AN ENGLER CONDITION WITH DERIVATION
FOR LEFT IDEALS

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Abstract. We generalize a number of results in the literature by proving the following theorem: Let \( R \) be a semiprime ring, \( D \) a nonzero derivation of \( R \), \( L \) a nonzero left ideal of \( R \), and let \([x, y] = xy - yx\). If for some positive integers \( t_0, t_1, \ldots, t_n \), and all \( x \in L \), the identity \([\ldots[[D(x^{t_0}), x^{t_1}], x^{t_2}]], \ldots], x^{t_n}] = 0\) holds, then either \( D(L) = 0 \) or else the ideal of \( R \) generated by \( D(L) \) and \( D(R)L \) is in the center of \( R \). In particular, when \( R \) is a prime ring, \( R \) is commutative.

In this paper we prove a theorem generalizing several results, principally [20] and [9], which combine derivations with Engel type conditions. Before stating our theorem we discuss the relevant literature. If one defines \([x, y]_0 = x \) and \([x, y]_1 = [x, y] = xy - yx\), then an Engel condition is a polynomial \([x, y]_n + 1 = \ldots[[x, y], y]] = 0\) in noncommuting indeterminates. A commutative ring satisfies any such polynomial, and a nilpotent ring satisfies one if \( n \) is sufficiently large. The question of whether a ring is commutative, or nilpotent, if it satisfies an Engel condition goes back to the well known work of Engel on Lie algebras [15, Chapter 2], and has been considered, with various modifications, by many since then (e.g. [2] or [7]). The connection of Engel type conditions and derivations appeared in a well known paper of E. C. Posner [23] which showed that for a nonzero derivation \( D \) of a prime ring \( R \), if \([D(x), x] \) is central for all \( x \in R \), then \( R \) is commutative. This result has led to many others (see [19] for various references), and in particular to a result of J. Vukman [25] showing that if \([D(x), x]_2 \) is central for all \( x \in R \), a prime ring with char \( R \neq 2, 3 \), then again \( R \) is commutative. We extended this result [20] by proving that if \([D(x), x]_n = 0 \) for all \( x \in I \), an ideal of the prime ring \( R \), then \( R \) is commutative, and if instead, this Engel type condition holds for all \( x \in U \), a Lie ideal of \( R \), then \( R \) embeds in \( M_2(F) \) for \( F \) a field with char \( F = 2 \). Recently, [9] proved that for a left ideal \( L \) of a semiprime ring \( R \), either \( D(L) = 0 \) or \( R \) contains a nonzero central ideal if either: \( R \) is 6-torsion free and \([D(x), x]_2 \) is central for all \( x \in L \); or if \([D(x), x^n] \) is central for all \( x \in L \) and \( R \) is \( n! \)-torsion free. The first of these conditions generalized [1, Theorem 3, p. 99], which assumed that \([D(x), x] \) is central for all \( x \in L \), with no restriction on torsion. The second, involving powers, is related to both [12], which showed that a prime ring \( R \) is commutative if \( D(x^k) = 0 \) for all \( x \in R \), and to [8], a significant extension of [12], showing that \( R \) is commutative if it contains no nonzero nil ideal and \([D(x^k), x^k]_n = 0 \) on
$R$. Other results and conditions involving the image of a derivation on a one-sided ideal of $R$ have been appearing with increased frequency (e.g. [3], [4], [21], [24]).

Our result here combines a variant of the Engel condition and the action of a derivation on a left ideal in a semiprime ring. It generalizes or extends a number of the results mentioned above and eliminates all torsion assumptions.

**Main Theorem.** Let $R$ be a semiprime ring, $D$ a nonzero derivation of $R$, and $L$ a nonzero left ideal of $R$. If for some positive integers $t_0, t_1, \ldots, t_n$, and all $x \in L$, the identity $[x, [x, \ldots, [D(x^{t_0}), x^{t_1}], x^{t_2}], \ldots], x^{t_n}] = 0$ holds, then either $D(L) = 0$ or else $D(L)$ and $D(R)L$ are contained in a nonzero central ideal of $R$. In particular, when $R$ is a prime ring, $R$ is commutative.

Note that the statement about prime rings does follow from the semiprime case since if $I$ is a central ideal in a prime ring $R$, then the identity $[xy, z] = x[y, z] + [x, z]y$ shows that $0 = [IR, R] = I[R, R]$, so $I = 0$ or $R$ is commutative. Also, when $R$ is prime and $D(L) = 0$, then $D(R)L = D(RL) = 0$, and $D = 0$ results. Something like the conclusion that when $R$ is a prime ring, its extended centroid $C(R) = C$ is a field which is the center of the symmetric quotient ring $Q = Q(R)$ of $R$. For our purposes it suffices to know that $RC$ and $Q$ are prime overrings of $R$, for each $q \in Q$ there is a nonzero ideal $I_q$ of $R$ with $qI_q + I_qq \subseteq R$, and if $qI_q = 0$, then $q = 0$. Any derivation $D$ of $R$ extends uniquely to $Q$, and if on $Q$, $D(q) = qA - Aq$ for $A \in Q$, then $D$ is called inner; otherwise $D$ is outer. An important result of W. S. Martindale [22] is that $R$ satisfies a generalized polynomial identity exactly when $H = \text{soc } RC \neq 0$ and for each minimal left ideal $RCe$ of $RC$ with $e^2 = e$, $eRCe$ is a finite dimensional division algebra over $C$.

**Theorem 1.** Let $R$ be a prime ring, $D$ a nonzero derivation of $R$, and $L$ a nonzero left ideal of $R$. If for integers $k, n + 1 \geq 1$, $D(x^k), x^k] = 0$ for all $x \in L$, then $R$ is commutative.

**Proof.** It is easy to see that if $L \subseteq R \cap C$, then $R$ must be commutative [14, Corollary, p. 7], so we may choose $a \in L - C$. For any $r \in R$, $D((ra)^k), (ra)^k] = 0$, and it follows that

$$[D((Xa)^k), (Xa)^k] = \sum_{i=0}^{k-1} (Xa)^i(XD(a) + XD(a))(Xa)^{k-i-1}, (Xa)^k]_n$$

is an identity with derivation which is satisfied by $R$. If $D$ is an outer derivation, a direct application of [17, Theorem 2, p. 65] or [6, Main Theorem, p. 251], together with [5, Theorem 2, p. 725] show that $[\sum_{i=0}^{k-1} (Xa)^i(Ya + XD(a))(Xa)^{k-i-1}, (Xa)^k]_n$ is an identity for $Q$, which yields easily that $[\sum_{i=0}^{k-1} (Xa)^i(Ya)(Xa)^{k-i-1}, (Xa)^k]_n$ is an identity for $Q$ by first setting $Y = 0$. Since $a \notin C$, this identity is a nonzero identity for $Q$. Therefore, $Q$ is a field, and hence so is $R$. This completes the proof.
generalized polynomial identity for $R$, so by Martindale’s theorem [22, Theorem 3, p. 579] $H = \text{soc } RC \neq 0$. Clearly the identity holds on $H \subseteq Q$. If $H$ is commutative, then so is $R$ and we are finished. Otherwise, since $Ha \subseteq H$ [18, Lemma 7, p. 779], there is a minimal left ideal $He \subseteq Ha$ with $e^2 = e \in H$ and $Hta = He$ for some $t \in H$. Consequently, $He$ satisfies $[\sum_{i=0}^{k-1} X^iYX^{k-i-1}, X^k]_n = 0$. Evaluating this expression with $X = he$ and $Y = (1-e)ye$ for arbitrary $h, y \in H$, and using $he(1-e)ye = 0$ results in $(1-e)ye(he)^{k(n+1)-1} = 0$. Because $He$ is minimal, if $(1-e)ye \neq 0$, it follows that $He = H(1-e)ye$, so $(he)^{k(n+1)} = 0$ results. This means that $He$ is a nil left ideal of bounded index and Levitzki’s theorem [13, Lemma 1.1, p. 1] forces $R$ to contain a nonzero nilpotent ideal. This contradiction shows that $R$ must be commutative when $D$ is outer.

We may now take $D(q) = [q, A]$ with $A \in Q - C$, since $D \neq 0$. As above, if we choose $a \in L - C$, then our assumption yields the identity $[A, (ra)^k]_{n+1} = 0$ for $R$. This is a nonzero generalized polynomial identity because $A \notin C$, so Martindale’s theorem [22, Theorem 3, p. 579] shows that $H = \text{soc } RC \neq 0$ and $eHe$ is finite dimensional over $C$ for $e^2 = e$ a minimal idempotent in $H$. Now the identity $[A, (Xa)^k]_{n+1}$ is also satisfied by $Q$ [5, Theorem 2, p. 725] and hence by $H$. As in the case above, $R$ is commutative if $H$ is, so we proceed with the assumption that $H$ is not commutative to get the contradiction $D = 0$.

We want to replace $R$ with $H$ and be able to assume that for any minimal idempotent $e \in H$, $Ce = eHe$. We note that $C = C(H), CH = H$ and $D(H) \subseteq H$ [18, Lemma 7, p. 59], and $C$ centralizes $H$, so it is clear that $Ce \subseteq Z(eHe)$ for any idempotent $e \in H$. Assume first that $C$ is a finite field. From the finite dimensionality of $eHe$ over $Ce$ it follows that $eHe$ is a finite field, so for $z \in eHe$ and any $h \in H$, $zehe = chez$, which forces $ze = ce$ for $c \in C(H) = C$ [22, Theorem 1, p. 577]. Therefore $Ce = Z(eHe) = eHe$ when $C$ is a finite field. If $C$ is infinite, then a Vandermonde determinant argument, for example that in [20, Lemma 2, p. 732], shows that $[A, (Xa)^k]_{n+1}$ is satisfied by any extension $H \otimes_C F$ of $H$, for $F$ a field extension of $C$. In particular we can take $F$ to be an algebraic closure of $C$. Now $C(H \otimes_C F) = F$ [10, Theorem 3.5, p. 59], $\text{soc } (RC \otimes_C F) = H \otimes_C F$, and for any minimal idempotent $e \in H \otimes_C F$, $e(H \otimes_C F)e$ is finite dimensional over $eF$, again by [22], so $e(H \otimes_C F)e = eF$ because $F$ is algebraically closed. Consequently, regardless of $\text{card } C$, we may assume that $H = R$ and $eC = eHe$ for any minimal idempotent $e \in H$.

Since $H$ satisfies the identity $[A, (Xa)^k]_{n+1}$, as for the case above when $D$ was assumed to be outer, for some minimal idempotent $e \in H$ and some $t \in H$, $He = Hte$ satisfies the identity $[A, X^k]_{n+1}$. In particular if $X = e$ we obtain $[A, e]_{n+1} = 0$ and also $[A, e]_{n+2} = 0$. Since one of $n + 1$ or $n + 2$ is odd and $[A, e] = [A, e]_3$, it follows immediately that $[A, e] = 0$, and we may write $A = eAe + (1-e)A(1-e)$. But $eAe = e(Ae)e \in eHe = Ce$, so $A = ce + (1-e)A(1-e)$. For any $h \in H$ we evaluate $[A, (he)^k]_{n+1} = 0$ using the identities $[y, x]_{n+1} = \sum_{i=0}^{n+1} (1)^i \binom{n+1}{i} x^i y x^{n+1-i}$ and $[x + y, z]_s = [x, z]_s + [y, z]_s$ to obtain

$$0 = (1-e)A(1-e)(he)^{k(n+1)} + \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} (he)^{ki} ec(he)^{k(n+1-i)}$$

$$= (1-e)A(1-e)(he)^{k(n+1)} + ec(he)^{k(n+1)}$$

$$+ ec \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} (he)^{ki} e(he)^{k(n+1-i)}$$
commutative, so the first two possibilities each force hypothesis carries over from $D$ we need to prove that for each prime ideal $D$ is semiprime. Consequently, to prove the existence of a nonzero central ideal, it is easy to see that $M$ is semiprime, and $M/R$, $R$ are fixed but need not assume that $R$ has no nil ideal. Before proving our Main Theorem, it will be helpful to collect a few observations together into a lemma.

**Lemma.** Let $R$ be a semiprime ring and $M$ the maximal central ideal of $R$.

1. $M = \text{ann}([R, R])$ is a semiprime ideal of $R$;
2. if $a \in R$ and $Ra$ is central, then $a \in M$; and
3. if $D$ is a derivation of $R$, then $D(M) \subseteq M$.

**Proof.** Since any annihilator ideal in a semiprime ring is a semiprime ideal, it suffices to show that $M = \text{ann}([R, R])$ to prove (1). Let $A = \text{ann}([R, R])$ and note that $0 = [MR, R] = M[R, R]$, so $M \subseteq A$. But $[A, R] \subseteq A \cap ([R, R]) = 0$ since $R$ is semiprime, and $A = M$. Next observe that $R/M$ has no nonzero central ideal. If $M \subseteq I$ is an ideal of $R$ with $I/M$ central in $R/M$, then $[I, R] \subseteq M$ implies that $[I, R], R] = 0$, so $[I, [R, R]] = 0$ and $I$ is central by [14, Lemma 1.1.8, p. 8] forcing $I = M$. Consequently, if $Ra + M$ is central in $R/M$, then $Ra \subseteq M$, which results in $a \in M$ by (1). Finally, for any derivation $D$ it is easy to see that $D(Z(R)) \subseteq Z(R)$, the center of $R$, and then that $M + D(M)$ is an ideal of $R$ in $Z(R)$. Thus $D(M) \subseteq M$ by the maximality of $M$. □

**Proof of Main Theorem.** Our assumption that $[[[..., [D(x^t_0), x^t_1, x^t_2], \ldots, x^t_n] = 0$ for all $x \in L$ implies that $[D(x^k), x^k]_n = 0$ for $k = t_0t_1 \cdots t_n$ since powers of $x$ commute, so we may as well assume that all $t_j = k$. We claim that $RD(R)L$ is a central ideal of $R$, and is not zero unless $D(L) = 0$. Should $D(R)L = 0$, then $L \subseteq \text{ann}(D(R))$, the left or right annihilator of $(D(R))$, the ideal $D(R)$ generates. It is easy to see that $D(L) \subseteq D(\text{ann}(D(R))) \subseteq D(R) \cap \text{ann}(D(R)) = 0$, since $R$ is semiprime. Consequently, to prove the existence of a nonzero central ideal, it suffices to assume that $D(L) \neq 0$ and show that $RD(R)L$ is central. Equivalently, we need to prove that for each prime ideal $P$ of $R$, the image of $RD(R)L$ is central in $R/P$. This is clear if $D(R)L \subseteq P$, so we need only consider those prime ideals with $D(R)L \not\subseteq P$.

Let $P$ be a prime ideal of $R$ so that $D(R)L \not\subseteq P$, and suppose that $D(P) \subseteq P$. In this case, $D$ induces a derivation $E$ on $R/P$ via $E(r + P) = E(r) + P$ and our hypothesis carries over from $R$ to $R/P$ using $E$ and the left ideal $L + P \subseteq R/P$. Applying Theorem 1 gives either $E = 0$, $L + P \subseteq P$, or $R/P$ commutative. Since the first two possibilities each force $D(R)L \subseteq P$, we must conclude that $R/P$ is commutative, so $RD(R)L + P$ is central in $R/P$. □
We may assume now that $D(R)L \not\subset P$ and $D(P) \not\subset P$. It is straightforward to check that $D(P) + P = B \subseteq R/P$ is a nonzero ideal. For any $t \in P$ and $y \in L$ our assumption that $[D((ty + y)^k), (ty + y)^k]_n = 0$, taken modulo $P$ becomes $[\sum_{i=0}^{k-1} y^i (D(t)y + D(y))y^{k-i-1}, y^k]_n = 0$ in $R/P$. But
\[
\left[ \sum_{i=0}^{k-1} y^i D(y)y^{k-i-1}, y^k \right]_n = [D(y^k), y^k]_n = 0,
\]
so $[\sum_{i=0}^{k-1} y^i D(t)y^{k-i}, y^k]_n = 0$ in $R/P$, which means that the expression $f(X, y) = [\sum_{i=0}^{k-1} X^i Y^{k-i}, Y^k]_n$ yields $O_{R/P}$ when elements of $B$ replace $X$ and elements of $L + P$ replace $Y$. If for some $y \in (L + P) - P, yw = 0$ in $R/P$ for $w \in R/P - O_{R/P}$, then for any $b \in B$ and $r \in R$, $O_{R/P} = f(wb, ry) = wb(ry)^{k(n+1)}$. Thus $wB(ry)^{k(n+1)} = 0$ in $R/P$, and since $B$ is a nonzero ideal and $R/P$ is prime, we must conclude that $Ry + P$ is a nil left ideal of bounded index in $R/P$, forcing the contradiction $y \in P$ by Levitzki’s theorem [13, Lemma 1.1, p. 1]. Therefore, we may assume that each nonzero $y \in L + P$ has no right annihilator in $R/P$.

To simplify notation, we assume that $R$ is a prime ring with a nonzero ideal $B$ and nonzero left ideal $L$ whose nonzero elements are left regular, that $f(X, y)$ is an identity for $B$ for each $y \in L$, and show that $R$ is commutative. Expanding $f(X, y)$ for $y \in L - 0$, yields the identity $\sum_{v=0}^n n_j y^j Xy^{v-j}$ for $B$, where $n_j$ are integers, $n_0 = 1$, and $v = k(n + 1)$. This is a generalized linear identity for $B$, so by [18, Lemma 1, p. 766], $\{1, y, \ldots, y^k\}$ must be $C(R)$ dependent. Let $m(y) = y^n + \cdots + c_1 y + c_0 = 0$ with $c_i \in C(R)$ and $s$ minimal. The definition of $Q$ allows us to choose a nonzero ideal $I$ of $R$ so that all $c_i I \subseteq R$. Thus if $c_0 = 0$ and $m(y) = yg(y)$, then $g(y)I \subseteq R$, so $g(y)I$ is in the right annihilator of $y$, and $g(y)I = 0$ forces $g(y) = 0$, contradicting the minimality of $s$. Therefore $c_0 \neq 0$ and $J = c_0 I = Ic_0 \subseteq R$. Now $f(X, y)$ is a polynomial identity for $B \cap J \subseteq L$, and so for its central localization, a finite dimensional central simple algebra [16, Theorem 2, p. 57]. Applying [20, Lemma 2, p. 732] shows that $B \cap J$ is commutative or that $f(X, y)$ is an identity for some $M_d(F)$ for $F$ a field and $d > 1$. But $f(e_{12}, e_{22}) = e_{12} \neq 0$, for $e_{12}$ and $e_{22}$ matrix units in $M_d(F)$, so $B \cap J$ is commutative, forcing $R$ to be commutative [14, Corollary, p. 7], and showing that our original semiprime ring must contain the nonzero central ideal $RD(R)L$.

Finally, we must show that $D(L), D(R)L \subseteq M$, the maximal central ideal of our semiprime ring $R$. We have just proven that $RD(R)L \subseteq M$, so by the Lemma $D(R)L \subseteq M$ and $D(R)D(L) \subseteq D(D(R)L) + D^2(R)L \subseteq D(M) + M = M$. Hence
\[
D(L)RD(L) \subseteq D(LR)D(L) + M \subseteq M,
\]
and the semiprimeness of $M$ by the Lemma forces $D(L) \subseteq M$. Therefore, the proof of the Main Theorem is complete.

It is clear that the Main Theorem generalizes both [9] and [20], and in the way we mentioned after Theorem 1, [8] as well. We end the paper with another consequence of the Main Theorem by giving an extension to one-sided ideals of [2, Theorem 3, p. 385] and [7, Theorem 2, p. 120].

**Theorem 2.** Let $R$ be a semiprime ring and $L$ a nonzero left ideal of $R$. If for integers $n, k \geq 1$, and some $a \in R, [a, x^k]_n = 0$ for all $x \in L$, then $[a, L] = 0$. When $R$ is a prime ring, then $a \in Z(R)$, the center of $R$. 
Proof. Define a derivation \( D \) of \( R \) by \( D(r) = [r, a] \). Then for all \( x \in L \),

\[-[D(x^k), x^k]_{n-1} = D(x^k, x^k)_{n-1} = [a, x^k]_{n} = 0.
\]

By the Main Theorem, either \( D = 0 \) or \( D(L) \subseteq Z(R) \). When \( D = 0 \), \( a \in Z(R) \) is immediate, and when \( D(L) \subseteq Z(R) \), \([a, L, R] = 0 \). In particular, if \( y \in L \) and 

\[0 = [a, ay, r] = [a[a, y], r] = [a, r][a, y],\]

so letting \( r = ys \) for \( s \in R \) shows that \([a, y]R[a, y] = 0 \). Since \( R \) is semiprime we are forced to conclude that \([a, L] = 0 \). When \( R \) is prime, \( 0 = [a, RL] = [a, R]L \), so \( a \in Z(R) \), proving the theorem. \( \square \)

References


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