrow, and then cancelling the 0-framed handle pair; Figure 1 is obtained from Figure 2 by blowing down the two $(-1)$-framed 2-handles. Now by the usual “blowing up and down” process (e.g. [A] figures 9-19) we can turn two $(-1)$-framed trefoil knots of Figure 4 into two $E_8$’s and obtain Figure 5. This process turns the $(+1)$-framed 2-handle into a connected sum of two right-handed trefoil knots. By introducing two hyperbolic pairs as in Figure 6 we can make the $(+1)$-framed knot of Figure 5 slice, which we can blow down.
This gives a smooth spin manifold $Q$ with intersection form $E_8 \oplus E_8 \oplus 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\partial Q = \partial M$. Hence if $\partial M$ were to bound a contractible manifold $W$, then $Q \leftarrow (-W)$ would be a manifold violating Donaldson’s Theorem-C in [D].

An interesting fact: by blowing down one of the $-1$ spheres of Figure 3 we see that the manifold $\partial M$ is also obtained by $-1$ surgery to 0-double of the left handed trefoil knot (Figure 7). Another interesting side fact which the reader can check is that the manifold $M$ is a 2-fold branched covering of the cusp manifold (i.e. 4-ball with a 2-handle attached along the left handed trefoil knot with either $(\pm 1)$-framing, e.g. Figure 8) along a properly imbedded 2-disc (the obvious disc bounded by $\gamma$ in Figure 8).
REFERENCES

