A NOTE ON THE BOGOMOLOV-TYPE SMOOTHNESS
ON DEFORMATIONS OF THE REGULAR PARTS
OF ISOLATED SINGULARITIES

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ABSTRACT. We apply the Tian-Todorov method, proving the Bogomolov smoothness theorem (for deformations of compact Kähler manifolds) to deformations of the regular part of a Stein space with a finite number of isolated singular points. By the argument based on the Hodge structure on a strongly pseudo-convex Kähler domain or on a punctured Kähler space, we obtain an unobstructed subspace of the infinitesimal deformation space.

INTRODUCTION

In this paper, we will consider Bogomolov-type smoothness on deformations of the regular part of a normal isolated singularity. In [B], F. Bogomolov showed the following new smoothness theorem for the versal family of n-dimensional compact Kähler manifold: *If there exists a non-vanishing holomorphic (n,0)-form, or if there exists a non-degenerate holomorphic (2,0)-form and $b_1 = 0$, then the versal deformation space of that Kähler manifold is smooth.* G. Tian and A. Todorov rephrased it in order to show the smoothness of the moduli space of compact Calabi-Yau manifolds ([Ti], [To]). Our aim is to consider smoothness of this type in the case of deformations of a normal isolated singularity. Since the smoothness of the versal family is a problem of formal deformation theory and the formal deformation theory of an isolated singularity is equivalent to that of its regular part if the depth of the singularity is greater than or equal to 3 (cf. [S]), we will consider formal deformations of the regular part of an isolated singularity. Let V be a normal Stein space with a finite number of isolated singular points $S := \{p_1, \ldots, p_k\}$, $\pi : X \to V$ a resolution of singularities, and denote $E := \pi^{-1}(S)$. Then X is a strongly pseudo-convex Kähler manifold with an exceptional subset E. Let $r : X \to [-\infty, +\infty)$ be a strictly plurisubharmonic exhaustion function such that $E = \bigcap_\epsilon \Omega_\epsilon$, where $\Omega_\epsilon := \{x \in X \mid r(x) < \epsilon\}$. We may assume that $dr \neq 0$ on $\Omega_a \setminus E$ and outside Ω_b for some $a \leq b$.

We consider deformations of $U := X \setminus E$. As was shown in [Ak-M], even if K_U is trivial, the (formal) versal family of deformations of U is not necessarily smooth. In fact, there might be an obstructed infinitesimal deformation class in $H^1(U, \Theta_U)$.

So, in this paper, we will consider unobstructedness of a subspace of $H^1(U, \Theta_U)$, where we call a subspace *unobstructed* if there exists a formal smooth family whose infinitesimal deformation space coincides with that subspace.

The main reason for the failure of the analogue of the Bogomolov smoothness theorem in the case of deformation of U is the non-existence of pure Hodge structure on $H^*(U, \mathbf{C})$. However, since there exists a pure Hodge structure of weight $\geq n+1$ on $H^*(\Omega_c, \mathbf{C})$, $n = \dim V$ (cf. [D], [O1], [O-T]), we can expect some effect on deformations of U. The following is the main theorem.

Theorem A. *Suppose that there exists an effective divisor $D \in | - K_X|$ with $SuppD \subset E$ or K_X is trivial. Then the subspace*

$$\mathrm{Im}(H^1(X, \Theta_X(-D)) \to H^1(U, \Theta_U))$$

is unobstructed, where we denote $\Theta_X(-D) = \Theta_X$ if $K_X = 1_X$.

Since any relatively compact Stein domain of V can be completed to a projective variety (cf. [Ar], [L]), we may assume that X is an open domain of a compact Kähler manifold $\bar{X}$. Using the pure Hodge structure of weight $\leq n - 2$ on the punctured Kähler space $X^{'} := \bar{X} \setminus E$ (cf. [O2]), we also have

Theorem B. *Suppose $\dim V \geq 4$ and there exists a non-degenerate holomorphic $(2,0)$-form ω on U which is extendable to a non-degenerate holomorphic $(2,0)$-form over $X^{'}$ or to a closed meromorphic $(2,0)$-form over $X^{'}$ such that the component of the local expression (ω_{ij}) of that meromorphic (2,0)-form has a holomorphic inverse. Then the subspace $Im(H^1(X^{'}, \Omega^1_{X^{'}}) \to H^1(U, \Omega^1_U) \simeq H^1(U, \Theta_U))$ is unobstructed.*

We will prove these theorems in §§2 and 3. Our proof is an analogue of the Tian-Todorov argument and depends on the Hodge structures stated above. Hence, in §1, we will derive the key properties for that argument from the Hodge structure on Ω_c or on $X^{'}$ (Corollary 1.4). In §4, we will mention briefly an example discussed in [M].

I would like to thank Professors T. Ohsawa and M. Tomari for helpful discussions in this work. In particular, the proof of Proposition 1.3 is due to Professor T. Ohsawa.

$$\S 1. \ \partial\bar{\partial}\text{-LEMMA AND } I_X^{n-1,1} = H_X^{n-1,1}$$

Let X be a complex manifold with $\dim_{\mathbf{C}} X = n$. Consider the double complex $(A_X^{p,q}, \partial, \bar{\partial})$. The Tian-Todorov method heavily depends on the Hodge structure about this double complex, in particular on the following two properties:

(1.1) $I_X^{n-1,1} = H_X^{n-1,1}$.

(1.2) ($\partial\bar{\partial}$-lemma at $A_X^{n-1,2}$) If an $(n-1,2)$-form is d-closed and ∂-exact, then it is $\partial\bar{\partial}$-exact.

Here we denote by $H_X^{p,q}$ the q-th Dolbeault cohomology group of $(A_X^{p,\bullet}, \bar{\partial})$, and $I_X^{p,q} := \mathrm{Ker}\bar{\partial} \cap Z_X^{p,q}/\bar{\partial}A_X^{p,q-1} \cap Z_X^{p,q}$ with $Z_X^{p,q} := \{\alpha \in A_X^{p,q}|\partial\alpha = 0\}$.

In this section, we will discuss the property $I_X^{p,q} = H_X^{p,q}$ and the $\partial\bar{\partial}$-lemma on a strongly pseudo-convex Kähler domain Ω_c or on a punctured Kähler space $X^{'}$. The Hodge structure on Ω_c (resp. on $X^{'}$), established in [O1] (see also [D], [O-T]) (resp. in [O2]), asserts the existence of $\varphi_\nu^{p,q} \in A_X^{p,q}$ ($p + q = k, 1 \leq \nu \leq h_X^{p,q} < +\infty$) satisfying the following ($k \geq n + 1$ for Ω_c and $k \leq n - 2$ for $X^{'}$):

(1.3) $d\varphi_\nu^{p,q} = 0,$

(1.4) $\{[\varphi_\nu^{p,q}]\}_{1\leq\nu\leq h_X^{p,q}}$ is a basis of $H^q(X,\Omega^p)$, where $[\varphi_\nu^{p,q}]$ denotes the cohomology class of $\varphi_\nu^{p,q}$ in $H^q(X,\Omega^p)$,

(1.5) $\{[\varphi_\nu^{p,q}]\}_{p+q=k,1\leq\nu\leq h_X^{p,q}}$ is a basis of $H^k(X,\mathbf{C})$, where $[\varphi_\nu^{p,q}]$ denotes the cohomology class of $\varphi_\nu^{p,q}$ in $H^k(X,\mathbf{C})$,

(1.6) $\{[\bar\varphi_\nu^{p,q}]\}_{1\leq\nu\leq h_X^{p,q}}$ is a basis of $H^p(X,\Omega^q)$.

Remark. The condition (1.6) is superfluous; in fact, it follows from (1.3)–(1.5).

Lemma 1.1. *Let $k\geq 1$ and suppose that there exist $\varphi_\nu^{p,q}\in A_X^{p,q}$ $(1\leq\nu\leq h_X^{p,q})$ satisfying (1.3)–(1.6) for all (p,q) with $p+q=k$. If $\alpha^{p,q}\in A_X^{p,q}$ $(p+q=k)$ is $\bar\partial$-closed and ∂-exact, then $\alpha^{p,q}\in\bigoplus_{p'+q'=k-1,q'\geq q}dA_X^{p',q'}$.*

Proof. Throughout the proof, we use the notation $\varphi^{s,t}$ as a linear combination of $\{\varphi_\nu^{s,t}\}_{1\leq\nu\leq h_X^{s,t}}$ and $y^{s,t}$ as an element of $A_X^{s,t}$. Note that $A_X^{s,t}=0$ unless $s(s-n)\leq 0$ and $t(t-n)\leq 0$. Let $\alpha^{p,q}=\partial y^{p-1,q}$. Since $\bar\partial\alpha^{p,q}=0$, we have $\partial y^{p-1,q}=\varphi^{p,q}-\bar\partial y^{p,q-1}$ by (1.4). Since $\bar\partial\partial y^{p,q-1}=0$, we have $\partial y^{p,q-1}=\varphi^{p+1,q-1}-\bar\partial y^{p+1,q-2}$ by (1.2) again. Repeating this argument, we have

$$\partial y^{p-1,q}+d(y^{p,q-1}+y^{p+1,q-2}+\cdots+y^{n,p+q-n-1})$$
$$=\varphi^{p,q}+\varphi^{p+1,q-1}+\cdots+\varphi^{n,p+q-n}.$$

On the other hand, applying the above argument to $\partial\overline{y^{p-1,q}}$, we have

$$\bar\partial y^{p-1,q}+d(y^{p-2,q+1}+y^{p-3,q+2}+\cdots+y^{p+q-n-1,n})$$
$$=\varphi^{p-1,q+1}+\varphi^{p-2,q+2}+\cdots+\varphi^{p+q-n,n}.$$

Hence we have $[\varphi^{p+q-n,n}+\cdots+\varphi^{p,q}+\cdots+\varphi^{n,p+q-n}]=0$ in $H^k(X,\mathbf{C})$. By (1.5), we have $\varphi^{p-1,q+1}=\varphi^{p-2,q+2}=\cdots=\varphi^{p+q-n,n}=0$ and

$$\alpha^{p,q}=d(y^{p-1,q}+y^{p-2,q+1}+\cdots+y^{p+q-n-1,n}).$$

$\square$

Proposition 1.2. *Under the assumption of Lemma 1.1, the $\partial\bar\partial$-lemma holds at $A_X^{p,q}$ for all (p,q) with $p+q=k$ and $r\leq q\leq n$ if and only if $I_X^{s,t}=H_X^{s,t}$ holds for all (s,t) with $s+t=k-1$ and $r\leq t\leq n$.*

Proof. We use the same notation as in the proof of Lemma 1.1. Let $\alpha^{p,q}\in A_X^{p,q}$ and $\beta^{p-1,q}\in A_X^{p-1,q}$ $(p+q=k)$ satisfy $\bar\partial\alpha^{p,q}=0$ and $\alpha^{p,q}=\partial\beta^{p-1,q}$. Then by Lemma 1.1,

$$\alpha^{p,q}=d(y^{p-1,q}+y^{p-2,q+1}+\cdots+y^{p+q-n-1,n}).$$

Since $I_X^{p+q-n-1,n}=H_X^{p+q-n-1,n}$ holds, we have

$$y^{p+q-n-1,n}=\varphi^{p+q-n-1,n}+\bar\partial\gamma^{p+q-n-1,n-1}.$$

Since $d\varphi^{p+q-n-1,n}=0$, if we set

$$y'^{p+q-n-2,n-1}:=y^{p+q-n-2,n-1}-\partial\gamma^{p+q-n-1,n-1},$$

we have $\bar\partial y'^{p+q-n-2,n-1}=0$ and $\partial y'^{p+q-n-2,n-1}=\partial y^{p+q-n-2,n-1}$. Repeating this argument, we have a $\gamma^{p-1,q-1}\in A_X^{p-1,q-1}$ such that

$$\partial(y^{p+q-n-2,n-1}-\bar\partial\gamma^{p-1,q-1})=0.$$

Therefore we have $\alpha^{p,q}=-\partial\bar\partial\gamma^{p-1,q-1}$. The only if part is proved by a standard argument. $\square$

About $I^{s,t}_{\Omega_c} = H^{s,t}_{\Omega_c}$ for $s + t = n$, we have

Proposition 1.3. $I^{s,t}_{\Omega_c} = H^{s,t}_{\Omega_c}$ *holds for* (s,t) *with* $s + t = n$ *and* $t \geq 1$.

Proof (due to T.Ohsawa). We use the same notation as in the proof of Lemma 1.1. We introduce a complete Kähler metric over Ω_c associated to the Kähler form $\omega - \sqrt{-1}\partial\bar{\partial}\log(c - r)$, where ω denotes the Kähler form on X. Then it is proved in [O1] that the following hold:

(i) For $k \geq n + 1$, the restriction $\beta_{|\Omega_c}$ of a $\beta \in L^k_{loc}(\Omega_{c+\epsilon})$ ($\epsilon > 0$) is an L^2-form on Ω_c.

(ii) For $k \geq n + 1$, the natural homomorphism $H^k(\Omega_c, \mathbf{C}) \to H^k_{(2)}(\Omega_c, \mathbf{C})$ is an isomorphism, where $H^k_{(2)}(\Omega_c, \mathbf{C})$ denotes the L^2-cohomology group.

(iii) The image of $\bar{\partial} : L^{p,q}_{\Omega_c} \to L^{p,q+1}_{\Omega_c}$ is closed, where $L^{p,q}_{\Omega_c}$ denotes the space of L^2 (p, q)-forms on Ω_c.

Let $\alpha^{p,q} \in A^{p,q}_{\Omega_c}$ ($p + q = n, q \geq 1$) be a $\bar{\partial}$-closed form. It is enough to find a $\gamma^{p,q-1} \in A^{p,q-1}_{\Omega_c}$ satisfying $\partial\bar{\partial}\gamma^{p,q-1} = \partial\alpha^{p,q}$. Since $H^q(X, \Omega^p) \to H^q(\Omega_c, \Omega^p)$ is an isomorphism, we may assume that $\alpha^{p,q}$ can be extendable to X. By (i) and (ii), we have that $\partial\alpha^{p,q}$ is an L^2-form, and $\partial\alpha^{p,q} = dy$ for some C^∞ and L^2-form y on Ω_c. Let $y = y^{0,n} + y^{1,n-1} + \cdots + y^{n,0}$, $y^{s,t} \in A^{s,t}_{\Omega_c} \cap L^{s,t}_{\Omega_c}$. By (iii), we can find $\gamma^{0,n-1} \in A^{0,n-1}_{\Omega_c} \cap L^{0,n-1}_{\Omega_c}$ such that $\bar{\partial}^*(y^{0,n} - \bar{\partial}\gamma^{0,n-1}) = 0$ and $\bar{\partial}^*\gamma^{0,n-1} = 0$ hold. Hence we have $\Box(y^{0,n} - \bar{\partial}\gamma^{0,n-1}) = 0$. This implies $\partial(y^{0,n} - \bar{\partial}\gamma^{0,n-1}) = 0$, since $\Box = \frac{1}{2}\Delta$. If we set $y^{'1,n-1} := y^{1,n-1} - \partial\gamma^{0,n-1} \in A^{1,n-1}_{\Omega_c} \cap L^{1,n-1}_{\Omega_c}$, we have $\bar{\partial}y^{'1,n-1} = 0$ and $\partial y^{'1,n-1} = \partial y^{1,n-1}$. Here we remark that $\partial\gamma^{0,n-1} \in L^{1,n-1}_{\Omega_c}$ holds, since $\gamma^{0,n-1}$, $\bar{\partial}\gamma^{0,n-1}$ and $\bar{\partial}^*\gamma^{0,n-1}$ are all L^2-forms. Repeating this argument, we finally have a $\gamma^{p,q-1}_1 \in A^{p,q-1}_{\Omega_c} \cap L^{p,q-1}_{\Omega_c}$ such that $\partial y^{p,q} = \partial\bar{\partial}\gamma^{p,q-1}_1$ holds. Next, by applying the same argument to $\overline{y^{n,0}}, \ldots, \overline{y^{p+1,q-1}}$, we have a $\gamma^{p,q-1}_2 \in A^{p,q-1}_{\Omega_c} \cap L^{p,q-1}_{\Omega_c}$ such that $\bar{\partial}y^{p+1,q-1} = \partial\bar{\partial}\gamma^{p,q-1}_2$ holds. Therefore, we have

$$\partial\bar{\partial}(\gamma^{p,q-1}_1 - \gamma^{p,q-1}_2) = \partial y^{p,q} + \bar{\partial}y^{p+1,q-1} = \partial\alpha^{p,q}.$$

$\square$

Therefore we have

Corollary 1.4. (1) $I^{p,q}_{\Omega_c} = H^{p,q}_{\Omega_c}$ *holds if* $p + q \geq n + 1$, *or if* $p + q = n$ *and* $q \geq 1$. *The* $\partial\bar{\partial}$-*lemma holds at* $A^{p,q}_{\Omega_c}$ *if* $p + q \geq n + 1$,

(2) $I^{p,q}_{X'} = H^{p,q}_{X'}$ *holds if* $p + q \leq n - 2$. *The* $\partial\bar{\partial}$-*lemma holds at* $A^{p,q}_{X'}$ *if* $p + q \leq n - 2$.

§2. Proof of Theorem A

In this section, we consider deformations of Ω_c and their effect on deformations of $U := \Omega_c \setminus E$. We remark that, since the (formal) deformation theories of Ω_c and $\Omega_c \setminus E$ are both independent of $c \in (-\infty, a) \cup [b, \infty)$ and we consider V as a germ of complex space at S, we may consider Ω_c instead of X; hence we may assume that Corlllary 1.4 (1) holds on X. Then, the following analogue of the Bogomolov smoothness theorem is clear.

Theorem 2.1. *If* K_X *is trivial, then the space* $H^1(X, \Theta_X)$ *is unobstructed for deformations of* X.

Theorem A for the case of $K_X = 1_X$ immediately follows from Theorem 2.1. Next, we consider the situation that there exists a meromorphic $(n, 0)$-form ω with only poles D supported in E, and prove the following theorem:

Theorem 2.2. *Suppose that there exists an effective divisor $D \in | - K_X|$ with $SuppD \subset E$. Then the subspace $Im(H^1(X, \Theta_X(-D)) \to H^1(U, \Theta_U))$ is unobstructed.*

Proof. Since $\iota(\phi) = \omega \rfloor \phi$, we have an isomorphism

$$\iota : A_X^{0,q}(T'X(-D)) \longrightarrow A_X^{n-1,q}.$$

Then by direct calculations, we have

(2.1) If $\phi \in A_X^{0,1}(T'X(-D))$, then $\bar{\partial}\phi \in A_X^{0,2}(T'X(-D))$.

(2.2) If $\phi, \psi \in A_X^{0,1}(T'X(-D))$, then $[\phi, \psi] \in A_X^{0,2}(T'X(-D))$.

And by the same calculations as in [Ti] or [To], we have

(2.3) $\bar{\partial}\iota = \iota\bar{\partial}$.

(2.4) If $\partial\alpha = \partial\beta = 0$, then $\iota[\iota^{-1}\alpha, \iota^{-1}\beta] = \partial(\iota^{-1}\alpha \rfloor \beta)$.

By (2.3), we have an isomorphism $H^q(X, \Theta_X(-D)) \cong H^q(X, \Omega_X^{n-1})$. By Corollary 1.4 (1), we can choose $\{\alpha_\nu^{n-1,1}\}_{1 \le \nu \le d} \subset Z_X^{n-1,1}$ such that $\{[\iota^{-1}\alpha_\nu^{n-1,1}]_{|U}\}_{1 \le \nu \le d}$ is a basis of $Im\left(H^1(X, \Theta_X(-D)) \to H^1(U, \Theta_U)\right)$. Then, by the same argument of [Ti] or [To], we can construct a formal power series $\alpha(s) \in A_X^{n-1,1}[[s_1, \ldots, s_d]]$ satisfying

(2.5) $\bar{\partial}\alpha(s) - \frac{1}{2}\partial(\iota^{-1}\alpha(s) \rfloor \alpha(s)) = 0$,

(2.6) $\alpha(s) \equiv \sum_{\nu=1}^d \alpha_\nu^{n-1,1} s_\nu \mod \mathfrak{m}^2$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{C}[[s_1, \ldots, s_d]]$.

Hence if we set $\phi(s) := \iota^{-1}\alpha(s)$, then it satisfies

(2.7) $\bar{\partial}\phi(s) - \frac{1}{2}[\phi(s), \phi(s)] = 0$,

(2.8) $\phi(s) \equiv \sum_{\nu=1}^d \iota^{-1}\alpha_\nu^{n-1,1} s_\nu \mod \mathfrak{m}^2$.

By restricting $\phi(s)$ on U, we have a formal family of deformations of U whose infinitesimal deformation space is $Im(H^1(X, \Theta_X(-D)) \to H^1(U, \Theta_U))$. Hence that space is unobstructed.

This completes the proof of Theorem 2.2. $\square$

§3. PROOF OF THEOREM B

In this section, we consider deformations of X' and their effect on deformations of U. By Corollary 1.4 (2), the following analogue of the Bogomolov smoothness theorem is clear. (Note that all holomorphic (2,0)-forms on X' are d-closed by Corollary 1.4 (2).)

Theorem 3.1. *If there exists a non-degenerate holomorphic $(2, 0)$-form on X', then the space $H^1(X', \Theta_{X'})$ is unobstructed for deformations of X'.*

Next, we suppose that there exists a non-degenerate holomorphic $(2, 0)$-form ω on U which is extendable to a closed meromorphic $(2, 0)$-form on X' such that (ω_{ij}) has a holomorphic inverse, where we denote the meromorphic $(2,0)$-form by $\omega = \frac{1}{2}\sum_{i,j}\omega_{ij} dz^i \wedge dz^j$, with local meromorphic functions ω_{ij}, with respect to a local coordinate $(z^1, \ldots, z^n)$ of X'. Then we have

Theorem 3.2. *Suppose that there exists a non-degenerate holomorphic $(2,0)$-form ω on U as above. Then the subspace $Im\left(H^1(X',\Omega^1_{X'}) \to H^1(U,\Omega^1_U) \simeq H^1(U,\Theta_U)\right)$ is unobstructed.*

Proof. Since $\iota(\phi) = \omega\rfloor\phi$, we have homomorphisms

$$\iota^{-1} : A^{1,q}_{X'} \longrightarrow A^{0,q}_{X'}(T'X')$$

and

$$\iota : \iota^{-1}(A^{1,q}_{X'}) \longrightarrow A^{1,q}_{X'}.$$

$\square$

Lemma 3.3. (1) $\bar{\partial}\iota^{-1} = \iota^{-1}\bar{\partial}$.
  (2) *If $\partial\xi = \partial\eta = 0$, then $\iota[\iota^{-1}\xi, \iota^{-1}\eta] = \partial(\iota^{-1}\xi\rfloor\eta)$. In particular, $[\iota^{-1}\xi, \iota^{-1}\eta] \in \iota^{-1}(A^{1,2}_{X'})$.*

Proof.   (1) is clear.
 (2) Let $\left(h^{\alpha,i}\right)$ be a holomorphic (n,n)-matrix satisfying $\left(h^{\alpha,i}\right)\left(\omega_{i,\gamma}\right) = \left(\delta^\alpha_\gamma\right)$ and $\left(\omega_{i,\gamma}\right)\left(h^{\gamma,j}\right) = \left(\delta^j_i\right)$. Suppose $\xi = \sum_{i,\beta}\xi_{i,\bar\beta}dz^i \wedge d\bar{z}^\beta$ and $\eta = \sum_{j,\delta}\eta_{j,\bar\delta}dz^j \wedge d\bar{z}^\delta$ satisfy $\partial\xi = \partial\eta = 0$. Since

$$
\begin{aligned}
[\iota^{-1}\xi, \iota^{-1}\eta] = \frac{1}{2}\sum_{\alpha,\beta,\delta}\sum_{i,j,\gamma}\Bigg\{ & \left(h^{\gamma i}h^{\alpha j}\xi_{i\bar\beta}\frac{\partial\eta_{j\bar\delta}}{\partial z^\gamma} - h^{\alpha i}h^{\gamma j}\eta_{j\bar\delta}\frac{\partial\xi_{i\bar\beta}}{\partial z^\gamma}\right) \\
& - \left(h^{\gamma i}h^{\alpha j}\xi_{i\bar\delta}\frac{\partial\eta_{j\bar\beta}}{\partial z^\gamma} - h^{\alpha i}h^{\gamma j}\eta_{j\bar\beta}\frac{\partial\xi_{i\bar\delta}}{\partial z^\gamma}\right) \\
& + \left(h^{\gamma i}\frac{\partial h^{\alpha j}}{\partial z^\gamma} - h^{\gamma j}\frac{\partial h^{\alpha i}}{\partial z^\gamma}\right)(\xi_{i\bar\beta}\eta_{j\bar\delta} - \xi_{i\bar\delta}\eta_{j\bar\beta})\Bigg\}\frac{\partial}{\partial z^\alpha} \otimes d\bar{z}^\beta \wedge d\bar{z}^\delta
\end{aligned}
$$

and $\partial\xi = \partial\eta = 0$, we have

$$
\begin{aligned}
\iota[\iota^{-1}\xi, \iota^{-1}\eta] = \frac{1}{2}\sum_{k,\beta,\delta}\sum_{i,j,\alpha,\gamma}\Bigg\{ & h^{\gamma i}\left(\xi_{i\bar\beta}\frac{\partial\eta_{\gamma\bar\delta}}{\partial z^k} - \xi_{i\bar\delta}\frac{\partial\eta_{\gamma\bar\beta}}{\partial z^k}\right) - h^{\gamma j}\left(\frac{\partial\xi_{\gamma\bar\beta}}{\partial z^k}\eta_{j\bar\delta} - \frac{\partial\xi_{\gamma\bar\delta}}{\partial z^k}\eta_{j\bar\beta}\right) \\
& + \omega_{k\alpha}\left(h^{\gamma i}\frac{\partial h^{\alpha j}}{\partial z^\gamma} - h^{\gamma j}\frac{\partial h^{\alpha i}}{\partial z^\gamma}\right)(\xi_{i\bar\beta}\eta_{j\bar\delta} - \xi_{i\bar\delta}\eta_{j\bar\beta})\Bigg\} dz^k \wedge d\bar{z}^\beta \wedge d\bar{z}^\delta.
\end{aligned}
$$

On the other hand, since $\iota^{-1}\xi\rfloor\eta = \frac{1}{2}\sum_{\beta,\delta}\sum_{i,\alpha}h^{\alpha i}(\xi_{i\bar\beta}\eta_{\alpha\bar\delta} - \xi_{i\bar\delta}\eta_{\alpha\bar\beta})d\bar{z}^\beta \wedge d\bar{z}^\delta$ and $h^{i\alpha} = -h^{\alpha i}$, we have

$$
\begin{aligned}
\partial(\iota^{-1}\xi\rfloor\eta) = \frac{1}{2}\sum_{k,\beta,\delta}\sum_{i,\alpha}\Bigg\{ & h^{\alpha i}\left(\xi_{i\bar\beta}\frac{\partial\eta_{\alpha\bar\delta}}{\partial z^k} - \xi_{i\bar\delta}\frac{\partial\eta_{\alpha\bar\beta}}{\partial z^k}\right) \\
& - h^{i\alpha}\left(\frac{\partial\xi_{i\bar\beta}}{\partial z^k}\eta_{\alpha\bar\delta} - \frac{\partial\xi_{i\bar\delta}}{\partial z^k}\eta_{\alpha\bar\beta}\right) + \frac{\partial h^{\alpha i}}{\partial z^k}(\xi_{i\bar\beta}\eta_{\alpha\bar\delta} - \xi_{i\bar\delta}\eta_{\alpha\bar\beta})\Bigg\} dz^k \wedge d\bar{z}^\beta \wedge d\bar{z}^\delta.
\end{aligned}
$$

Hence (2) follows from the following

Claim.

$$\frac{\partial h^{ji}}{\partial z^k} = \sum_{\alpha,\gamma}\omega_{k\alpha}\left(h^{\gamma i}\frac{\partial h^{\alpha j}}{\partial z^\gamma} - h^{\gamma j}\frac{\partial h^{\alpha i}}{\partial z^\gamma}\right).$$

Proof. By $h^{ji} = \sum_{\alpha,\gamma} h^{j\gamma}\omega_{\gamma\alpha}h^{\alpha i}$ and $d\omega = 0$, we have

$$\frac{\partial h^{ji}}{\partial z^k} = -\sum_{\alpha,\gamma} h^{j\gamma}\left(\frac{\partial\omega_{k\alpha}}{\partial z^\gamma} + \frac{\partial\omega_{\gamma k}}{\partial z^\alpha}\right)h^{\alpha i}.$$

Hence the claim follows from $\sum_\alpha \omega_{k\alpha}h^{\alpha i} = \delta_k^i$ and $\sum_\gamma h^{j\gamma}\omega_{\gamma k} = \delta_k^j$. □

Then by the same argument as in §2, we have a formal power series $\alpha(s) \in A_{X'}^{1,1}[[s_1,\dots,s_d]]$ satisfying

(3.1) $\bar\partial\alpha(s) - \frac{1}{2}\partial(\iota^{-1}\alpha(s)\rfloor\alpha(s)) = 0$,
(3.2) the linear part of $\iota^{-1}\alpha(s)_{|U}$ spans

$$Im\left(H^1(X',\Omega_{X'}^1) \to H^1(U,\Omega_U^1) \simeq H^1(U,\Theta_U)\right),$$

where $d = \dim_{\mathbf{C}}Im\left(H^1(X',\Omega_{X'}^1) \to H^1(U,\Omega_U^1) \simeq H^1(U,\Theta_U)\right)$.

Hence, if we set $\phi(s) := \iota^{-1}\alpha(s)_{|U}$, then we have

(3.3) $\bar\partial\phi(s) - \frac{1}{2}[\phi(s),\phi(s)] = 0$,
(3.4) the linear part of $\phi(s)$ spans

$$Im\left(H^1(X',\Omega_{X'}^1) \to H^1(U,\Omega_U^1) \simeq H^1(U,\Theta_U)\right).$$

This completes the proof of Theorem 3.2. □

§4. Examples (Deformations of quasi-Gorenstein cone singularities)

We consider deformations of quasi-Gorenstein cone singularities as an example of our argument. For detailed discussion, see [M]. Let Y be a projective algebraic manifold and $\pi : F \to Y$ a negative line bundle. We denote by X the total space of F, by D_0 the 0-section of F and $U := X \setminus D_0$.

Example 4.1. We suppose that the canonical line bundle K_U is trivial. Then $K_Y \simeq F^\mu$ for some $\mu \in \mathbf{Z}$.

(1) The case of $K_Y \simeq F^\mu$ ($\mu > 1$): Since $H^2(X,\Theta_X) \simeq H^2(X,\Omega_X^{n-1}(K_X^{-1})) = 0$ ([N]), the subspace $Im(H^1(X,\Theta_X) \to H^1(U,\Theta_U))$ is unobstructed.
(2) The case of $K_Y \simeq F^{-\mu}$ ($\mu \geq -1$): Since $K_X \simeq \pi^*(F^{-\mu-1})$, by Theorem A, the space $Im(H^1(X,\Theta_X(-(\mu+1)D_0)) \to H^1(U,\Theta_U))$ is unobstructed, where we denote $\Theta_X(-(\mu+1)D_0) := \Theta_X$ if $\mu = -1$.

Example 4.2. Let $Y := \mathbf{P}(T^*\mathbf{P}^n)$ ($n \geq 2$) and F the dual tautological line bundle. We denote by $\bar{X}$ the associated $\mathbf{P}^1$-bundle and by D_∞ the ∞-section. Then $X := \bar{X} \setminus D_\infty$ is a special case of Example 4.1 (1) and $X' := \bar{X} \setminus D_0$ is a punctured Kähler space. Since there exists a closed meromorphic (2,0)-form on X' which is non-degenerate and holomorphic on U and having a holomorphic inverse on X', by Theorem B, the space $Im(H^1(X',\Omega_{X'}^1) \to H^1(U,\Theta_U))$ is unobstructed.

Remark. In either case of these examples, the above unobstructed subspace of $H^1(U,\Theta_U)$ induces an unobstructed subspace of $\text{Ext}^1(\Omega_V^1,\mathcal{O}_V)$, where we denote by V the germ of a normal complex space obtained by contracting D_0.

References

[Ak-M] Akahori,T. and Miyajima,K., *A note on the analogue of the Bogomolov type theorem on deformations of CR-structures*, Canadian Math. Bull. **37** (1994), 8–12. MR **95d:**32018

[Ar] Artin,M., *Algebraic approximation of structures over complete local rings*, Publ. Math. I.H.E.S. **36** (1969), 23-58. MR **42:**3087

[B] Bogomolov,F.A., *Hamiltonian Kähler manifolds*, Dokl. Akad. Nauk. SSSR **243** (1978), 1101–1104; English transl., Soveit Math. Dokl. **19** (1978), 1462–1465. MR **80c:**32024

[D] Demailly,J.-P., *Cohomology of q-convex spaces in top degrees*, Math. Z. **204** (1990), 283–295. MR **91e:**32014

[L] Lempert,L., *Algebraic approximations in analytic geometry*, Invent. Math. **121** (1995), 335–354.

[M] Miyajima,K., *A Bogomolov-type smoothness on deformations of quasi-Gorenstein cone singularities of dim ≥ 4*, "Geometric Complex Analysis", ed. J. Noguchi et al., World Scientific, 1996, pp. 461–465.

[N] Nakano,S., *Vanishing theorems for weakly 1-complete manifolds, II*, Publ. R.I.M.S., Kyoto Univ. **10** (1974), 101–110. MR **52:**3617

[O1] Ohsawa,T., *A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds*, Inv. Math. **63** (1981), 335–354. MR **82k:**32032

[O2] ———, *Hodge spectral sequence on compact Kähler spaces*, Publ. R.I.M.S., Kyoto Univ. **23** (1987), 265–274. MR **92f:**32050

[O-T] Ohsawa,T. and Takegoshi,K., *Hodge spectral sequence on pseudoconvex domains*, Math. Z. **197** (1988), 1–12. MR **89d:**32042

[S] Schlessinger,M., *On rigid singularities*, Conference on Complex Analysis, Rice University Studies **59** (1973), no. 1, 147–162. MR **49:**9258

[Ti] Tian,C., *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, Mathematical Aspects of String Theory , S.T. Yau ed., World Scientific, 1987, pp. 629–646. MR **89m:**81001

[To] Todorov,A.N., *The Weil-Petersson geometry of the moduli space of SU(n ≥ 3) (Calabi-Yau)Manifolds I*, Commun. Math. Phys. **126** (l989), 325–346. MR **91f:**32022

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