IRREDUCIBLE REPRESENTATIONS
OF THE ALTERNATING GROUP IN ODD CHARACTERISTIC

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ABSTRACT. We use the recently-proved conjecture of Mullineux to determine which modular irreducible representations of the symmetric group Σ_n split on restriction to A_n, and which remain irreducible (everything taking place over a splitting field for A_n of characteristic $p > 2$). An indexing of the absolutely irreducible representations of A_n is thus obtained. A modular analogue of the Frobenius symbol for a partition is introduced, which makes the Mullineux map somewhat more intuitive.

1. INTRODUCTION AND NOTATION

Parameterizations of the irreducible representations of a class of groups by some combinatorial objects provide some of the most appealing results and useful tools in group theory — labeled Dynkin diagrams (groups of Lie type), partitions and tableaux (symmetric groups), etc. It is Alfred Young's parameterization of the ordinary representations of the symmetric group Σ_n by partitions of n (see [9] for an overview) that began the line of research we continue here. In particular, Young's constructions were extended by James ([6]) to representations of Σ_n over an arbitrary field K of characteristic p, and we investigate these modular representations which James constructed.

As soon as one has a result about the symmetric group, it is natural to ask what the result implies for the alternating group A_n. Young's constructions can be used to give a nice parameterization of the ordinary irreducible representations of the alternating group A_n by describing those irreducible representations of Σ_n which split upon restriction to A_n (see [7, §2.5]). But James's parameterization of the modular irreducible representations of Σ_n has proved much more difficult to use to obtain information about A_n. Benson in [3] used a reduction modulo two of the spin representations to determine which 2-modular irreducible representations of Σ_n are irreducible upon restriction to A_n, and which split as the sum of two (non-isomorphic) irreducible modules (that each Σ_n-module must do one of these things is clear as A_n is normal of index 2 in Σ_n).

With the Mullineux conjecture ([8]) in the representation theory of the symmetric group now proved ([5]), a reasonably nice parameterization of the absolutely

irreducible representations of the alternating group A_n over a field of odd charac-teristic is now available. We will assume that K is a splitting field for A_n $(n > 1)$, of characteristic $p > 2$.

The standard approach in descending from Σ_n to A_n is to use Clifford's theory (see Curtis and Reiner [4, §49]), of which we need only an easy application since Σ_n/A_n has order 2.

First some notation: Let λ be a partition of n (written $\lambda \vdash n$); that is,

$$\lambda = (l_1 \geq l_2 \geq \cdots \geq l_m > 0), \quad \sum l_i = n.$$

We usually collect like parts together and write

$$\lambda = (\lambda_1^{a_1}, \ldots, \lambda_k^{a_k}),$$

with $\lambda_i > \lambda_{i+1} > 0$ for $i = 1, \ldots, k-1$; $a_i \neq 0$; and $\sum a_i \lambda_i = n$. That the number of partitions of n is equal to the number of possible cycle structures for an element of Σ_n, and thus to the number of conjugacy classes in Σ_n, is clear. But the number of conjugacy classes in Σ_n is the same as the number of ordinary irreducible representations. Young showed how to construct (over $\mathbf{Q}$) an irreducible module S^λ for each partition λ, such that if μ and λ are distinct partitions, then S^λ and S^μ are not isomorphic.

We use the symbol $[\lambda]$ to represent the partition's *Young diagram*, which consists of n nodes $\bullet$ placed in decreasing rows, as illustrated by the following example. Let $\lambda = (5^2, 3, 1)$ (so $n = 14$); then the Young diagram is:

$$[\lambda] = \begin{matrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet \end{matrix}$$

By the *rim* of $[\lambda]$ we shall mean the collection of nodes which are either at the bottom of a column, at the right end of a row, or both.

Given a partition λ, one obtains another partition λ', called the *conjugate* partition to λ, by transposing the Young diagram $[\lambda]$ about its main diagonal. So if $\lambda = (5^2, 3, 1)$ as above, then the Young diagram for λ' is

$$[\lambda'] = \begin{matrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{matrix}$$

and $\lambda' = (4, 3^2, 2^2)$.

Finally, If H is a subgroup of a group G, and M is an FG-module for some field F, by $M \downarrow_H$ we shall mean M restricted to H, that is, M considered as a module over $FH \subseteq FG$. If N is an FH-module, then $N \uparrow^G$ will be the induced module $FG \otimes_{FH} N$ (see [4, §38] for more).

2. MULLINEUX: A MODULAR λ'

Of course, if λ is a partition of n, then so is λ', and so λ and λ' correspond to (perhaps different, if $\lambda \neq \lambda'$) irreducible $\mathbf{Q}$-representations of Σ_n. The crucial fact in descending to A_n is that $S^\lambda \otimes \mathrm{sgn}_n = S^{\lambda'}$, where for $\sigma \in \Sigma_n$, $\mathrm{sgn}_n(\sigma) = 1$ if σ is an even permutation, and $\mathrm{sgn}_n(\sigma) = -1$ if σ is odd. Thus $S^\lambda \downarrow_{A_n} = S^{\lambda'} \downarrow_{A_n}$ (as

the two representations differ only on odd elements of Σ_n). Clifford's theory then allows us to tell fairly easily when $S^\lambda \downarrow_{A_n}$ is irreducible (this happens if $\lambda \neq \lambda'$), and when $S^\lambda \downarrow_{A_n}$ is the sum of two inequivalent irreducible A_n-modules (if $\lambda = \lambda'$).

If we now consider representations over K instead of over $\mathbf{Q}$, things don't work as nicely. Assume λ is a p-regular partition, and let D^λ be the irreducible $K\Sigma_n$-module corresponding to λ. It is still true that $(D^\lambda \otimes \mathrm{sgn}_n) \downarrow_{A_n} = D^\lambda \downarrow_{A_n}$, but it is no longer true that $D^\lambda \otimes \mathrm{sgn}_n = D^{\lambda'}$. Mullineux's algorithm helps out here and gives a modular analogue of λ'. That is, Mullineux in [8] defined a bijection m on the set of p-regular partitions of n, and his conjecture that $D^\lambda \otimes \mathrm{sgn}_n = D^{m(\lambda)}$ has now been verified ([5]). With this in hand, we can describe the irreducible K-representations of A_n.

Clearly $D^\lambda \downarrow_{A_n} = D^{m(\lambda)} \downarrow_{A_n}$, as in the characteristic zero case. Also, $D^\lambda \downarrow_{A_n}$ is completely reducible (since A_n is a normal subgroup of Σ_n). Consider an irreducible summand D of $D^\lambda \downarrow_{A_n}$ ($= D^{m(\lambda)} \downarrow_{A_n}$). If we consider the action of an odd element of Σ_n, say (12), on D, the possibilities are that $D(12) = D$ (the inertia group of D is Σ_n), or $D(12) \neq D$ (the inertia group of D is A_n).

In the first case, we have that $D = D^\lambda \downarrow_{A_n}$ is irreducible, since D^λ is an irreducible Σ_n-module and D is an invariant submodule. Then it is easy to see that D^λ and $D^\lambda \otimes \mathrm{sgn}_n = D^{m(\lambda)}$ are *not* isomorphic as Σ_n-modules: Assume that σ is an Σ_n-isomorphism. Then σ is also an A_n-isomorphism; but D is an irreducible A_n-module and K is a splitting field for A_n. Thus σ must be a scalar multiple of the identity map; but this contradicts the assumption that it is an isomorphism between D^λ and $D^\lambda \otimes \mathrm{sgn}_n$.

If, on the other hand, $D(12) \neq D$, then Clifford's theorem says

$$D^\lambda \downarrow_{A_n} = D \oplus D(12) = D^{m(\lambda)} \downarrow_{A_n},$$

and this implies (by Mackey's subgroup theorem; see for example [4, §44]) that $D^\lambda \cong D \uparrow^{\Sigma_n} \cong D^{m(\lambda)}$.

Summarizing, we have:

Theorem 2.1.　1. *If $\lambda \neq m(\lambda)$, then $D^\lambda \downarrow_{A_n}$ is irreducible.*
　2. *If $\lambda = m(\lambda)$, then $D^\lambda \downarrow_{A_n}$ is the sum of two irreducible, non-equivalent, representations of A_n, say D^λ_+ and D^λ_-.*
　Finally,

$$\{D^\lambda \downarrow_{A_n} \mid \lambda \neq m(\lambda)\} \cup \{D^\lambda_+, D^\lambda_- \mid \lambda = m(\lambda)\}$$

is a complete system of inequivalent irreducible KA_n-modules.

3. Fixed points and a modular Frobenius symbol

So the issue is identifying the fixed points of the Mullineux map. Andrews and Olsson in [2] counted the fixed points of m and described them to some extent. Unfortunately, the fixed points are not easy to "see" from the diagram or the usual presentation of the partitions, as they are in the ordinary case. We shall describe (briefly) the Mullineux map and describe its fixed points. We will use an alternative method of recording the data involved, which makes the map a bit more appealing.
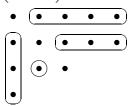
First let us recall the *Frobenius symbol* of a partition (see for example [1]), a method of representing a partition to make the conjugation action obvious. Given

a partition λ, we draw its Young diagram $[\lambda]$, and then construct a $2 \times k$ array

$$\mathrm{Fr}(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix},$$

where k is the number of nodes on the main diagonal of the diagram, a_i is the number of nodes to the right of the ith node on the diagonal, and b_i is the number of nodes below the ith node on the diagonal. For example, the Frobenius symbol for $\lambda = (5^2, 3, 1)$ is $\mathrm{Fr}(\lambda) = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & 0 \end{pmatrix}$, as we see from the diagram below.

Notice that n is the sum of all the entries in the array, plus the number of columns. Also notice that another way to obtain the symbol is to set

$a_1 =$ (the length of the rim of $[\lambda]$) $-$ (the number of rows in $[\lambda]$),

$b_1 =$ (the number of rows in $[\lambda]$) $- 1$;

then remove the rim and repeat to obtain a_2, b_2, etc.

The advantage of the notation is that $\mathrm{Fr}(\lambda')$ is obtained from $\mathrm{Fr}(\lambda)$ by simply interchanging the two rows. In particular, the fixed points of the conjugation map become obvious. We shall see that with an appropriately defined modular analogue of $\mathrm{Fr}(\lambda)$, the same is true of the Mullineux map.

Given a partition λ, we define the p-rim of the associated Young diagram $[\lambda]$ as follows (see [5] for a more rigorous definition): Beginning at the top right-hand corner of $[\lambda]$, the first p nodes of the rim are in the p-rim. Then skip to the next row, and take the next p nodes of the rim. Continue until we reach the end of the rim; the last of these "p-segments" may contain fewer than p nodes. For example, the 3-rim and the 5-rim of the diagram for $\lambda = (5^2, 3, 1)$ are shown as open circles:

Now let h_1 be the number of nodes in the p-rim of λ, and let r_1 be the number of rows in λ. Delete the p-rim and repeat to obtain sequences $h_1, h_2, \ldots$ and $r_1, r_2, \ldots$. Let k be such that $h_{k+1} = r_{k+1} = 0$, but $h_k \neq 0 \neq r_k$. The standard way of keeping track of this data is via the *Mullineux symbol*

$$M(\lambda) = \begin{pmatrix} h_1 & h_2 & \cdots & h_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix};$$

but we want to give a different formulation which encodes the same information but makes the desired map more intuitive.

The *p-modular Frobenius symbol* for λ, $\mathrm{Fr}_p(\lambda)$, will be a $3 \times k$ array

$$\mathrm{Fr}_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k \end{pmatrix},$$

constructed as follows:

$$\varepsilon_i = \begin{cases} 0 \text{ if } p | h_i, \\ 1 \text{ if } p \nmid h_i, \end{cases}$$
$$a_i = h_i - r_i,$$
$$b_i = r_i - \varepsilon_i.$$

Notice that n is the sum of all the entries in $\mathrm{Fr}_p(\lambda)$.

The fact that the Mullineux symbol uniquely determines λ (proved by Mullineux in [8]) means that the modular Frobenius symbol uniquely determines λ (as it is possible to reconstruct $M(\lambda)$ from $\mathrm{Fr}_p(\lambda)$).

Finally, if λ has modular Frobenius symbol

$$\mathrm{Fr}_p(\lambda) = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k \end{pmatrix},$$

then the Mullineux map m is defined by

$$\mathrm{Fr}_p(m(\lambda)) = \begin{pmatrix} b_1 & b_2 & \cdots & b_k \\ a_1 & a_2 & \cdots & a_k \\ \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k \end{pmatrix}.$$

So to perform the p-analogue of conjugation, we just interchange the top two rows of the modular Frobenius symbol for the partition — just as in the ordinary case, we switch the two rows of $\mathrm{Fr}(\lambda)$.

Now it is easy to see the fixed points of m: $\lambda = m(\lambda)$ if and only if the top two rows of $\mathrm{Fr}_p(\lambda)$ are the same. For example, if $\lambda = (5^2, 3, 1)$, then

$$\mathrm{Fr}_3(\lambda) = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \mathrm{Fr}_5(\lambda) = \begin{pmatrix} 2 & 2 & 0 \\ 3 & 3 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

Thus if char $K = 3$, $D^\lambda \cong D^{m(\lambda)}$ and thus $D^\lambda \downarrow_{A_n}$ splits as a sum of two irreducible modules; while if char $K = 5$, $D^\lambda \not\cong D^{m(\lambda)}$, and so $D^\lambda \downarrow_{A_n}$ is irreducible.

References

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