CONTINUOUS FELL BUNDLES
ASSOCIATED TO MEASURABLE TWISTED ACTIONS

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Abstract. Given a measurable twisted action of a second-countable, locally compact group $G$ on a separable $C^*$-algebra $A$, we prove the existence of a topology on $A \times G$ making it a continuous Fell bundle, whose cross-sectional $C^*$-algebra is isomorphic to the Busby–Smith–Packer–Raeburn crossed product.

1. Introduction

Let $A$ be a $C^*$-algebra and $G$ be a locally compact group. According to [2], a twisted action of $G$ on $A$ is a pair $(\theta, w)$ of maps $\theta : G \to \text{Aut}(A)$, and $w : G \times G \to \text{UM}(A)$, where $\text{Aut}(A)$ denotes the automorphism group of $A$ and $\text{UM}(A)$ is the set of unitary elements in the multiplier algebra $\mathcal{M}(A)$, satisfying

(i) $\theta_e$ is the identity automorphism of $A$,
(ii) $\theta_r(\theta_s(a)) = w(r, s)\theta_{rs}(a)w(r, s)^*$,
(iii) $w(e, t) = w(t, e) = 1$,
(iv) $\theta_r(w(s, t))w(r, st) = w(r, s)w(rs, t),$

for $r, s, t$ in $G$ and $a$ in $A$.

Also following [2], we will say that $(\theta, w)$ is measurable if $\theta$ is strongly measurable in the sense that for each $a$ in $A$, the map

$$t \in G \mapsto \theta_t(a) \in A$$

is Borel measurable, and if $w$ is strictly measurable in the sense that, for each $a$ in $A$, the maps

$$r, s \in G \times G \mapsto aw(r, s) \in A,$$

$$r, s \in G \times G \mapsto w(r, s)a \in A$$

are Borel measurable.

For convenience, we will restrict ourselves to the treatment of separable $C^*$-algebras and second countable groups, as in [2], so that, in particular, the various notions of measurability for $A$ valued maps coincide.

Finally, $(\theta, w)$ will be termed continuous if the maps (1)–(3) above are continuous. In this case, it is easy to see that the conditions of [3, Definition 3.8] are
satisfied, and hence that we can construct the associated *semi-direct product bundle* of \( A \) by \( G \) [3, Theorem 3.10].

Fell bundles, also frequently referred to as \( C^* \)-algebraic bundles, were introduced by J. M. G. Fell [4] (see also [5]) in the 60’s. Among the many interesting features surrounding this concept, we would like to point out its relevance for the study of crossed product \( C^* \)-algebras. In fact Fell bundles can (and should) be viewed as intermediate steps in the construction of crossed products; the procedure being to start by constructing the associated semi-direct product bundle [5, VIII.4], [3] and then to consider its cross sectional algebra [5, VIII.17.2].

The available theory of Fell bundles, of which [5] is one of the most authoritative accounts, does not include, as far as we know, a systematic study of measurable (as opposed to continuous) bundles. However, crossed products by measurable twisted actions have been profitably studied by Packer and Raeburn [8], where they play an important role in the theory of group actions on \( C^* \)-algebras. Therefore, it seems plausible that this latter crossed product construction could be obtained in a similar two step procedure, involving, as the intermediate step, the construction of a “measurable” Fell bundle.

Our main point, however, is that, given a measurable twisted action of a second-countable group on a separable \( C^* \)-algebra, the associated \( L^1 \) algebra studied by Busby–Smith [2], as well as the crossed product of Packer–Raeburn [8], can both be obtained from a *continuous* Fell bundle. This result bears a certain degree of similarity with the result of S. Banach, according to which a measurable homomorphism between complete metric groups is necessarily continuous [1, Theorem I.4].

The stabilisation trick of Packer and Raeburn can be used to obtain information on the representation theory of twisted crossed products. Indeed, Kaliszewski [6, 7], has developed a theory of induced representations and Mackey normal subgroup analysis suitable for measurable twisted systems, as had been suggested in the introduction to [8]. Since the Mackey machine has, to a large extent, been generalized to Fell bundles, (e.g. [5, chapters XI and XII]), our result gives a new way of making it available to the study of twisted crossed products by establishing that they are isomorphic to cross sectional algebras of Fell bundles.

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2. The main results

Let us fix, throughout, a measurable twisted action \( (\theta, w) \) of a locally compact, second-countable group \( G \) on a separable \( C^* \)-algebra \( A \). If we let \( G^d \) denote the discrete group obtained by replacing the given topology of \( G \) by the discrete topology, then, following [3, Theorem 2.7], we get a Fell bundle structure on \( A \times G^d \), by means of the following operations, where we use the notation \( a\delta_t \) for \( (a, t) \) in \( A \times G^d \):

\[
(a\delta_r)(b\delta_s) = a\theta_r(b)w(r, s)\delta_{rs}, \quad a, b \in A, \quad r, s \in G;
\]

and

\[
(a\delta_t)^* = \theta_t^{-1}(a^*)w(t^{-1}, t)^*\delta_{t^{-1}}, \quad a \in A, \quad t \in G^d.
\]
We will refer to this bundle as $\mathcal{B}(A, G^d, \theta, w)$. Of course it bears no relationship, whatsoever, with the topological nature of $G$. Our main result is precisely intended to fill this gap. We will therefore show that there exists a topology on $A \times G$ for which $\mathcal{B}(A, G^d, \theta, w)$ is a continuous Fell bundle over $G$. (Whenever $G$ occurs without the superscript ‘$d$’, it is to be thought of as carrying its originally given topology.) In addition, we will show that the Banach algebra of $L^1$ sections of $\mathcal{B}(A, G^d, \theta, w)$ is isomorphic to $L^1(G, A, \theta, w)$ (see [2]) and its cross sectional $C^*$-algebra is isomorphic to the Packer–Raeburn crossed product $A \rtimes_{\theta, w} G$, both isomorphisms being canonical.

One of the main ingredients in the proof of this result is the “Packer–Raeburn stabilization trick” [8, Theorem 3.4], which we briefly describe below, mainly to fix our notation.

2.1. Definition ([8], Definition 3.1). The measurable twisted actions $(\alpha, u)$ and $(\beta, w)$ of $G$ on $A$ are exterior equivalent if there exists a strictly measurable map $v : G \to UM(A)$ such that

(i) $\beta_t(a) = v_t \alpha_t(a) u_t^*$, \quad $a \in A$, \quad $t \in G$.

(ii) $w(r, s) = v_{t-r} \alpha_t(r) u_t u_s^*$, \quad $r, s \in G$.

2.2. Proposition. If $(\alpha, u)$ and $(\beta, w)$ are exterior equivalent then the associated bundles $\mathcal{B}(A, G^d, \alpha, u)$ and $\mathcal{B}(A, G^d, \beta, w)$ are isomorphic.

Proof. One checks that the map $\phi : A \times G \to A \times G$ defined by $\phi(a, t) = (av_t^*, t)$ is an isomorphism for the respective bundle structures. In proving this, the identity $\beta_t^{-1}(v_t) = \alpha_t^{-1}(v_t)$, which follows from 2.1(i) with $a = \alpha_t^{-1}(v_t)$, comes in handy. □

The Packer–Raeburn stabilization trick asserts that, if $(\theta, w)$ is a measurable twisted action of the second-countable group $G$ on the separable $C^*$-algebra $A$, then there exists a strongly continuous (untwisted) action $\beta$ of $G$ on $A \otimes K$ (where $K$ denotes the algebra of compact operators on a separable Hilbert space), such that $(\theta \otimes 1, w \otimes 1)$ is exterior equivalent to $(\beta, 1)$. Therefore, by (2.2), the bundles

$$\mathcal{B}_0 := \mathcal{B}(A \otimes K, G^d, \theta \otimes 1, w \otimes 1)$$

and

$$\mathcal{B}_1 := \mathcal{B}(A \otimes K, G^d, \beta, 1)$$

are isomorphic. Now, since $\beta$ is strongly continuous, the product topology on $(A \otimes K) \times G$ makes $\mathcal{B}_1$ a continuous Fell bundle over $G$ [3, Theorem 3.10]. Thus we can make $\mathcal{B}_0$ into a continuous Fell bundle over $G$ by transferring the topology from $\mathcal{B}_1$ to $\mathcal{B}_0$ via the isomorphism given by (2.2).

On the other hand, if $p$ denotes a minimal projection in $K$, then $\mathcal{B}(A, G^d, \theta, w)$ sits naturally in $\mathcal{B}_0$ as the sub-bundle $(A \otimes p) \times G^d$.

The crucial point in our argument is to show that this sub-bundle, with the inherited topology, is a continuous Fell bundle.

In order to verify this claim, let us fix some notation. First of all, given the two distinct bundle structures on $(A \otimes K) \times G^d$, let us agree to denote the element $(x, t)$ in $(A \otimes K) \times G^d$ by $x \partial_t$ when it is viewed as an element of $\mathcal{B}_1$ while retaining the notation $x \theta_t$ when $\mathcal{B}_0$ is concerned. Secondly, let us denote by $v$ a given Borel map $v : G \to UM(A \otimes K)$ implementing the equivalence (2.1) between $(\theta \otimes 1, w \otimes 1)$ and $(\beta, 1)$, which exists by the stabilization trick.
By Proposition (2.2), the map
\[ \phi(x\delta_t) = xv^*_t \partial_t, \quad x \in A \otimes K, \quad t \in G, \]
is an isomorphism from \( \mathcal{B}_0 \) to \( \mathcal{B}_1 \).

Recall that any multiplier of the unit fiber algebra of a Fell bundle extends to a multiplier of order \( e \) (where \( e \) denotes the group unit) of the bundle concerned [5, VIII.3.8]. In particular, \( 1 \otimes p \), viewed as a multiplier of \( A \otimes K \), extends to a multiplier of \( \mathcal{B}_1 \), which we will denote by \( \pi \).

**2.3. Lemma.** For each \( t \) in \( G \) one has
\[ \phi\left((A \otimes p)\delta_t\right) = \pi\left((A \otimes K)\partial_t\right)\pi. \]

**Proof.** We have
\[
\pi\left((A \otimes K)\partial_t\right)\pi = (1 \otimes p)\partial_t(A \otimes K)(1 \otimes p)\partial_e = (1 \otimes p)(A \otimes K)v_t(1 \otimes p)v^*_t \partial_t = (A \otimes p)v^*_t \partial_t = \phi((A \otimes p)\delta_t). \]

\( \Box \)

This brings us to our first main result.

**2.4. Theorem.** Let \( (\theta, w) \) be a measurable twisted action of the locally compact, second-countable group \( G \) on the separable \( C^* \)-algebra \( A \). Also let \( \mathcal{B}_1 \) be as above. Then \( \mathcal{B}(A, G^d, \theta, w) \), equipped with the topology induced by the embedding
\[ \mathcal{B}(A, G^d, \theta, w) \rightarrow \mathcal{B}_1 \]
given by
\[ a\delta_t \mapsto (a \otimes p)v^*_t \partial_t, \]
is a continuous Fell bundle over \( G \).

**Proof.** The proof consists in showing that the collection of subspaces
\[ (A \otimes p)v^*_t \partial_t \subseteq (A \otimes K)\partial_t, \quad t \in G, \]
forms a continuous bundle of Banach spaces over \( G \) [5, II.13.1], in the sense that one can find a continuous section \( \gamma \) passing through any preassigned element \( (a \otimes p)v^*_0 \partial_0 \), and such that \( \gamma(t) \in (A \otimes p)v^*_t \partial_t \) for all \( t \) in \( G \) [5, II.13.18], [3, Proposition 3.3]. For this, it suffices to take a section \( \sigma \) of \( \mathcal{B}_1 \) such that \( \sigma(t_0) = (a \otimes p)v^*_0 \partial_0 \). Since the left and right actions of multipliers are continuous maps on the bundle [5, VIII.2.14]), \( \gamma(t) = \pi\sigma(t)\pi \) gives the desired section. \( \Box \)

Let us denote by \( \mathcal{B}(A, G, \theta, w) \) (omitting the superscript in \( G^d \)), the continuous bundle over \( G \), arising from (2.4)

**2.5. Theorem.** Let \( A, G, \theta \) and \( w \) be as in (2.4). Then the formula \( \psi(f)(t) = f(t)\delta_t \), for \( f \) in \( L^1(G, A) \) and \( t \) in \( G \), gives a Banach \(*\)-algebra isomorphism
\[ \psi : L^1(G, A, \theta, w) \rightarrow L^1(\mathcal{B}(A, G, \theta, w)), \]
which, in turn, induces an isomorphism between the crossed product \( A \rtimes_{\theta, w} G \) and the cross sectional \( C^* \)-algebra of \( \mathcal{B}(A, G, \theta, w) \).
Proof. Under the identification provided by Theorem (2.4) we may write \( \psi(f)(t) = (f(t) \otimes p)v^*_t \partial_t. \) Observe that, if \( f \) is in \( L^1(G, A, \theta, w) \) then \( \psi(f) \) is an integrable section of \( \mathfrak{B}(A, G, \theta, w) \) because \( v \) is strictly Borel measurable. The same reasoning applies to prove the converse and hence \( \psi \) gives an isometric linear isomorphism \( \psi : L^1(G, A, \theta, w) \to L^1(\mathfrak{B}(A, G, \theta, w)). \)

It is now easy to show that \( \psi \) is also a Banach *-algebra isomorphism.

The description of \( A \rtimes_{\theta, w} G \) which better suits our purposes is that given in [8, Remark 2.6], where \( A \rtimes_{\theta, w} G \) is described as the enveloping \( C^* \)-algebra of \( L^1(G, A, \theta, w) \). Since, on the other hand, the cross sectional \( C^* \)-algebra of \( \mathfrak{B}(A, G, \theta, w) \) is the enveloping \( C^* \)-algebra of the algebra of integrable sections of this bundle, we see that the last part of the statement follows by taking the enveloping \( C^* \)-algebras of the corresponding Banach algebras.

References


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