

DIFFERENTIABLE SELECTION OF OPTIMAL SOLUTIONS IN PARAMETRIC LINEAR PROGRAMMING

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ABSTRACT. In the present paper we prove that if the data of a parametric linear optimization problem are smooth, the solution map admits a local smooth selection “almost” everywhere. This in particular shows that the set of points where the marginal function of the problem is nondifferentiable is nowhere dense.

1. INTRODUCTION

Subdifferential calculus for marginal functions defined by parametric optimization problems has recently become a subject of much attention because of its potential applications to the solution of various questions arising in optimization, control theory and other areas of applied mathematics. “Outer” estimation for the subdifferentials of marginal functions has been studied in numerous papers (see e.g. [1], [10], [17], [19], [21]). However, how to calculate at least one element from these subdifferentials remains a difficult part of nonsmooth optimization (see [6], [9], [12], [20]). In most existing results an important condition to require is that the solution map admits a locally Lipschitz selection, which is differentiable at a point of our consideration.

Motivated by the above condition, we set our aim to investigate differentiability properties of the marginal function and the solution map of a parametric linear optimization problem. The main result of the paper states that if the objective and the constraints of the problem are smooth, there is an open dense set in the parameter space where the feasible set is continuous (Section 2) and the solution map admits a smooth local selection (Section 3).

This result is remarkable in part for the following reason. It is known (Michael’s selection theorem [2]) that under suitable conditions, a set-valued map admits a continuous (or Lipschitz continuous) selection. We have a natural question: do differentiable selections exist? The result of Section 3 provides a sufficient condition for a positive answer.

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2. STABILITY OF LINEAR SYSTEMS

Throughout the paper we shall make use of the following notation: p, q, r, s are positive integers; c, a_1, \dots, a_{p+q} denote continuous vector functions from R^r to R^s ; $\alpha_1, \dots, \alpha_{p+q}$ stand for continuous scalar functions on R^r ; and $\langle \cdot, \cdot \rangle$ stands for the inner product in R^s . The marginal function we are going to study is defined as

$$m(\omega) := \min\{\langle c(\omega), x \rangle \mid x \in M(\omega)\},$$

where

$$M(\omega) := \{x \in R^s \mid \langle a_i(\omega), x \rangle + \alpha_i(\omega) \leq 0, i = 1, \dots, p, \\ \langle a_j(\omega), x \rangle + \alpha_j(\omega) = 0, j = p + 1, \dots, p + q\},$$

and the solution map is defined by

$$S(\omega) := \{x \in M(\omega) \mid \langle c(\omega), x \rangle = m(\omega)\}.$$

By a selection of the solution map we mean any (single-valued) function $g(\omega)$ from R^r to R^s with $g(\omega) \in S(\omega)$ for all $\omega \in R^r$. Let us recall that a set-valued map A from R^r to R^s is said to be closed at a point $\omega_0 \in R^r$ if, whenever a sequence $\{(\omega_n, x_n)\} \subseteq R^r \times R^s$ with $x_n \in A(\omega_n)$ converges to (ω_0, x_0) , one has $x_0 \in A(\omega_0)$. A is said to be lower continuous at $\omega_0 \in R^r$ if for every neighborhood U in R^s with $A(\omega_0) \cap U \neq \emptyset$ there exists a neighborhood W of ω_0 in R^r such that $A(\omega) \cap U \neq \emptyset$ for all $\omega \in W$ (here we use “lower continuous” instead of “lower semicontinuous” to avoid the confusion with lower semicontinuity of point-valued real functions). The map A is said to be continuous at ω_0 (on a nonempty set $W \subseteq R^r$) if it is closed and lower continuous at ω_0 (at every point of W , resp.). There exist different definitions of continuity in the literature (see [2], [3], [4], [11], [14]). However, they are all equivalent to the one given above if the values of A are compact in a bounded set. In this event, A is continuous at ω_0 if and only if the Hausdorff distance $h(A(\omega), A(\omega_0))$ between $A(\omega)$ and $A(\omega_0)$ tends to 0 as ω tends to ω_0 . As to the concept of upper (semi-)continuity in the case of noncompact values, we do not pay our attention because in that case generally the map M is never upper continuous. We shall adopt the convention that A is continuous at ω_0 with $A(\omega_0)$ empty if $A(\omega)$ is empty for all ω in some neighborhood of ω_0 .

Denote

$$P(\omega) := \{x \in R^s \mid \langle a_i(\omega), x \rangle + \alpha_i(\omega) \leq 0, i = 1, \dots, p\}, \\ Q(\omega) := \{x \in R^s \mid \langle a_j(\omega), x \rangle + \alpha_j(\omega) = 0, j = p + 1, \dots, p + q\}.$$

Furthermore, define, for any subset I of $\{1, \dots, p\}$,

$$P_I(\omega) := \{x \in R^s \mid \langle a_i(\omega), x \rangle + \alpha_i(\omega) \leq 0, i \in I\}, \\ \widehat{P}_I(\omega) := \{x \in R^s \mid \langle a_i(\omega), x \rangle + \alpha_i(\omega) < 0, i \in I\}, \\ \widehat{P}^I(\omega) := \{x \in R^s \mid \langle a_i(\omega), x \rangle + \alpha_i(\omega) < 0, i \in \{1, \dots, p\} \setminus I\},$$

and, for any subset J of $\{1, \dots, p + q\}$,

$$Q_J(\omega) := \{x \in R^s \mid \langle a_j(\omega), x \rangle + \alpha_j(\omega) = 0, j \in J\}.$$

To date important results have been obtained on the continuity of the map $M(\cdot)$ (see for instance [3], [8], [15]). However, the following seems to be new.

Proposition 2.1. *For every open set $W_0 \subseteq R^r$, there exists an open subset $W \subseteq W_0$ such that M is continuous on W . Consequently, the set of discontinuity of M is nowhere dense.*

To prove this proposition the following lemmas are needed.

Lemma 2.1. *Let ω_0 be a fixed point in R^r . Assume that the system $\{a_j(\omega_0), j \in J\}$ is linearly independent. Then, there exists a neighborhood W of ω_0 in R^r such that the set-valued map $\omega \mapsto Q_J(\omega)$ is continuous on W .*

Proof. It is clear that $Q_J(\cdot)$ is a closed map (which is true without the linear independence assumption). The lower continuity can be derived from Robinson's general criterion ([15]), however we provide here a direct proof. Let $\nu := |J|$ and $A(\omega_0)$ denote the $(\nu \times s)$ -matrix whose rows are $a_j(\omega_0), j \in J$. It is evident that $\nu \leq s$ and $A(\omega_0)$ possesses a nonsingular $(\nu \times \nu)$ -submatrix, say $C(\omega_0)$. The complementary part of $C(\omega_0)$ in $A(\omega_0)$ is denoted by $D(\omega_0)$. Without loss of generality we may assume that

$$A(\omega_0) = (C(\omega_0)|D(\omega_0)).$$

For $x \in R^s$, we write $x = (x^1, x^2)$, where x^1 consists of the first ν components of x and x^2 is the rest. Then

$$Q_J(\omega_0) = \{x \in R^s | x^1 = [C(\omega_0)]^{-1}(-\alpha(\omega_0) - D(\omega_0)x^2), x^2 \in R^{s-\nu}\},$$

where $[C(\omega_0)]^{-1}$ denotes the inverse matrix of $C(\omega_0)$ and $\alpha(\cdot)$ stands for the vector function whose components are $\alpha_j(\cdot), j \in J$. One can find a neighborhood W of ω_0 such that $C(\omega)$ is nonsingular whenever $\omega \in W$. Hence, we also have

$$Q_J(\omega) = \{x \in R^s | x^1 = [C(\omega)]^{-1}(-\alpha(\omega) - D(\omega)x^2), x^2 \in R^{s-\nu}\}.$$

From this we easily see that $Q_J(\cdot)$ is lower continuous on W . The lemma is proved. \square

Lemma 2.2. *Suppose that the system $\{a_j(\omega), j \in J\}$ is linearly dependent for every ω on an open set $W_0 \subseteq R^r$. Then there exist an open subset $W \subseteq W_0$ and a proper subset $J' \subseteq J$ such that $\{a_j(\omega), j \in J'\}$ is a maximal linearly independent subsystem of the system $\{a_j(\omega), j \in J\}$ for every $\omega \in W$.*

Proof. For every $\omega \in W_0$, the rank of the system $\{a_j(\omega), j \in J\}$ cannot exceed $\nu - 1$ where $\nu = |J|$, hence one can find a point $\omega_0 \in W_0$ such that the rank of the system $\{a_j(\omega_0), j \in J\}$ is maximal on W_0 and equals, say $\mu < \nu$. Let $\{a_j(\omega_0), j \in J'\}$ with $|J'| = \mu, J' \subseteq J$, be a linearly independent subsystem of the system $\{a_j(\omega_0), j \in J\}$. The matrix $A(\omega_0)$ whose rows are $a_j(\omega_0), j \in J'$, possesses a nonsingular $(\mu \times \mu)$ -submatrix, say $C(\omega_0)$. There exists, then, a neighborhood $W \subseteq W_0$ of ω_0 such that $C(\omega)$ is nonsingular whenever $\omega \in W$. Hence, the subsystem $\{a_j(\omega), j \in J'\}$ has rank equal to μ , for every $\omega \in W$. In other words $\{a_j(\omega), j \in J\}$ is linear independent. It is maximal because the rank of the system $\{a_j(\omega), j \in J\}$ does not exceed μ . The proof is complete. \square

Lemma 2.3. *For every open set $W_0 \subseteq R^r$ there exists an open subset $W \subseteq W_0$ such that the set-valued map $\omega \mapsto Q(\omega)$ is continuous on W .*

Proof. If there exists a point $\omega_0 \in W_0$ such that the system $\{a_j(\omega_0), j = p + 1, \dots, p + q\}$ is linearly independent, then in view of Lemma 2.1, Q is continuous on some neighborhood $W \subseteq W_0$ of ω_0 . If the system $\{a_j(\omega), j = p + 1, \dots, p + q\}$

is linearly dependent for every $\omega \in W_0$, we apply Lemma 2.2 to get an open subset $W_1 \subseteq W_0$ and a proper subset $J \subseteq \{p + 1, \dots, p + q\}$ such that $\{a_j(\omega), j \in J\}$ is a maximal linearly independent subsystem of $\{a_j(\omega), j = p + 1, \dots, p + q\}$, $\omega \in W_1$. We claim that there is an open set $W \subseteq W_1$ such that either

$$Q(\omega) = Q_J(\omega), \quad \text{for all } \omega \in W,$$

or

$$Q(\omega) = \emptyset, \quad \text{for all } \omega \in W.$$

In fact, take an arbitrary index $j_0 \in \{p + 1, \dots, p + q\} \setminus J$. If the system $\{a_j(\omega), j \in J \cup \{j_0\}\}$ is linearly dependent, one can find $\lambda_j(\omega) \in R^r$ such that

$$a_{j_0}(\omega) = \sum_{j \in J} \lambda_j(\omega) a_j(\omega),$$

where $\lambda_j(\omega), j \in J$, are uniquely determined and continuous with respect to ω (because $\{a_j(\omega), j \in J\}$ is linearly independent) on some neighborhood of ω . If at some $\omega \in W_1$ one has $\alpha_{j_0}(\omega) \neq \sum_{j \in J} \lambda_j(\omega) \alpha_j(\omega)$, then by the continuity of $\lambda_j(\cdot)$ and $\alpha_j(\cdot)$ there is a neighborhood $W \subseteq W_1$ of ω such that

$$\alpha_{j_0}(\omega') \neq \sum_{j \in J} \lambda_j(\omega') \alpha_j(\omega'), \quad \text{for all } \omega' \in W.$$

This shows that $Q(\omega') = \emptyset$ for every $\omega' \in W$. If $\alpha_{j_0}(\omega) = \sum_{j \in J} \lambda_j(\omega) \alpha_j(\omega)$ for every $\omega \in W_1$, we have

$$Q_J(\omega) = Q_{J \cup \{j_0\}}(\omega), \quad \omega \in W_1.$$

Continuing this procedure for other indices from the set $\{p + 1, \dots, p + q\} \setminus J$, we arrive at the conclusion that either $Q(\omega) = \emptyset$ for all $\omega \in W$, or $Q(\omega) = Q_J(\omega)$ for all $\omega \in W$. In both cases, Q is continuous on W . \square

Lemma 2.4. *Given a point $\omega_0 \in R^r$ and a subset $I \subseteq \{1, \dots, p\}$. Assume that $\widehat{P}_I(\omega_0)$ is nonempty. Then, the map $\omega \mapsto P_I(\omega)$ is continuous on some neighborhood of ω_0 . Moreover, for every $x_0 \in \widehat{P}_I(\omega_0)$ there exist a neighborhood U of x_0 in R^s and a neighborhood W of ω_0 in R^r such that $U \subseteq \widehat{P}_I(\omega)$, for every $\omega \in W$.*

Proof. As before, the closedness of P_I is obvious. To prove the lemma it suffices to show the second part. We have

$$\langle a_i(\omega_0), x_0 \rangle + \alpha_i(\omega_0) < 0, \quad \text{for all } i \in I.$$

One can find a positive ε such that

$$\langle a_i(\omega_0) + a_i, x_0 + x \rangle + \alpha_i(\omega_0) + \alpha_i < 0, \quad \text{for all } i \in I,$$

whenever $\|a_i\| < \varepsilon, \|x\| < \varepsilon, |\alpha_i| < \varepsilon$. By the continuity of $a_i(\cdot)$ and $\alpha_i(\cdot)$, there exists a neighborhood W of ω_0 such that

$$\|a_i(\omega) - a_i(\omega_0)\| < \varepsilon, \quad |\alpha_i(\omega_0) - \alpha_i(\omega)| < \varepsilon,$$

for every $\omega \in W$. Taking $U := \{x \in R^s \mid \|x - x_0\| < \varepsilon\}$, we conclude that

$$\langle a_i(\omega), x \rangle + \alpha_i(\omega) < 0, \quad \text{for all } i \in I, \omega \in W, x \in U.$$

This means that $U \subseteq \widehat{P}_I(\omega)$ for every $\omega \in W$, and the lemma is proved. \square

Lemma 2.5. *For every open set W_0 , there exist an open subset $W \subseteq W_0$ and a subset $I \subseteq \{1, \dots, p\}$ such that for every $\omega \in W$,*

$$P(\omega) = P_I(\omega) \cap Q_J(\omega), \quad \text{where } J := \{1, \dots, p\} \setminus I,$$

and $\widehat{P}_I(\omega) \neq \emptyset$.

Proof. Observe first that $\widehat{P}_i(\omega) = \emptyset$ implies $\alpha_i(\omega) = 0, a_i(\omega) = 0$, and that $\widehat{P}_i(\omega_0) \neq \emptyset$ for some ω_0 implies $\widehat{P}_i(\omega) \neq \emptyset$ for every ω in a sufficiently small neighborhood of ω_0 . Hence without loss of generality we may assume that

$$\widehat{P}_i(\omega) \neq \emptyset, \quad \text{for all } i \in \{1, \dots, p\}, \text{ for all } \omega \in W_0.$$

Let $I(\omega)$ denote the maximal subset of $\{1, \dots, p\}$ with $\widehat{P}_I(\omega) \neq \emptyset$ (with respect to the inclusion preorder), i.e.

$$\widehat{P}_I(\omega) \cap \widehat{P}_i(\omega) = \emptyset, \quad \text{for all } i \in J.$$

Since $|I(\omega)| \leq p$ for every $\omega \in R^r$, one can find a point $\omega_0 \in W_0$ such that

$$|I(\omega_0)| = \max\{|I(\omega)| \mid \omega \in W_0\}.$$

We show that there is a neighborhood W of ω_0 such that $I(\omega) = I(\omega_0)$ for every $\omega \in W$. In fact, as in the proof of Lemma 2.4, $\widehat{P}_{I(\omega_0)}(\omega_0) \neq \emptyset$ implies that there exists a neighborhood W of ω_0 such that $\widehat{P}_{I(\omega_0)}(\omega) \neq \emptyset$ for every $\omega \in W$. Hence $I(\omega_0) \subseteq I(\omega)$ whenever $\omega \in W$. We have, in fact, equality because $|I(\omega_0)|$ is maximal. Set $I = I(\omega_0)$. For every $\omega \in W$, one sees that

$$P_I(\omega) \cap Q_J(\omega) \subseteq P(\omega).$$

On the other hand, $x \in P(\omega)$ implies

$$\langle a_j(\omega), x \rangle + \alpha_j(\omega) = 0, \quad j \in J.$$

Indeed, if this is not the case, for some $j \in J$, one has

$$\langle a_j(\omega), x \rangle + \alpha_j(\omega) < 0.$$

Hence, there is a neighborhood U of x in R^s such that

$$\langle a_j(\omega), x' \rangle + \alpha_j(\omega) < 0, \quad \text{for all } x' \in U.$$

It is evident that $P_I(\omega)$ is the closure of $\widehat{P}_I(\omega)$ which is nonempty, hence one can find in U some $x' \in \widehat{P}_I(\omega)$. Consequently, $x' \in \widehat{P}_I(\omega) \cap \widehat{P}_J(\omega)$ that contradicts the maximality of I . In this way, $P(\omega) = P_I(\omega) \cap Q_J(\omega)$, with $\widehat{P}_I(\omega) \neq \emptyset$ for every $\omega \in W$. This completes the proof. \square

Let M_1, M_2 be two set-valued maps from R^r to R^s . It is known that $M_1 \cap M_2$ is closed if M_1 and M_2 are closed, and that $M_1 \cap M_2$ is not necessarily lower continuous if M_1 and M_2 are. The following exception is useful (see also [13]).

Lemma 2.6. *Assume that M_1 is lower continuous at $\omega_0 \in R^r$ and M_2 has the property that for every x_0 from a subset $M' \subseteq M_2(\omega_0)$ one can find neighborhoods W of ω_0 and U of x_0 such that $U \subseteq M_2(\omega)$ for every $\omega \in W$. Then the intersection map $M_1 \cap M_2$ is lower continuous at ω_0 provided that $M_1(\omega_0) \cap M_2(\omega_0)$ coincides with the closure of the set $M' \cap M_1(\omega_0)$.*

Proof. Let $x \in M_1(\omega_0) \cap M_2(\omega_0)$, and let U_0 be any neighborhood of x in R^s . We may find a point $x_0 \in M' \cap M_1(\omega_0)$ with $x_0 \in U_0$. Then, there exist neighborhoods U and W as stated in the lemma. Take $U_1 = U_0 \cap U$. By the lower continuity of M_1 , one can find a neighborhood $W_1 \subseteq W$ of ω_0 such that $M_1(\omega) \cap U_1 \neq \emptyset$ for every $\omega \in W_1$. Consequently,

$$\emptyset \neq U_1 \cap M_1(\omega) \subseteq U_0 \cap U \cap M_1(\omega) \subseteq U_0 \cap [M_1(\omega) \cap M_2(\omega)]$$

for every $\omega \in W_1$, and the lower continuity of $M_1 \cap M_2$ follows. □

Now we are ready to give

Proof of Proposition 2.1. We have $M(\omega) = P(\omega) \cap Q(\omega)$, for every $\omega \in R^r$. In view of Lemma 2.5, there exists an open subset $W_1 \subseteq W_0$ such that $P(\omega) = P_I(\omega) \cap Q_J(\omega), \omega \in W_1$, where $\hat{P}_I(\omega) \neq \emptyset$ and $J = \{1, \dots, p\} \setminus I$. Hence we may assume that $\hat{P}(\omega) \neq \emptyset$ for all $\omega \in W_1$ (by considering P_I instead of P and $Q \cap Q_J$ instead of Q). By Lemma 2.3, there is a neighborhood $W \subseteq W_1$ such that Q is continuous on W . Using Lemma 2.4 and Lemma 2.6, we conclude that M is continuous on W . □

3. CONTINUITY AND DIFFERENTIABILITY OF THE SOLUTION MAPPING

In this section we study the marginal function m and the solution map S defined at the beginning of the previous section.

Proposition 3.1. *Assume that M is continuous on an open set $W_0 \subseteq R^r$ and $M(\omega_0)$ is compact for some $\omega_0 \in W_0$. Then there is a neighborhood $W \subseteq W_0$ of ω_0 such that the marginal function is continuous on W .*

Proof. It is evident that there exists a neighborhood $W \subseteq W_0$ of ω_0 such that $M(\omega)$ is compact for $\omega \in W$. Direct verification shows that $m(\cdot)$ is continuous at every point $\omega \in W$ (see also [2], [4]). □

Results similar to Proposition 3.1 can be found in several papers on marginal functions. In the rest of this note we shall prove a stronger result; namely, we shall show that it is possible to find a solution $\bar{x}(\omega)$ of the problem

$$\min\{ \langle c(\omega), x \rangle \mid x \in M(\omega) \}$$

which depends smoothly on ω in W , provided that $c(\cdot), a_i(\cdot), \alpha_i(\cdot)$ are smooth. From now on, for the sake of simple presentation, we presume that $M(\omega)$ is compact for every ω . A more general case will be treated in a forthcoming paper. Recall that the solution set is defined as

$$S(\omega) := \{x \in M(\omega) \mid \langle c(\omega), x \rangle = m(\omega)\}.$$

Let I be any subset of $\{1, \dots, p + q\}$. As before,

$$Q_I(\omega) := \{x \in R^s \mid \langle a_i(\omega), x \rangle + \alpha_i(\omega) = 0, i \in I\}.$$

Lemma 3.1. *Assume that the map $Q_I \cap M$ and the function $m(\cdot)$ are continuous on an open set $W_0 \subseteq R^r$ and, for some $\omega_0 \in W_0$,*

(1)
$$Q_I(\omega_0) \cap S(\omega_0) = \emptyset.$$

Then, there exists a neighborhood $W \subseteq W_0$ of ω_0 such that

(2)
$$Q_I(\omega) \cap S(\omega) = \emptyset, \quad \text{for all } \omega \in W.$$

Proof. Condition (1) implies that

$$\langle c(\omega_0), x \rangle - m(\omega_0) > 0, \quad \text{for all } x \in Q_I(\omega_0) \cap M(\omega_0).$$

Hence, there is $\varepsilon > 0$ such that

$$(3) \quad \langle c(\omega), x' \rangle - m(\omega) > 0$$

provided

$$(4) \quad \|c(\omega) - c(\omega_0)\| < \varepsilon, \quad |m(\omega) - m(\omega_0)| < \varepsilon,$$

and

$$(5) \quad \|x' - x\| < \varepsilon,$$

for some $x \in Q_I(\omega_0) \cap M(\omega_0)$. The continuity of $c(\cdot)$ and $m(\cdot)$ implies the existence of a neighborhood $W_1 \subseteq W_0$ of ω_0 such that (4) holds whenever $\omega \in W_1$. The continuity of $Q_I \cap M$ shows that for some neighborhood $W \subseteq W_1$ of ω_0 ,

$$h(Q_I(\omega) \cap M(\omega), Q_I(\omega_0) \cap M(\omega_0)) < \varepsilon, \quad \text{for all } \omega \in W.$$

This means that, for every $x' \in Q_I(\omega) \cap M(\omega)$, one can find $x \in Q_I(\omega_0) \cap M(\omega_0)$ such that (5) holds. Consequently, (3) is satisfied whenever $\omega \in W$, and in fact (2) is true. \square

For every $\omega \in R^r$, denote by $V(\omega) = \{v_i(\omega), i \in \{1, \dots, k(\omega)\}\}$ the set of all the vertices of $M(\omega)$ and by $J(v_i(\omega))$ the index set of all equalities determining the vertex $v_i(\omega)$, i.e.

$$\begin{aligned} \langle a_j(\omega), v_i(\omega) \rangle + \alpha_j(\omega) &= 0, \quad j \in J(v_i(\omega)), \\ \langle a_{j'}(\omega), v_i(\omega) \rangle + \alpha_{j'}(\omega) &< 0, \quad j' \in J^c(v_i(\omega)) := \{1, \dots, p+q\} \setminus J(v_i(\omega)). \end{aligned}$$

Lemma 3.2. *For every open set $W_0 \subseteq R^r$, there exists an open subset $W \subseteq W_0$ such that the cardinality of $V(\omega)$ (denoted by $k(\omega)$) is constant on W , say k . Moreover, the vertices of $M(\omega)$ can be indexed so that $J(v_i(\omega))$ remains unchanged with respect to ω on W , for $i = 1, \dots, k$.*

Proof. In view of Proposition 2.1, without loss of generality we may assume that M is continuous on W_0 . Since $p+q$ is fixed, the number of vertices of $M(\omega)$ is majorized by a fixed integer. Hence, there is a point $\omega_0 \in W_0$ such that

$$k(\omega_0) = \max\{k(\omega) | \omega \in W_0\}.$$

We shall write $k := k(\omega_0)$. Our aim now is to show that for every vertex $v_i(\omega_0)$ of $M(\omega_0)$ and for every $\delta > 0$, there exist a neighborhood $W_i \subseteq W_0$ of ω_0 and a positive $\varepsilon < \delta$ such that for every $\omega \in W_i$ one can find a vertex of $M(\omega)$, say $v_i(\omega)$, such that $\|v_i(\omega) - v_i(\omega_0)\| < \varepsilon$. In fact, if not, one can find a sequence $\{\omega_n\} \subseteq W_0$ converging to ω_0 and a positive ε_0 such that

$$(6) \quad \|v_j(\omega_n) - v_i(\omega_0)\| \geq \varepsilon_0, \quad \text{for all } j \in \{1, \dots, k(\omega_n)\}.$$

Since $M(\cdot)$ is continuous, there exists $x_n \in M(\omega_n)$ with $\lim_{n \rightarrow \infty} x_n = v_i(\omega_0)$. By (6), x_n can be expressed as

$$(7) \quad x_n = \sum_{j=1}^{k(\omega_n)} \lambda_j^n v_j(\omega_n)$$

for some $\lambda_j^n \geq 0$ such that $\sum_j \lambda_j^n = 1$. Since $k(\omega)$ is majorized, by taking a subsequence if necessary, we may assume that $\{v_j(\omega_N)\}, \{\lambda_j^n\}$ converge respectively

to v_j^0, λ_j^0 . Again, the continuity of $M(\cdot)$ implies that $v_j^0 \in M(\omega_0)$. In view of (6) and (7),

$$v_i(\omega_0) = \sum_j \lambda_j^0 v_j^0$$

with

$$\|v_j^0 - v_i(\omega_0)\| \geq \varepsilon_0, \quad \text{for all } j.$$

It follows that $v_i(\omega_0)$ belongs to the convex hull of the points v_j^0 's and $v_i(\omega_0) \neq v_j^0$, for all j . Therefore, $v_i(\omega_0)$ is not a vertex of this convex hull, which is contained in $M(\omega_0)$. This contradicts the fact that $v_i(\omega_0)$ is a vertex of $M(\omega_0)$. Now let us take $\delta > 0$ so small that

$$\|v_i(\omega_0) - v_j(\omega_0)\| \geq 4\delta, \quad \text{for all } i, j \in \{1, \dots, k\}, j \neq i.$$

For this δ , as before, we obtain W_1, \dots, W_k for the vertices $v_1(\omega_0), \dots, v_k(\omega_0)$. We claim that $k(\omega) = k$ for every $\omega \in W' := W_1 \cap \dots \cap W_k$. In fact, for two arbitrary distinct vertices $v_i(\omega_0), v_j(\omega_0)$ of $M(\omega_0)$ and for every $\omega \in W'$ one can find two vertices, denoted with the same indices $v_i(\omega), v_j(\omega)$, of $M(\omega)$ such that

$$(8) \quad \|v_i(\omega) - v_i(\omega_0)\| < \delta, \quad \|v_j(\omega) - v_j(\omega_0)\| < \delta.$$

One can see that $v_i(\omega) \neq v_j(\omega)$ because

$$\begin{aligned} \|v_i(\omega) - v_j(\omega)\| &\geq \|v_i(\omega_0) - v_j(\omega_0)\| - \|v_i(\omega) - v_i(\omega_0)\| - \|v_j(\omega) - v_j(\omega_0)\| \\ &\geq 4\delta - 2\delta > \delta. \end{aligned}$$

Consequently, $M(\omega)$ has at least k vertices, i.e. $k(\omega) \geq k$. Actually we have equality because k is maximal. In sum, we have shown that for $\omega \in W'$, $M(\omega)$ has exactly k vertices $\{v_1(\omega), \dots, v_k(\omega)\}$ which satisfy (8). For every fixed index $i \in \{1, \dots, k\}$, the number $|J(v_i(\omega))|$ is majorized by $p + q$. One can find a point $\omega_* \in W'$ such that

$$|J(v_i(\omega_*))| = \min\{|J(v_i(\omega))| \mid \omega \in W'\}.$$

We want to find a neighborhood $W'_i \subseteq W'$ of ω_* such that $J(v_i(\omega)) = J(v_i(\omega_*))$, for all $\omega \in W'_i$. In fact, if $j \notin J(v_i(\omega_*))$, then

$$\langle a_j(\omega_*), v_i(\omega_*) \rangle + \alpha_j(\omega_*) < 0.$$

Since a_i, α_i are continuous, by (8) with δ sufficiently small, we can find a neighborhood $W''_j \subseteq W'$ of ω_* such that

$$\langle a_j(\omega), v_i(\omega) \rangle + \alpha_j(\omega) < 0, \quad \text{for all } \omega \in W''_j.$$

Take $W'_i = \bigcap \{W''_j \mid j \in \{1, \dots, p + q\} \setminus J(v_i(\omega_*))\}$ to see that

$$J(v_i(\omega)) \subseteq J(v_i(\omega_*)), \quad \text{for all } \omega \in W'_i.$$

Since $|J(v_i(\omega_*))|$ is minimal, the inclusion becomes equality. By applying the above argument for $i = 1, 2, \dots, k$, we arrive at an open subset $W = W'_k \subseteq W'_{k-1} \subseteq \dots \subseteq W'_1 \subseteq W'$ which satisfies the requirements of the lemma. \square

Let us denote

$$J_0(\omega) := \bigcup \{J(v_i(\omega)) \mid v_i(\omega) \in S(\omega)\}.$$

Lemma 3.3. *For every open set $W_0 \subseteq R^r$, there exists an open subset $W \subseteq W_0$ such that $J_0(\omega)$ remains unchanged for every $\omega \in W$.*

Proof. Without loss of generality we may assume that M as well as $Q_I \cap M$ are continuous on W_0 (where I is any subset of $\{1, \dots, p + q\}$). Let $\omega_0 \in W_0$ be a point such that $|J_0(\omega_0)| = \min\{|J_0(\omega)| \mid \omega \in W_0\}$. Applying Lemma 3.1 for $I = \bigcup\{J(v_j(\omega_0)) \mid v_j(\omega_0) \notin S(\omega_0)\}$, one can find a neighborhood $W \subseteq W_0$ of ω_0 such that

$$J_0(\omega) \subseteq J_0(\omega_0), \quad \text{for all } \omega \in W.$$

The minimality of $|J_0(\omega_0)|$ implies that $J_0(\omega) = J_0(\omega_0), \omega \in W$, and the proof is complete. \square

Proposition 3.2. *Assume that $a_i(\cdot), \alpha_i(\cdot), c(\cdot)$ are differentiable. Then, for any open set $W_0 \subseteq R^r$, there exists an open subset $W \subseteq W_0$ such that $S(\omega)$ admits a differentiable selection on W . Consequently, the set of nondifferentiability of $m(\cdot)$ is nowhere dense.*

Proof. By Proposition 2.1, there exists an open set $W_1 \subseteq W_0$ such that the maps M and $Q_I \cap M$, where I is any subset of $\{1, \dots, p + q\}$, are continuous on W_1 . In view of Lemma 3.3, $J_0(\omega)$ does not change on some open subset $W_2 \subseteq W_1$. Let $W_3 \subseteq W_2$ be an open set as stated in Lemma 3.2. We fix any point $\omega_0 \in W_3$ and consider vertex $v_i(\omega_0)$ of $S(\omega_0)$. By Lemmas 3.2–3.3, for any $\omega \in W_3$, the solution of the system

$$\langle a_j(\omega), x \rangle + \alpha_j(\omega) = 0, \quad j \in J(v_i(\omega_0)),$$

exists uniquely and is a vertex of $S(\omega)$. Let $A(\omega_0)$ be any nonsingular $(r \times r)$ -matrix, whose columns are taken from $\{a_j(\omega_0), j \in J(v_i(\omega_0))\}$. Then, $v_i(\omega_0) = [A(\omega_0)]^{-1}(-\alpha(\omega_0))$, where $\alpha(\omega_0)$ is the vector of the values $\alpha_j(\omega_0)$ corresponding to the columns of $A(\omega_0)$. One can find a neighborhood $W \subseteq W_3$ of ω_0 such that $A(\omega)$ is nonsingular whenever $\omega \in W$. It is evident that $v_i(\omega) = [A(\omega)]^{-1}(-\alpha(\omega))$ and it is a differentiable selection of $S(\omega)$ on W . The marginal function $m(\omega) = \langle c(\omega), v_i(\omega) \rangle$, which is differentiable on W as well. The proof is complete. \square

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