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# NONCOMMUTATIVE $H^2$ SPACES

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra with a faithful, finite, normal tracial state $\tau$, and let $\mathcal{A}$ be a finite, maximal subdiagonal algebra of $\mathcal{M}$. Let $H^2$ be the closure of $\mathcal{A}$ in the noncommutative Lebesgue space $L^2(\mathcal{M},\tau)$. Then $H^2$ possesses several of the properties of the classical Hardy space on the circle, including a commutant lifting theorem, some results on Toeplitz operators, an $H^1$ factorization theorem, Nehari's Theorem, and harmonic conjugates which are $L^2$ bounded.

## 1. Introduction

In [1] Arveson introduced the concept of a subdiagonal algebra in order to unify the analysis of several broad classes of nonselfadjoint operator algebras. Arveson drew a close analogy between a subdiagonal algebra and the classical Hardy space $H^\infty$, the boundary values of the bounded analytic functions on the disk. Roughly, a subdiagonal algebra $\mathcal{A}$ stands in relation to its von Neumann algebra $\mathcal{M}$ as $H^\infty$ stands in relation to the Lebesgue space $L^\infty$ of the unit circle. Subsequently, several authors studied the invariant subspaces of $\mathcal{A}$ acting on the noncommutative Lebesgue space $L^p$ [6], [8], [10]. There has also been considerable investigation of analytic crossed products, which are a type of subdiagonal algebra introduced by McAsey, Muhly and Saito, including their invariant subspace structure [8], [10], maximality among weak* closed subalgebras of $\mathcal{M}$ [8], associated Toeplitz operators [11] and Hankel operators [5]. We shall study the closure of $\mathcal{A}$ in $L^2$ as an analogue of the classical Hardy space $H^2$, and so obtain analogues of several classical results including a commutant lifting theorem, some results on Toeplitz operators, an $H^1$ factorization theorem, Nehari's Theorem on the norm of a Hankel operator, and the existence and $L^2$ boundedness of the harmonic conjugate.

Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal finite tracial state $\tau$. For $1 \le p < \infty$, let $L^p = L^p(\mathcal{M},\tau)$ denote the noncommutative Lebesgue space which is associated with $\mathcal{M}$ and $\tau$ (cf. [3], [9], [12]). For $t \in \mathcal{M}, x \in L^2$, let $L_t(x) = tx$ and $R_t(x) = xt$. Then $\mathcal{L} = \{L_t\colon\ t \in \mathcal{M}\}$ and $\mathcal{R} = \{R_t\colon\ t \in \mathcal{M}\}$ are von Neumann algebras on $L^2$ which are each other's commutants. Furthermore, the map $t \to L_t$ (resp. $t \to R_t$) is a normal, $^*$-isomorphism (resp. $^*$-anti-isomorphism) of $\mathcal{M}$ onto $\mathcal{L}$ (resp. $\mathcal{R}$), and the identity 1 is a cyclic and separating vector for $\mathcal{L}$ and $\mathcal{R}$. The map $x \to x^*$ on $\mathcal{M}$ extends to a conjugate linear isometry on $L^p$. As is customary, we identify $\mathcal{M}$ with $L^\infty$ while the ultraweak topology on $\mathcal{M}$ will

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be identified with the weak* topology on $L^\infty$ regarded as the dual of $L^1$. We now introduce $\mathcal{A}$, the noncommutative analogue of $H^\infty$ (cf. [1], [8]).

**Definition.** Let $\mathcal{A}$ be a w*-closed unital subalgebra of $\mathcal{M}$, and let $\Phi$ be a faithful, normal expectation from $\mathcal{M}$ onto the diagonal $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Then $\mathcal{A}$ is a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ with respect to $\Phi$ if:

(1) $\mathcal{A} + \mathcal{A}^*$ is w*-dense in $\mathcal{M}$,
(2) $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{A}$,
(3) $\mathcal{A}$ is maximal among those subalgebras satisfying (1) and (2), and
(4) $\tau \circ \Phi = \tau$.

For $\mathcal{S} \subset L^p$, $1 \le p < \infty$, let $[\mathcal{S}]_p$ denote the closure of $\mathcal{S}$ in $L^p$. Let $H^p = [\mathcal{A}]_p$, $H^p_0 = [\{x \in \mathcal{A} \colon \Phi(x) = 0\}]_p$, and $P$ be the orthogonal projection of $L^2$ onto $H^2$. Then $\Phi$ extends to the orthogonal projection of $L^2$ onto $[\mathcal{D}]_2$ and $L^2 = H^2 \oplus (H^2_0)^* = H^2_0 \oplus [\mathcal{D}]_2 \oplus (H^2_0)^*$ by [8, Proposition 1.1]. For $t \in \mathcal{M}$ we define the (left) Toeplitz operator with symbol $t$ by $T_t = PL_tP$. We define the (left) Hankel operator with symbol $t$ by $H_t = (1-P)L_tP$.

## 2. A commutant lifting theorem

Let $\mathcal{T}$ denote the algebra $\{L_a \colon a \in \mathcal{A}\}$ considered as an algebra of operators on $H^2$. In this section we identify $\mathcal{T}'$, the operators on $H^2$ that commute with $\mathcal{T}$. In particular, we show that every element of $\mathcal{T}'$ lifts to an element of $\mathcal{R}$. Our result is analogous to Yoshino's Theorem [13], which describes the commutant of a rationally cyclic subnormal operator. Indeed, if $\mu$ is a measure on the complex plane with compact support $K$, $R$ is the algebra of rational functions with poles off of $K, R^2$ is the $L^2(\mu)$ closure of $R$, and $S$ is multiplication by $z$ restricted to $R^2$, then the commutant of $S$ is $\{M_\phi \colon \phi \in R^2 \cap L^\infty(\mu)\}$, where $M_\phi f = \phi f$ is the usual multiplication operator. It follows that if $X$ is an operator on $R^2$ which commutes with multiplications by elements of $R$, then $X = M_\phi$ for some $\phi \in L^\infty(\mu)$ with $M_\phi(R^2) \subset R^2$. So the commutant of $R$ acting on its $L^2(\mu)$ closure lifts to the von Neumann algebra $L^\infty(\mu)$ acting on $L^2(\mu)$. For $\mathcal{T}$ acting on $H^2$ we have the following result.

**Theorem 1.** *If $X \in \mathcal{T}'$, then there exists $b \in \mathcal{M}$ such that $H^2$ is invariant for $R_b$, $X = R_b$ on $H^2$, and $\|X\| = \|b\|$.*

*Proof.* Let $X(1) = h$. First, we will show that $\|rh\|_2 \le \|X\|\,\|r\|_2$ for all $r \in \mathcal{M}$. Let $\varepsilon > 0$. Then $r^*r + \varepsilon 1$ is an invertible positive operator in $\mathcal{M}$. So by [1, Corollary 4.2.4] there is an $a \in \mathcal{A}$ such that $r^*r + \varepsilon 1 = a^*a$. Thus

$$\begin{aligned}\langle (r^*r + \varepsilon 1)h, h\rangle &= \langle a^*ah, h\rangle = \|ah\|_2^2 = \|Xa\|_2^2 \\ &\le \|X\|^2\|a\|_2^2 = \|X\|^2\langle a^*a, 1\rangle = \|X\|^2\langle r^*r + \varepsilon 1, 1\rangle.\end{aligned}$$

Letting $\varepsilon$ go to 0, we obtain $\|rh\|_2^2 \le \|X\|^2\|r\|_2^2$.

Define a map $Y$ via $Y(r) = rh$. Clearly $Y$ extends to a bounded operator on $L^2$ which commutes with $\mathcal{L}$, and $\|Y\| \le \|X\|$. Thus there is a $b \in \mathcal{M}$ such that $Y = R_b$. Because $h \in H^2$, $Y$ leaves $H^2$ invariant. Obviously $Y$ agrees with $X$ on $H^2$, and so $\|b\| = \|Y\| = \|X\|$.

## 3. Factorizations, Toeplitz and Hankel operators

Our next theorem is a useful factorization result for elements of $L^2$, which might be expressed as $L^2 = L^\infty H^2 = H^2 L^\infty$. The proof in the classical case is an application of the fact that if $\log|f|$ is integrable for an $L^2$ function $f$, then $|f| = |h|$ for some $H^2$ function $h$. However, this fact is a consequence of Szegő's Theorem, and it is not yet known if Szegő's Theorem extends to the subdiagonal algebra setting. Nevertheless [10, Proposition 1] is a sufficient substitute. As consequences we obtain some facts about Toeplitz operators, the factorization of elements of $H^1$ as products of $H^2$ elements (previously obtained by Saito), and Nehari's Theorem.

**Theorem 2.** *For every $\varepsilon > 0$ and $z \in L^2$, there exist $h_1, h_2 \in H^2$ and $v_1, v_2 \in \mathcal{M}$, such that $z = v_1h_1 = h_2v_2$, $\|v_i\| \le 1$, $\|h_i\|_2 < (1+\varepsilon)\|z\|_2$, and $h_i^{-1} \in \mathcal{A}$ for $i = 1, 2$.*

*Proof.* We prove the existence of $v_1$ and $h_1$. The proof for $v_2$ and $h_2$ is similar. Choose $\delta < \sqrt{2\varepsilon + \varepsilon^2}\|z\|_2$. Consider the positive weak$^*$ continuous linear functional $\omega_{\delta 1}+\omega_z$ on $\mathcal{R}$. By [7, Theorem 7.2.3] there is a vector $y \in L^2$ such that $\omega_y = \omega_{\delta 1}+\omega_z$ on $\mathcal{R}$. Clearly, $\omega_{\delta 1} \le \omega_y$ and $\omega_z \le \omega_y$ on $\mathcal{R}$. It follows that there exist $r, s \in \mathcal{M}$ such that $\delta 1 = ry$ and $z = sy$. Because $\mathcal{M}$ is finite, $\delta^{-1}r$ has an inverse in $L^2$.

By [10, Proposition 1] there exist a unitary $u \in \mathcal{M}$ and $g \in \mathcal{A}$ such that $\delta^{-1}r = gu$, and $h = g^{-1} \in H^2$. Thus $y = u^*h, z = su^*h$, and $\|h\|_2 = \|y\|_2 = \sqrt{\delta^2 + \|z\|_2^2} < (1+\varepsilon)\|z\|_2$. Now let $v_1 = su^*$ and $h_1 = h$.

In the classical setting, one has that the map $t \to T_t$ is an isometry on $L^\infty$, but the proof relies heavily on the fact that $L^\infty$ is commutative. We do not know if this map is isometric in our noncommutative setting. We do have the following partial results, including the fact that the map is very well behaved on $\mathcal{A}$.

**Corollary 3.** *For every $t \in \mathcal{M}$, $\|L_tP\| = \|t\|$.*

*Proof.* We have $\|L_t\| = \|t\|$. So given $\varepsilon > 0$, choose $z$ such that $\|z\|_2 < 1$ and $\|tz\|_2 \ge \|t\| - \varepsilon$. By Theorem 2 we can write $z = hv$ where $v \in \mathcal{M}$, $\|v\| \le 1$, $h \in H^2$, and $\|h\|_2 < 1$. Thus $\|tz\|_2 = \|thv\|_2 \le \|th\|_2$, so $\|th\|_2 \ge \|t\| - \varepsilon$.

**Corollary 4.** *For every $t \in \mathcal{M}$, $\|T_{t^*t}\| = \|t^*t\|$.*

*Proof.* We have $\|T_{t^*t}\| = \|PL_{t^*t}P\| = \|PL_t^*L_tP\| = \|L_tP\|^2 = \|t\|^2 = \|t^*t\|$.

**Corollary 5.** *The map $a \to T_a$ is an isometric algebra isomorphism and a weak$^*$ homeomorphism of $\mathcal{A}$ onto $\{T_a \colon a \in \mathcal{A}\}$.*

*Proof.* For $a \in \mathcal{A}$, $T_a = L_aP$, so by Corollary 3 the map is an isometry. Because the map $a \to L_a$ is a weak$^*$ continuous isomorphism and $H^2$ is invariant for $\{L_a \colon a \in \mathcal{A}\}$, the map $a \to T_a$ is a weak$^*$ continuous isomorphism. It follows from [2, Theorem 1.20] that $\{T_a \colon a \in \mathcal{A}\}$ is weak$^*$ closed and the map is a weak$^*$ homeomorphism.

Let $\mathcal{A}_*$ denote the space of weak$^*$ continuous linear functionals on $\mathcal{A}$. The next result shows that the weak$^*$ continuous functionals on the algebra $\{T_a \colon a \in \mathcal{A}\}$ have a rank one structure. In the language of the theory of dual algebras [2] the algebra has property $\mathbf{A}_1(1)$.

**Corollary 6.** *For every $\varepsilon > 0$ and for every $\phi \in \mathcal{A}_*$, there exist $g, h \in H^2$ such that $\phi(a) = \langle ag, h\rangle$ for $a \in \mathcal{A}$, and $\|g\|_2\|h\|_2 < (1+\varepsilon)\|\phi\|$.*

*Proof.* For $\varepsilon > 0$, choose $\eta > 0$ such that $(1+\eta)^2 < 1+\varepsilon$. There is a functional $\hat{\phi}$ in $\mathcal{M}_*$ such that $\hat{\phi}$ extends $\phi$ and $\|\hat{\phi}\| < (1+\eta)\|\phi\|$ by [4, Lemma 2.4]. Because of the duality between $L^1$ and $\mathcal{M}$, there is an $f \in L^1$ such that $\hat{\phi}(t) = \tau(tf)$ for $t \in \mathcal{M}$ and $\|\hat{\phi}\| = \|f\|_1$. So there exist vectors $x, y \in L^2$ such that $f = xy$ and

$\|x\|_2\|y\|_2 = \|f\|_1 = \|\hat{\phi}\| < (1+\eta)\|\phi\|$. By Theorem 2 $x = gv$ with $g \in H^2$, $v \in \mathcal{M}$, $\|v\| \leq 1$, and $\|g\|_2 < (1+\eta)\|x\|_2$. Thus, for all $a \in \mathcal{A}$,

$$\phi(a) = \langle L_a gv, y\rangle = \langle L_a g, yv^*\rangle = \langle L_a g, P(yv^*)\rangle.$$

Let $h = P(yv^*)$, and the result follows.

We now proceed to establish an analogue of the factorization theorem for $H^1$. Recall that any function $f$ in the classical $H^1$ space can be written $f = gh$ where $g, h$ are in $H^2$ and $\|f\|_1 = \|g\|_2\|h\|_2$. The following result was previously obtained by Saito [11, Lemma 5.5], but we include it for completeness.

**Corollary 7.** *For every $\varepsilon > 0$ and for every $f \in H^1$, there exist $g, h \in H^2$ such that $f = gh$ and $\|g\|_2\|h\|_2 < (1+\varepsilon)\|f\|_1$. If $f \in H_0^1$, then $h$ is in $H_0^2$.*

*Proof.* Because $f \in L^1$, there exist vectors $x, y \in L^2$ such that $f = xy$ and $\|x\|_2\|y\|_2 = \|f\|_1$. By Theorem 2 we can find $g \in H^2$ and $v \in \mathcal{M}$ such that $x = gv$, $\|v\| \leq 1$, $\|g\|_2 < (1+\varepsilon)\|x\|_2$ and $g^{-1} \in \mathcal{A}$. Now by [10, Lemma 3] $H^1 = \{x \in L^1 \colon \tau(xy) = 0 \text{ for all } y \in H_0^\infty\}$. Thus, for $a \in \mathcal{A}_0$,

$$\tau(af) = \tau(axy) = \tau(agvy) = \langle ag, (vy)^*\rangle.$$

Let $h = vy$. Then $h^* \in [\mathcal{A}_0 g]_2^\perp$. But $[\mathcal{A}_0 g]_2 = [\mathcal{A}_0]_2$, because $g^{-1} \in \mathcal{A}$. Thus $h^* \in [\mathcal{A}_0]_2^\perp = (H^2)^*$, so $h \in H^2$ and $f = xy = gvy = gh$. Finally, $\|g\|_2\|h\|_2 < (1+\varepsilon)\|x\|_2\|y\|_2 = (1+\varepsilon)\|f\|_1$.

Note that if $f \in H_0^1$, then $\tau(af) = 0$ for all $a \in \mathcal{A}$. So $h^* \in (H^2)^\perp = (H_0^2)^*$, and thus $h \in H_0^2$.

As a consequence of the previous factorization theorem, we can obtain an exact analogue of Nehari's Theorem on the norm of a Hankel operator. In the classical setting, the norm of a Hankel operator $H_f$ is given by the distance in $L^\infty$ from the symbol $f$ to $H^\infty$. This remains true in the noncommutative setting.

**Corollary 8.** *For every $t \in \mathcal{M}$, $\|H_t\| = \operatorname{dist}(t, \mathcal{A})$.*

*Proof.* By [10, Lemma 4] $H_0^1 = \{f \in L^1 \colon \tau(fy) = 0 \text{ for all } y \in \mathcal{A}\}$. By Corollary 7 $\{f \in H_0^1 \colon \|f\|_1 < 1\} = \{gh \colon g \in H^2, h \in H_0^2, \|g\|_2\|h\|_2 < 1\}$. Because $\mathcal{A}$ is w*-closed,

$$\begin{aligned}
\operatorname{dist}(t, \mathcal{A}) &= \sup\{|\tau(tf)| \colon f \in L^1, \tau(yf) = 0 \text{ for all } y \in \mathcal{A}, \|f\|_1 < 1\} \\
&= \sup\{|\tau(tf)| \colon f \in H_0^1, \|f\|_1 < 1\} \\
&= \sup\{|\tau(tgh)| \colon g \in H^2, h \in H_0^2, \|g\|_2\|h\|_2 < 1\} \\
&= \sup\{|\langle L_t g, h^*\rangle| \colon g \in H^2, h \in H_0^2, \|g\|_2\|h\|_2 < 1\} \\
&= \|H_t\|.
\end{aligned}$$

## 4. Harmonic conjugates

Every harmonic function $u$ on the unit disk has an associated harmonic function $\tilde{u}$, the harmonic conjugate of $u$, such that $u + i\tilde{u}$ is analytic and $\tilde{u}(0) = 0$. The map $u \to \tilde{u}$ is a real linear transformation which is $L^2$ bounded. We will establish an analogue of this result by thinking of Re $\mathcal{A}$, the real parts of the operators in $\mathcal{A}$, as a noncommutative version of the space of bounded harmonic functions. We first construct a real linear map from Re $\mathcal{A}$ to $\mathcal{A}$ which is the analogue of the Herglotz transform, i.e. integration against the kernel $(e^{i\theta} + z)(e^{i\theta} - z)^{-1}$.

**Theorem 9.** *There is a real linear transformation* $T\colon \operatorname{Re}\mathcal{A} \to \mathcal{A}$ *such that* $u = \operatorname{Re}(Tu)$, $\Phi(\operatorname{Im} Tu) = 0$, *and* $\|Tu\|_2 \le \sqrt{2}\|u\|_2$ *for all* $u \in \operatorname{Re}\mathcal{A}$. *Thus* $T$ *extends to a bounded operator from* $[\operatorname{Re}\mathcal{A}]_2$ *to* $H^2$.

*Proof.* Let $u \in \operatorname{Re}\mathcal{A}$. So $u = \operatorname{Re} g$ for some $g \in \mathcal{A}$. Let $a = g - \frac{1}{2}\Phi(g - g^*)$. Then $a \in \mathcal{A}$, $u = \operatorname{Re} a$, and $\Phi(\operatorname{Im} a) = 0$. Thus there exists an $a \in \mathcal{A}$ with the desired properties. We now show that such an element of $\mathcal{A}$ is unique. Suppose that $u = \operatorname{Re} a = \operatorname{Re} h$ for $a, h \in \mathcal{A}$, and $\Phi(\operatorname{Im} a) = \Phi(\operatorname{Im} h) = 0$. Then $a + a^* = h + h^*$ implies that $a - h = h^* - a^*$, so $(a-h)^* = h - a$. Furthermore, $\Phi(\operatorname{Im} a) = \Phi(\operatorname{Im} h) = 0$ implies that $\Phi(a) = \Phi(a^*)$ and $\Phi(h) = \Phi(h^*)$. Consequently, $\Phi(a) = \Phi(h)$, because $\operatorname{Re} a = \operatorname{Re} h$. Thus

$$\Phi((a-h)(a-h)^*) = \Phi((a-h)(h-a)) = \Phi(a-h)\Phi(h-a) = 0.$$

So $a = h$, because $\Phi$ is faithful. Thus we can define $Tu = a$, where $a$ is the unique element of $\mathcal{A}$ with $u = \operatorname{Re} a$ and $\Phi(\operatorname{Im} a) = 0$. It is easy to see that $T$ is real linear.

We now proceed to establish the inequality $\|Tu\|_2 \le \sqrt{2}\|u\|_2$. Let $Tu = d + b$, where $d \in \mathcal{D}$, $b \in \mathcal{A}_0$. Now $\Phi(\operatorname{Im} Tu) = 0$ and $\Phi(b) = \Phi(b^*) = 0$, so we have $d = \Phi(d) = \Phi(d^*) = d^*$. Thus

$$\|Tu\|_2^2 = \tau((d+b)(d+b^*)) = \|d\|_2^2 + \tau(db^*) + \tau(db) + \|b\|_2^2 = \|d\|_2^2 + \|b\|_2^2.$$

By a simple computation we have

$$0 = \tau(b^2) = \tau((\operatorname{Re} b)^2) - \tau((\operatorname{Im} b)^2) + 2i\tau((\operatorname{Re} b)(\operatorname{Im} b)).$$

Note that

$$\overline{\tau((\operatorname{Re} b)(\operatorname{Im} b))} = \tau((\operatorname{Im} b)(\operatorname{Re} b)) = \tau((\operatorname{Re} b)(\operatorname{Im} b)).$$

By equating real and imaginary parts we obtain

$$\|(\operatorname{Re} b)\|_2^2 = \|(\operatorname{Im} b)\|_2^2 \quad \text{and} \quad \tau((\operatorname{Re} b)(\operatorname{Im} b)) = 0.$$

So

$$\|b\|_2^2 = \|(\operatorname{Re} b)\|_2^2 + \|(\operatorname{Im} b)\|_2^2 = 2\|(\operatorname{Re} b)\|_2^2.$$

Thus

$$\|Tu\|_2^2 = \|\operatorname{Re} d\|_2^2 + 2\|\operatorname{Re} b\|_2^2.$$

Now because $b$ is orthogonal to $\mathcal{D}$, we have $\tau((\operatorname{Re} d)(\operatorname{Re} b)) = 0$. Consequently,

$$\|Tu\|_2^2 \le 2(\|\operatorname{Re} d\|_2^2 + \|\operatorname{Re} b\|_2^2) = 2\|\operatorname{Re}(d+b)\|_2^2 = 2\|u\|_2^2.$$

**Corollary 10.** *For each* $u \in [\operatorname{Re}\mathcal{A}]_2$ *there is a unique* $\tilde{u}$ *in* $[\operatorname{Re}\mathcal{A}]_2$ *such that* $u + i\tilde{u} \in H^2$ *and* $\Phi(\tilde{u}) = 0$. *Furthermore,* $\|\tilde{u}\|_2 \le \sqrt{2}\|u\|_2$.

*Proof.* Let $\tilde{u} = \operatorname{Im} Tu$.

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