SHIFT-INvariant SPACES ON THE REAL LINE

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Abstract. We investigate the structure of shift-invariant spaces generated by a finite number of compactly supported functions in $L^p(\mathbb{R})$ ($1 \leq p \leq \infty$). Based on a study of linear independence of the shifts of the generators, we characterize such shift-invariant spaces in terms of the semi-convolutions of the generators with sequences on $\mathbb{Z}$. Moreover, we show that such a shift-invariant space provides $L^p$-approximation order $k$ if and only if it contains all polynomials of degree less than $k$.

1. Introduction

The purpose of this paper is to investigate the structure of shift-invariant spaces on the real line. In particular, we are interested in those properties of shift-invariant spaces on the real line which are not shared by shift-invariant spaces on higher dimensional spaces $\mathbb{R}^s$, $s > 1$.

Finitely generated shift-invariant subspaces of $L_2(\mathbb{R}^s)$ were studied in [4] by de Boor, DeVore, and Ron, who gave a simple characterization for such spaces in terms of the Fourier transforms of their generators. However, when $p \neq 2$, few results have been known for shift-invariant subspaces of $L^p(\mathbb{R}^s)$.

In this paper, we are mainly concerned with shift-invariant spaces generated by a finite number of compactly supported functions in $L^p(\mathbb{R})$ ($1 \leq p \leq \infty$). We will give a characterization for such spaces in terms of the semi-convolutions of their generators with sequences on $\mathbb{Z}$. The result is then applied to give a characterization of the approximation order provided by such shift-invariant spaces.

Let $\mathcal{S}$ be a linear space of distributions on $\mathbb{R}$. We say that $\mathcal{S}$ is shift-invariant if $f \in \mathcal{S} \Rightarrow f(\cdot - j) \in \mathcal{S}$ for all $j \in \mathbb{Z}$.

A mapping from $\mathbb{Z}$ to $\mathbb{C}$ is called a sequence. The linear space of all sequences on $\mathbb{Z}$ is denoted by $\ell(\mathbb{Z})$. Let $\phi$ be a compactly supported distribution on $\mathbb{R}$, and let $a : \mathbb{Z} \to \mathbb{C}$ be a sequence. The semi-convolution of $\phi$ with $a$, denoted $\phi *' a$, is defined by

$$\phi *' a := \sum_{j \in \mathbb{Z}} \phi(\cdot - j)a(j).$$

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Given a finite collection $\Phi$ of compactly supported distributions on $\mathbb{R}$, we denote by $S_0(\Phi)$ the linear span of $\{\phi(\cdot - j) : \phi \in \Phi, j \in \mathbb{Z}\}$, and by $S(\Phi)$ the linear space of all distributions of the form $\sum_{\phi \in \Phi} \phi \ast' a_\phi$ with $a_\phi$ being a sequence on $\mathbb{Z}$ for each $\phi \in \Phi$. The elements in $\Phi$ are called the generators for $S(\Phi)$.

Now suppose that $\Phi$ is a finite subset of $L_p(\mathbb{R})$ for some $p$ with $1 \leq p \leq \infty$. We denote by $S_p(\Phi)$ the closure of $S_0(\Phi)$ in $L_p(\mathbb{R})$. One of the main results of this paper is a characterization of $S_p(\Phi)$ in terms of semi-convolution. In Section 3, we shall prove that for $1 < p < \infty$, a function $f \in L_p(\mathbb{R})$ lies in $S_p(\Phi)$ if and only if

$$f = \sum_{\phi \in \Phi} \phi \ast' a_\phi$$

for some sequences $a_\phi \in \ell(\mathbb{Z})$. When $p = \infty$, a modified result will also be established.

We observe that this result is not valid for the case $p = 1$. To see this, let $\chi$ be the characteristic function of the interval $[0,1)$, and let $\phi := \chi - \chi(\cdot - 1)$. Then for any $f \in S_1(\phi)$ we have $\int f = 0$; hence $\chi \notin S_1(\phi)$. But $\chi = \sum_{j=0}^{\infty} \phi(\cdot - j)$.

Next, we consider approximation in $L_p(\mathbb{R})$ spaces ($1 \leq p \leq \infty$). For $f, g \in L_p(\mathbb{R})$, we write $\text{dist}_p(f, g)$ for $\|f - g\|_p$. Moreover, for a subset $G$ of $L_p(\mathbb{R})$, the distance from $f$ to $G$, denoted $\text{dist}_p(f, G)$, is defined by

$$\text{dist}_p(f, G) := \inf_{g \in G} \|f - g\|_p.$$

Let $\Phi$ be a finite collection of compactly supported functions in $L_p(\mathbb{R})$. The preceding result tells us that $S_p(\Phi) = S(\Phi) \cap L_p(\mathbb{R})$ for $1 < p < \infty$. Suppose $1 \leq p \leq \infty$. Let $S := S(\Phi) \cap L_p(\mathbb{R})$, and let $S^h := \{g(\cdot/h) : g \in S\}$ for $h > 0$. Given a real number $r \geq 0$, we say that $S(\Phi)$ provides $L_p$-approximation order $r$ if, for each sufficiently smooth function $f \in L_p(\mathbb{R})$,

$$\text{dist}_p(f, S^h) \leq Ch^r,$$

where $C$ is a positive constant independent of $h$ ($C$ may depend on $f$). We say that $S(\Phi)$ provides $L_p$-density order $r$ (see [3]) if, for each sufficiently smooth function $f \in L_p(\mathbb{R})$,

$$\lim_{h \to 0^+} \text{dist}_p(f, S^h)/h^r = 0.$$

In [7] Jia characterized the $L_\infty$-approximation order of $S(\Phi)$ in terms of the Strang-Fix conditions (see [16]). When $\Phi$ consists of a single generator $\phi$, Ron [13] proved that, for a positive integer $k$, $S(\phi)$ provides $L_\infty$-approximation order $k$ if and only if $S(\phi)$ contains $\Pi_{k-1}$, the set of all polynomials of degree $\leq k - 1$. Zhao [18] also gave a characterization for the $L_p$-approximation order ($1 < p < \infty$) provided by $S(\phi)$.

In Section 4, we shall prove that $S(\Phi)$ provides $L_p$-approximation order ($1 \leq p \leq \infty$) if and only if $S(\Phi)$ contains $\Pi_{k-1}$. This result is no longer true for shift-invariance spaces on $\mathbb{R}^s$, $s > 1$. See the counterexamples given in [5] and [6].

In our study of shift-invariant spaces linear independence plays a crucial role. Let $\Phi$ be a finite collection of compactly supported distributions on $\mathbb{R}$. The shifts of the elements in $\Phi$ are said to be linearly independent if

$$\sum_{\phi \in \Phi} \phi \ast' a_\phi = 0 \Rightarrow a_\phi = 0 \quad \forall \phi \in \Phi.$$
When the shifts of the elements in \( \Phi \) are linearly independent, we say that \( S(\Phi) \) has linearly independent generators.

In Section 2 we shall show that a finitely generated shift-invariant space always has linearly independent generators. More precisely, if \( \Phi \) is a finite collection of compactly supported distributions on \( \mathbb{R} \), then there exists a finite collection \( \Psi \) of compactly supported distributions on \( \mathbb{R} \) such that \( S(\Psi) = S(\Phi) \) and the shifts of the elements in \( \Psi \) are linearly independent. When \( \Phi \) consists of compactly supported continuous functions, this result was essentially known to de Boor and DeVore (see [2]). When \( \Phi \) consists of a single generator \( \phi \), Ron [12] showed that \( S(\phi) \) contains a linearly independent generator. Our contribution is to give a concrete construction for \( \Psi \) so that \( \Psi \) inherits most properties possessed by \( \Phi \). For instance, if \( \Phi \subset L^p(\mathbb{R}) \) for some \( p \) with \( 1 \leq p \leq \infty \), then \( \Psi \) can be chosen to be a subset of \( L^p(\mathbb{R}) \). Furthermore, for \( 1 < p \leq \infty \), \( \Psi \) can be chosen to be a subset of \( S_p(\Phi) \). These properties enable us to characterize shift-invariant subspaces of \( L^p(\mathbb{R}) \) and the approximation order provided by them.

2. LINEAR INDEPENDENCE

This section is devoted to a study of linear independence. Linear independence can be characterized in terms of the Fourier transforms of the generators. For a compactly supported integrable function \( f \) on \( \mathbb{R} \), the Fourier-Laplace transform of \( f \) is given by

\[
\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx, \quad \xi \in \mathbb{C}.
\]

The domain of the Fourier-Laplace transform can be extended to all compactly supported distributions. If \( f \) is a compactly supported distribution, then \( \hat{f} : \xi \mapsto \hat{f}(\xi) \) is an entire function on \( \mathbb{C} \). It is known (see [10] and the references cited there) that the shifts of the elements in \( \Phi \) are linearly independent if and only if for every \( \zeta \in \mathbb{C} \), the sequences \( (\hat{\phi}(\zeta + 2\pi k))_{k \in \mathbb{Z}} \), \( \phi \in \Phi \), are linearly independent.

For later use we introduce some concepts related to compactly supported distributions. Let \( \phi \) be a compactly supported distribution on \( \mathbb{R} \). Suppose \( \phi \neq 0 \). The support of \( \phi \), denoted \( \text{supp} \phi \), is a compact subset of \( \mathbb{R} \). Let \( [r_\phi, s_\phi] \) be the smallest integer-bounded interval containing \( \text{supp} \phi \). The length of the interval \( [r_\phi, s_\phi] \) is

\[
l(\phi) := s_\phi - r_\phi.
\]

We call \( l(\phi) \) the length of \( \phi \).

Let \( \Phi \) be a finite collection of compactly supported distributions on \( \mathbb{R} \). The length of \( \Phi \), denoted \( l(\Phi) \), is defined by

\[
l(\Phi) := \sum_{\phi \in \Phi} l(\phi).
\]

Also, we denote by \( \#\Phi \) the number of elements in \( \Phi \).

**Theorem 1.** Let \( \Phi \) be a finite collection of nontrivial distributions on \( \mathbb{R} \) with compact support. Then there exists a finite collection \( \Psi \) of compactly supported distributions on \( \mathbb{R} \) with the following properties:

(a) The shifts of the elements in \( \Psi \) are linearly independent;
(b) \( \#\Psi \leq \#\Phi \);
(c) \( \Phi \subset S_0(\Psi) \);
(d) \( S(\Psi) = S(\Phi) \).
If, in addition, $\Phi \subset L_p(\mathbb{R})$ for some $p$, $1 \leq p \leq \infty$, then $\Psi$ can be chosen to be a subset of $L_p(\mathbb{R})$. Furthermore, for $1 < p \leq \infty$, $\Psi$ can be chosen to be a subset of $S_p(\Phi)$.

Proof. It is sufficient to prove that, if the shifts of the elements in $\Phi$ are linearly dependent, then there exists $\Psi$ with $l(\Psi) \leq l(\Phi) - 1$ satisfying all the conclusions of the theorem, except perhaps (a). Suppose $\Phi = \{\phi_1, \ldots, \phi_m\}$. Let

$$K(\Phi) := \left\{ (b_1, \ldots, b_m) \in (\ell(\mathbb{Z}))^m : \sum_{j=1}^m \phi_j \ast' b_j = 0 \right\}.$$ 

Then the shifts of the elements in $\Phi$ are linearly independent if and only if $K(\Phi) = \{0\}$. If $K(\Phi) = \{0\}$, then we may take $\Psi = \Phi$. Suppose $K(\Phi) \neq \{0\}$. By [10, Theorem 3.3], $K(\Phi) \neq \{0\}$ implies that there exists some $\theta \in \mathbb{C}\setminus\{0\}$ and $(a_1, \ldots, a_m) \in \mathbb{C}^m \setminus \{0\}$ such that

$$\sum_{j=1}^m \sum_{k=-\infty}^\infty a_j \theta^k \phi_j(\cdot - k) = 0. \quad (2.1)$$

where $\theta^k$ denotes the sequence $k \mapsto \theta^k$, $k \in \mathbb{Z}$. It follows from (2.1) that

$$\sum_{j=1}^m \sum_{k=-\infty}^\infty a_j \theta^k \phi_j(\cdot - k) = 0. \quad (2.2)$$

For each $\phi_j$, let $r_j := r_{\phi_j}$ and $s_j := s_{\phi_j}$. After shifting the $\phi_j$ appropriately, we may assume that all $r_j = 0$. Then $s_j = l(\phi_j)$, the length of $\phi_j$. Let $l := \max \{l(\phi_j) : a_j \neq 0\}$. For simplicity, we assume that $a_1 \neq 0$ and $l(\phi_1) = l$. Let

$$\rho := \sum_{j=1}^m a_j \phi_j$$

and

$$\psi := \sum_{k=0}^\infty \theta^k \rho(\cdot - k). \quad (2.3)$$

By our choice of $\rho$, we deduce from (2.2) that

$$\sum_{k=-\infty}^\infty \theta^k \rho(\cdot - k) = 0. \quad (2.4)$$

Let $\Psi := \{\psi, \phi_2, \ldots, \phi_m\}$. We have

$$\psi - \theta \psi(\cdot - 1) = \sum_{k=0}^\infty \theta^k \rho(\cdot - k) - \sum_{k=0}^\infty \theta^{k+1} \rho(\cdot - k - 1) = \rho = a_1 \phi_1 + \cdots + a_m \phi_m.$$

Since $a_1 \neq 0$, we obtain $\phi_1 \in S_0(\psi, \phi_2, \ldots, \phi_m)$, and hence $\Phi \subset S_0(\psi)$. It follows that $S(\Phi) \subseteq S(\Psi)$.

Evidently, $\psi \in S(\Phi)$. If $f = \psi \ast' b$ for some sequence $b$ on $\mathbb{Z}$, then for any bounded open interval $E$ of $\mathbb{R}$, there exists an element $g \in S(\Phi)$ such that $g$ agrees with $f$ on $E$. Thus, by [8, Theorem 4], $f$ belongs to $S(\Phi)$. This shows $S(\Psi) \subseteq S(\Phi)$. Therefore $S(\Psi) = S(\Phi)$.

Let us show $l(\Psi) < l(\Phi)$. For this purpose we only have to prove $\text{supp } \psi \subseteq [0, l - 1]$. Clearly, $\text{supp } \psi \subseteq [0, \infty)$. Hence, it suffices to show that $\langle \psi, u \rangle = 0$ for
every \( u \in C_c^\infty(\mathbb{R}) \) with \( \text{supp } u \subset (l-1, \infty) \). Let \( u \) be such a test function. Note that for each \( j \), \( \phi_j(\cdot - k) \) is supported on \([k, l+k]\). Hence \( \langle \phi_j(\cdot - k), u \rangle = 0 \) for \( k \leq -1 \). It follows that \( \langle \rho(\cdot - k), u \rangle = 0 \) for \( k \leq -1 \). This in connection with (2.4) gives

\[
\langle \psi, u \rangle = \left( \sum_{k=0}^{\infty} \theta^k \rho(\cdot - k), u \right) = \left( \sum_{k=-\infty}^{\infty} \theta^k \rho(\cdot - k), u \right) = 0.
\]

Consequently, \( \text{supp } \psi \subseteq [0, l-1] \).

Now suppose \( \Phi \in L_p(\mathbb{R}) \) for some \( p, 1 \leq p \leq \infty \). Then \( \rho \in L_p(\mathbb{R}) \), and (2.3) tells us that for each integer \( k \), \( \psi \) is \( p \)th power integrable on the interval \([k, k+1]\). But \( \psi \) is compactly supported; hence \( \psi \in L_p(\mathbb{R}) \).

It remains to prove that \( \psi \in S_p(\Phi) \) if \( \Phi \in L_p(\mathbb{R}) \) for \( 1 < p \leq \infty \). If \( |\theta| < 1 \), then (2.3) implies \( \psi \in S_p(\Phi) \). If \( |\theta| > 1 \), then \( \psi - \theta \psi(\cdot - 1) = \rho \) implies

\[
\psi = \sum_{k=1}^{\infty} -\theta^{-k} \rho(\cdot + k) \in S_p(\Phi).
\]

When \( |\theta| = 1 \), we set

\[
f_n := \sum_{k=0}^{n-1} (1 - k/n) \theta^k \rho(\cdot - k),
\]

where \( n \) is an integer greater than \( l \). Then \( f_n \in S_0(\Phi) \). The desired result \( \psi \in S_p(\Phi) \) will be established if we can show

\[
\| f_n - \psi \|_p \to 0 \quad \text{as } n \to \infty.
\]

To prove (2.5) we observe that \( \rho \) is supported on \([0, l]\), \( \psi \) is supported on \([0, l-1]\), and \( f_n \) is supported on \([0, n+l-1]\). For \( x \in [0, l-1] \) we have

\[
\psi(x) - f_n(x) = \sum_{k=0}^{l-1} (k/n) \theta^k \rho(x - k).
\]

Hence

\[
\| \psi - f_n \|_{L_p([0, l-1])} \to 0 \quad \text{as } n \to \infty.
\]

For \( x \in [n-1, n+l-1] \), we have \( \psi(x) = 0 \) and

\[
\psi(x) - f_n(x) = \sum_{k=n-l}^{n-1} -(1 - k/n) \theta^k \rho(x - k).
\]

But \( |1 - k/n| \leq l/n \) for \( n - l \leq k \leq n - 1 \); hence

\[
\| \psi - f_n \|_{L_p([n-1, n+l-1])} \to 0 \quad \text{as } n \to \infty.
\]

It remains to prove

\[
\| \psi - f_n \|_{L_p([-l, -n-1])} \to 0 \quad \text{as } n \to \infty.
\]

For this purpose let \( j \) be an integer in \([l-1, n-2]\). We observe that for almost every \( x \in [j, j+1) \), \( \rho(x - k) = 0 \) for \( k \not\in (j-l, j+1) \), and hence by (2.4) we have

\[
\sum_{k=j-l+1}^{j} \theta^k \rho(x - k) = \sum_{k=-\infty}^{\infty} \theta^k \rho(x - k) = 0.
\]
Therefore, for almost every $x \in [j, j + 1]$, we have

$$
\psi(x) - f_n(x) = (1 - j/n) \sum_{k=j-l+1}^{j} \theta^k \rho(x - k) - \sum_{k=j-l+1}^{j} (1 - k/n) \theta^k \rho(x - k)
$$

$$
= \sum_{k=j-l+1}^{j} \frac{k-j}{n} \theta^k \rho(x - k).
$$

But $|k-j| \leq l$ for $j-l+1 \leq k \leq j$. Consequently, (2.8) holds true for $p = \infty$. If $1 < p < \infty$, then there exists a positive constant $C$ independent of $n$ such that

$$
\int_{[j,j+1]} |\psi(x) - f_n(x)|^p \, dx \leq C^p/n^p, \quad l - 1 \leq j \leq n - 2.
$$

It follows that

$$
\int_{[l-1,n-1]} |\psi(x) - f_n(x)|^p \, dx \leq nC^p/n^p = C^p/n^{p-1}.
$$

This verifies (2.8) for $1 < p < \infty$. Finally, (2.6), (2.7), and (2.8) together imply (2.5). We conclude that $\psi \in S_p(\Phi)$ for $1 < p \leq \infty$.

The results obtained so far can be summarized as follows: If the shifts of the elements in $\Phi$ are linearly dependent, then they are stable. Moreover, $\Psi$ meets the requirement of the theorem. Repeat the preceding process until $l(\Psi)$ achieves its minimum. The resulting set $\Psi$ has the property that the shifts of the elements in $\Psi$ are linearly independent. Moreover, $\Psi$ meets the requirement of the theorem.

### 3. Characterization of shift-invariant spaces

In this section we investigate the structure of shift-invariant spaces generated by a finite number of compactly supported functions in $L_p(\mathbb{R})$ ($1 \leq p \leq \infty$).

We use $\ell_0(\mathbb{Z})$ to denote the linear space of all finitely supported sequences on $\mathbb{Z}$. Then, for $1 \leq p < \infty$, $\ell_0(\mathbb{Z})$ is dense in $\ell_p(\mathbb{Z})$. For $p = \infty$, the closure of $\ell_0(\mathbb{Z})$ in $\ell_\infty(\mathbb{Z})$ is $c_0(\mathbb{Z})$, the linear space of all sequences $a$ on $\mathbb{Z}$ such that $\lim_{|k| \to \infty} a(k) = 0$.

For a measurable subset $E$ of $\mathbb{R}$ and a measurable function $f$ on $\mathbb{R}$, we denote by $\|f\|_\infty(E)$ the essential supremum of $f$ on $E$. Let $L_{\infty,0}(\mathbb{R})$ be the linear space of all functions $f \in L_\infty(\mathbb{R})$ for which $\lim_{r \to \infty} \|f\|_\infty(\mathbb{R}\setminus[-r,r]) = 0$.

Let $\Phi = \{\phi_1, \ldots, \phi_m\}$ be a finite collection of compactly supported functions in $L_p(\mathbb{R})$. We say that the shifts of the functions of $\Phi$ are stable, if there exist two positive constants $C_1$ and $C_2$ such that for any choice of sequences $a_1, \ldots, a_m \in \ell_p(\mathbb{Z})$,

$$
C_1 \sum_{j=1}^{m} \|a_j\|_{\ell_p(\mathbb{Z})} \leq \left\| \sum_{j=1}^{m} \phi_j \ast a_j \right\|_{L_p(\mathbb{R})} \leq C_2 \sum_{j=1}^{m} \|a_j\|_{\ell_p(\mathbb{Z})}.
$$

It was proved by Jia and Micchelli in [10] and [11] that the shifts of the functions in $\Phi$ are stable if and only if for every $\xi \in \mathbb{R}$, the sequences $(\hat{\phi}_j(\xi + 2\pi k))_{k \in \mathbb{Z}}$, $j = 1, \ldots, m$, are linearly independent. Thus, if the shifts of the functions in $\Phi$ are linearly independent, then they are stable.
Consider the linear mapping $T_\Phi$ from $(\ell_p(\mathbb{Z}))^m$ to $L_p(\mathbb{R})$ given by

$$T_\Phi(a_1, \ldots, a_m) = \sum_{j=1}^m \phi_j *^\ell a_j, \quad a_1, \ldots, a_m \in \ell_p(\mathbb{Z}).$$

If the shifts of the functions in $\Phi$ are stable, then $T_\Phi$ is a continuous mapping and the range of $T_\Phi$ is closed (see [14, p. 70]). Therefore, for $1 \leq p < \infty$, $S_p(\Phi)$ is the range of $T_\Phi$. In other words, for $1 \leq p < \infty$, $f$ lies in $S_p(\Phi)$ if and only if

$$f = \sum_{\phi \in \Phi} \phi *^\ell a_\phi$$

for some sequences $a_\phi \in \ell_p(\mathbb{Z})$. Suppose $\Psi = \{1\}$ and $\{17\}$. Let $\Phi = \{1\}$ and the shifts of the functions in $\Phi$ are linearly independent. Moreover, for $1 < p < \infty$,

$$T_s(\Phi) \cap L_p(\mathbb{R}) = S_p(\Phi).$$

In other words, for $1 < p < \infty$, a function $f$ lies in $S_p(\Phi)$ if and only if $f \in L_p(\mathbb{R})$ and

$$f = \sum_{\phi \in \Phi} \phi *^\ell a_\phi$$

for some sequences $a_\phi \in \ell(\mathbb{Z})$. In the case $p = \infty$, $f \in S_\infty(\Phi)$ if and only if $f \in L_\infty,0(\mathbb{R})$ and (3.2) holds true for some sequences $a_\phi \in \ell(\mathbb{Z})$.

**Proof.** By Theorem 1, there exists a finite collection $\Psi \subset L_p(\mathbb{R})$ such that $S(\Psi) = S(\Phi)$ and the shifts of the functions in $\Psi$ are linearly independent. Moreover, for $1 < p \leq \infty$, $\Psi$ can be so chosen that $S_p(\Psi) = S_p(\Phi)$.

We first show that $S(\Phi) \cap L_p(\mathbb{R})$ is closed in $L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$. This can be derived from [8, Theorem 4]. Here we establish this result by using the dual functionals discussed in [1] and [17]. Suppose $\Psi = \{\psi_1, \ldots, \psi_m\}$. Let $f \in S(\Psi) \cap L_p(\mathbb{R})$. Then

$$f = \sum_{j=1}^m \psi_j *^\ell a_j,$$

where $a_j \in \ell(\mathbb{Z}), j = 1, \ldots, m$. From [1] and [17] we see that there are functions $u_1, \ldots, u_m \in C^\infty_c(\mathbb{R})$ such that for $j, k = 1, \ldots, m$ and $\alpha \in \mathbb{Z}$,

$$\langle \psi_j, u_k(\cdot - \alpha) \rangle = \delta_{jk} \delta_{\alpha 0},$$

where $\delta_{jk}$ stands for the Kronecker sign: $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$.

It follows that

$$a_j(\alpha) = \langle f, u_j(\cdot - \alpha) \rangle, \quad \alpha \in \mathbb{Z}.$$  

Since $f \in L_p(\mathbb{R})$, we obtain $a_j \in \ell_p(\mathbb{Z})$ for $j = 1, \ldots, m$ (see [11, Theorem 3.1]). Thus, by the discussion at the beginning of this section, $S(\Phi) \cap L_p(\mathbb{R})$ is closed in $L_p(\mathbb{R})$. But $S(\Phi) = S(\Psi)$. Hence $S(\Phi) \cap L_p(\mathbb{R})$ is closed in $L_p(\mathbb{R})$.

Furthermore, for $1 < p < \infty$, $S(\Phi) \cap L_p(\mathbb{R}) = S_p(\Psi)$. But, for $1 < p \leq \infty$, we have $S_p(\Psi) = S_p(\Phi)$. Therefore, (3.1) is true for $1 < p < \infty$.

Finally, it is easily seen that $S_\infty(\Psi) \subset S(\Psi) \cap L_{\infty,0}(\mathbb{R})$. If $f \in S(\Psi) \cap L_{\infty,0}(\mathbb{R})$ has the expression as in (3.3), then it follows from (3.4) that $a_j \in c_0(\mathbb{Z})$ for $j = 1, \ldots, m$. Hence $f \in S_\infty(\Psi)$. This shows that $S_\infty(\Psi) = S(\Psi) \cap L_{\infty,0}(\mathbb{R})$. But $S(\Phi) = S(\Psi)$.
and $S_\infty(\Phi) = S_\infty(\Psi)$. We therefore conclude that $S_\infty(\Phi) = S(\Phi) \cap L_{\infty,0}(\mathbb{R})$. This verifies the last statement of the theorem. \hfill \qed

4. APPROXIMATION ORDER

We are now in a position to consider approximation in $L_p(\mathbb{R})$ spaces ($1 \leq p \leq \infty$).

**Theorem 3.** Let $\Phi$ be a finite collection of compactly supported functions in $L_p(\mathbb{R})$, $1 \leq p \leq \infty$. Let $k$ be a positive integer. Then the following statements are equivalent.

(a) $S(\Phi)$ provides $L_p$-approximation order $k$.
(b) $S(\Phi)$ provides $L_p$-density order $k - 1$.
(c) $S(\Phi)$ contains $\Pi_{k-1}$, the set of all polynomials of degree $\leq k - 1$.
(d) $S(\Phi)$ contains a compactly supported function $\psi$ such that

\[
\sum_{\beta \in \mathbb{Z}} q(\beta) \psi(\cdot - \beta) = q \quad \forall q \in \Pi_{k-1}.
\]

(4.1)

**Proof.** It is obvious that (a) implies (b). It was proved in [8] that (b) implies (c). The implication (d) $\Rightarrow$ (a) is well known. See [9] for an explicit $L_p$-approximation scheme. It remains to prove (c) $\Rightarrow$ (d). By Theorem 1, we may assume that the shifts of the functions in $\Phi$ are linearly independent. Suppose $\Phi = \{\phi_1, \ldots, \phi_m\}$.

Since the shifts of the functions in $\Phi$ are linearly independent, there exist test functions $u_1, \ldots, u_m \in C^\infty_c(\mathbb{R})$ such that

\[
\langle \phi_r(\cdot - \alpha), u_s(\cdot - \beta) \rangle = \delta_{rs} \delta_{\alpha \beta}, \quad r, s \in \{1, \ldots, m\}, \alpha, \beta \in \mathbb{Z}.
\]

(4.2)

By condition (c), $q \in S(\Phi)$ for $q \in \Pi_{k-1}$. Hence by (4.2) we have

\[
q = \sum_{j=1}^m \sum_{\alpha \in \mathbb{Z}} \phi_j(\cdot - \alpha) \langle \ell_r, u_j \rangle.
\]

Let $(\ell_r : r = 1, \ldots, k)$ be the Lagrange polynomials of degree $k - 1$ for the points $1, \ldots, k$. Then, for any $q \in \Pi_{k-1},$

\[
q = \sum_{\alpha \in \mathbb{Z}} \sum_{j=1}^m \phi_j(\cdot - \alpha) \left( \sum_{r=1}^k q(r + \alpha) \ell_r, u_j \right) = \sum_{\beta \in \mathbb{Z}} \psi(\cdot - \beta) q(\beta),
\]

with

\[
\psi := \sum_{j=1}^m \sum_{r=1}^k \phi_j(r + \cdot) \langle \ell_r, u_j \rangle
\]

certainly a compactly supported element of $S(\Phi)$. Therefore, (c) implies (d). \hfill \qed

It was proved by Schoenberg [15] that (4.1) is equivalent to the following conditions: $D^\alpha \psi(0) = \delta_{\alpha 0}$ and $D^\alpha \psi(2\pi j) = 0$ for $0 \leq \alpha < k$ and $j \in \mathbb{Z}\setminus\{0\}$. Now these conditions are referred to as the Strang-Fix conditions (see [16]).
REFERENCES


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