SIMPLE CONNECTEDNESS OF PROJECTIVE VARIETIES

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Abstract. A Lefschetz type theorem is proven relating the algebraic fundamental group of a smooth projective variety $X$ to the algebraic fundamental group of a subvariety set theoretically defined by $\leq \dim(X) - 2$ forms.

In this paper we prove a generalization of Grothendieck’s Lefschetz theorem for complete intersections (SGA2 XII 3.5). Our result is:

**Theorem 1.** Suppose that $k$ is a field, $W$ is a smooth, geometrically connected subvariety of $\mathbb{P}^m_k$ of dimension $n$ and $Z \subset W$ is a closed subscheme set theoretically defined by the vanishing of $r$ forms of $\mathbb{P}^m_k$ on $W$.

1. If $r \leq n - 1$ then $Z$ is geometrically connected and there is a surjection $\pi_1(Z) \to \pi_1(W)$.
2. If $r \leq n - 2$, then $\pi_1(Z) \cong \pi_1(W)$.

**Corollary 2.** Suppose that $k$ is a field and $Z \subset \mathbb{P}^n_k$ is a closed subscheme set theoretically defined by $r$ forms.

1. If $r \leq n - 1$ then $Z$ is geometrically connected.
2. If $r \leq n - 2$, then $\pi_1(Z) \cong \text{Gal}(\bar{k}/k)$ where $\bar{k}$ is an algebraic closure of $k$.

The corresponding theorem for the topological fundamental group of a complex projective variety follows from Hamm [H] and the Theorem of II 1.2 in [GM]. Their proofs use different methods (Morse theory) and do not extend to positive characteristic.

$\pi_1(X)$ will denote the algebraic fundamental group of a scheme $X$. If $k$ is a field, $\bar{k}$ will denote an algebraic closure of $k$.

**Simple Connectedness**

**Proof of the first half of 1. of Theorem 1.** Suppose that $r \leq n - 1$. Let $K$ be an extension field of $k$. Let $(A, m)$ be the local ring of the homogeneous coordinate ring of $W \otimes_k K$, $\hat{A}$ be completion at $m$. Suppose that $Z$ is set theoretically defined by $r$ forms $f_1, \ldots, f_r \in A$. $r \leq (n + 1) - 2$ implies $\text{spec}(\hat{A}/(f_1, \ldots, f_r)) - \hat{m}$ is connected by Corollary 4 to Theorem 1 [F], since $\hat{A}$ is a domain. Hence $Z$ is geometrically connected.

**Definition 3.** A morphism $f : X \to Y$ is separable if $f$ is flat and for any $y \in Y$, $X \times_Y k(y)$ is geometrically reduced over $k(y)$.
Lemma 4. Suppose that $Y$ is a noetherian scheme, $f : X \to Y$ is a finite type morphism. Let

$$A = \{ y \in Y \mid X_y \text{ is geometrically reduced over } k(y) \}.$$ 

Then

1. $A$ is a constructible subset of $Y$.
2. If $g : Y' \to Y$ is a morphism then

$$g^{-1}(A) = \{ y' \in Y' \mid X \times_Y Y'_y \text{ is geometrically reduced over } k(y') \}.$$ 

Proof. This is EGA IV.9.7.7 and IV.9.2.2 (iv).

Lemma 5. Suppose that $Y$ is a noetherian scheme and $Z \subset Y$ is a constructible subset. Let $Z$ be the closure of $Z$ in $Y$. If $Z$ contains no generic points of codimension one irreducible subschemes of $Y$, then $Z$ has codimension $\geq 2$ in $Y$.

Proof. This follows from the fact proved in EGA 0_{III} 9.2.3 that a constructible subset of an irreducible subset $W$ of $Y$ is dense in $W$ if and only if it contains a nonempty open subset.

We need to generalize to morphisms which are not separable the exact homotopy sequence for proper separable morphisms of SGA1 X 1.4 and Theorem 6.3.2.1 [M].

Theorem 6. Suppose that $Y$ is a connected regular scheme, $X$ is normal, $f : X \to Y$ is a proper morphism such that $f$ is separable in codimension one (in $Y$) and $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let $y \in Y$ be the generic point, $\overline{X}_y = X \times_Y \bar{k}(y)$. Then there is a natural right exact sequence

$$\pi_1(\overline{X}_y) \xrightarrow{\Phi} \pi_1(X) \xrightarrow{\Psi} \pi_1(Y) \to 0.$$ 

Proof. The proofs that $\Psi$ is surjective and $\Psi \circ \Phi = 0$ are exactly as in the proof of Theorem 6.3.2.1 [M]. We must prove that $\text{Image}(\Phi) \supset \text{kernel}(\Psi)$. By the criterion of 5.2.4 [M], we must show that if $g : X' \to X$ is a connected étale cover of $X$ and the base change $\overline{g} : X' \times_X \overline{X}_y \to \overline{X}_y$ has a section $\sigma$ over $\overline{X}_y$, then there exists a connected étale cover $Y'/Y$ such that $X' \cong X \times_Y Y'$.

Suppose that $g : X' \to Y$ is such a morphism. $f \circ g$ is proper and separable in codimension one by Lemma 6.3.2.2 [M]. Let $X' \xrightarrow{h} Y' \to Y$ be the stein factorization of $f \circ g$. By Theorem 6.2.1 [M] $Y' \to Y$ is étale in codimension one. $Y'$ is normal since $X'$ is. By purity of branch locus for regular schemes (c.f. SGA1 X 3.1) $Y' \to Y$ is étale.

It remains to show that the natural map $\alpha : X' \to X \times_Y Y'$ is an isomorphism. This is shown exactly as on pages 115-116 in the proof of Theorem 6.3.2.1 [M].

Corollary 7. Suppose that $Y$ is a connected regular excellent scheme, $X$ is normal, $f : X \to Y$ is a proper morphism such that $f$ is separable in codimension 1, and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let $z_0 \in Y$ be a point and $z_1 \in Y$ be the generic point. Let $\overline{X}_0 = X \times_Y \bar{k}(z_0)$, $\overline{X}_1 = X \times_Y \bar{k}(z_1)$. Then there is a natural surjection

$$\pi_1(\overline{X}_1) \to \pi_1(\overline{X}_0).$$ 

Proof. Let $A = \mathcal{O}_Y$, $z_0$, $\hat{A}$ be the completion of $A$ at its maximal ideal, $Y' = \text{spec}(\hat{A})$. Let $z'_1$ be the generic point and $z'_0$ be the closed point of $Y'$, $X' = X \times_Y Y'$, with natural morphism $f' : X' \to Y'$. $f'$ is proper and $f'_*\mathcal{O}_{X'} = \mathcal{O}_{Y'}$. $X'$ is normal since $X$ is excellent. If $P \subset \hat{A}$ is a height one prime then $\text{ht}(P \cap A) \leq 1$. By Lemma
4 \( f' \) is separable in codimension 1. Let \( \overline{X} = X' \times_Y' \overline{k}(z'_0), \overline{X}' = X' \times_Y' \overline{k}(z'_1) \). Let notation be as in the above paragraph. Given \( Y \subset W = k[x_0, \ldots, x_m] \) be the homogeneous ideal of \( W \). Suppose that \( r \leq n - 1 \) and \( k \) is algebraically closed.

Let \( H^1(P^m_k, \mathcal{O}(d)) \otimes I_W = H^1(P^m_k, \mathcal{O}(d)) \otimes I_W = 0 \) and we can take \( d_2 \) arbitrarily large relative to \( d_1 \) such that \( Z \) is defined set theoretically by the vanishing of \( r - 1 \) forms \( f_1, \ldots, f_{r-1} \) of degree \( d_1 \) and a form \( f_r \) of degree \( d_2 \). Let \( t_1 = h^0(W, \mathcal{O}_W(d_1)), t_2 = h^0(W, \mathcal{O}_W(d_2)). \) Let \( a_1^i, \ldots, a_{r-1}^i, b_r^i \) be \( (r-1)t_1 + t_2 \) indeterminates, where \( I \) indexes a basis \( \sigma_1, \ldots, \sigma_{t_1} \) of \( H^0(W, \mathcal{O}_W(d_1)) \) and \( J \) indexes a basis \( \tau_1, \ldots, \tau_{t_2} \) of \( H^0(W, \mathcal{O}_W(d_2)) \). Let

\[
F_1 = \sum_{l=1}^{t_1} a_l^i \sigma_1, \ldots, F_{r-1} = \sum_{l=1}^{t_1} a_{r-1}^i \sigma_{l,1}, F_r = \sum_{j=1}^{t_2} b_r^j \tau_j.
\]

\[
F_1, \ldots, F_r \in (k[x_0, \ldots, x_m]/I_W)[a_1^i, \ldots, a_{r-1}^i, b_r^j].
\]

Let \( Y = \text{Proj}(k[a_1^i, \ldots, a_{r-1}^i, b_r^j]), \)

\[
X = V(F_1, \ldots, F_r) \subset Y \times W,
\]

be the subscheme determined by \( F_1, \ldots, F_r \). There is a natural projective morphism \( f : X \to Y \). Let \( p \in Y \) be the closed point such that \( (X_p)_{\text{red}} \cong Z_{\text{red}} \).

**Proposition 8.** Let notation be as in the above paragraph.

1. \( X \) is smooth over \( k \).
2. \( f_* \mathcal{O}_X = \mathcal{O}_Y \).
3. Let \( E = \{ y \in Y \mid X_y \text{ is not geometrically reduced over } k(y) \} \), \( \overline{E} \) be the closure of \( E \) in \( Y \). Then \( \text{codim}_Y(\overline{E}) \geq 2 \).
4. Let \( F = \{ y \in Y \mid \text{there exists } x \in f^{-1}(y) \text{ such that } \mathcal{O}_{X,x} \text{ is not flat over } \mathcal{O}_{Y,y} \} \). Then \( \text{codim}_Y(F) \geq 2 \).

Note That \( E \) is constructible and \( F \) is closed.

**Proof.** \( X \) is smooth over \( k \) by the Jacobian criterion.

Let \( X \to Y' \to Y \) be the Stein factorization of \( f \). \( Y' \to Y \) is dominant, finite and \( Y' \) is normal. By Bertini’s theorem (cf. II 8.18 [Ha]), there exists a dense open subset \( U \) of \( Y \) such that if \( q \in Y \) is a closed point, then \( X_q \subset W \) is a smooth irreducible complete intersection of dimension \( \geq 1 \). Hence \( Y' \to Y \) is generically 1-1. Thus \( Y' = Y \) by Zariski’s Main Theorem, and \( f_* \mathcal{O}_X = \mathcal{O}_Y \).

Given \( a_l^i \in k \) (with at least one \( a_l^i \neq 0 \)), let

\[
V(a_l^i) = \text{Proj}(k[a_1^i, \ldots, a_{r-1}^i, b_r^j]/(a_l^i a_{l,k}^j - a_k^j a_l^i)) \subset Y.
\]
Each \( V(\overline{a_f^1}) \cong P^{d_2} \). The union of \( V(\overline{a_f^1}) \) over all choices of \( a_f^1 \) is \( Y \). Let
\[
D = \text{Proj}(k[a_1^f, \ldots, a_{(r-1)}^f, b_r^f]/(a_1^f, \ldots, a_{(r-1)}^f)) \cong \text{Proj}(k[b_r^f]).
\]
Let \( \overline{F_1}, \ldots, \overline{F_{r-1}} \in k[x_0, \ldots, x_m]/I_W \) be the corresponding specializations of \( F_1, \ldots, F_{r-1} \) over \( (a_f^1) \to (a_f^1) \).

We can choose \( a_f^1 \) such that if \( \Gamma \) is a codimension one integral component of \( F \cup E \) then \( \Gamma \cap V(\overline{a_f^1}) \) is not contained in \( D \) and if \( T \) is the subscheme of \( W \) determined by the vanishing of \( \overline{F}_1, \ldots, \overline{F}_{r-1} \), then \( T \) is a smooth subvariety of \( W \) of dimension \( n - (r - 1) \geq 2 \).

Let \( V = V(\overline{a_f^1}) - D \). \( V \) parametrizes the intersections of \( T \) with the zero locus of sections of \( H^0(W, \mathcal{O}_W(d_2)) \). Hence we have a natural identification
\[
V = \mathcal{V}(S(H^0(W, \mathcal{O}_W(d_2))^*)),
\]
where \(*\) denotes dual \( k \) vector space.

To show that \( \text{codim}_T(E) \geq 2 \) and \( \text{codim}_T(F) \geq 2 \), it suffices by our construction of \( V \) to show that \( \text{codim}_T(V \cap E) \geq 2 \) and \( \text{codim}_T(V \cap F) \geq 2 \).

Let \( I_T = (\overline{F}_1, \ldots, \overline{F}_{r-1}) + I_W \subset k[x_0, \ldots, x_m] \). \( I_T \) is the homogeneous ideal of \( T \). Recall that for fixed \( T \), we are free to choose \( d_2 \) arbitrarily large. Since \( \mathcal{O}(1) \) is ample, \( \dim T \geq 2 \), we can choose \( d_2 \) so that
\begin{enumerate}
  \item \( H^1(W, \mathcal{O}_W(d_2) \otimes I_T) = 0 \).
  \item \( h^0(W, \mathcal{O}_W(d_2) \otimes I_T) < h^0(W, \mathcal{O}_W(d_2)) - 1 \).
  \item If \( D, C \) are nonzero effective divisors on \( T \) such that \( \mathcal{O}_T(D + C) \cong \mathcal{O}_T(d_2) \), then
    \[
    h^0(T, \mathcal{O}_T(C)) < h^0(T, \mathcal{O}_T(d_2)) - \dim \text{Pic}(T) - 1.
    \]
\end{enumerate}

Assertions i) and ii) follow from Serre’s vanishing theorem, and since \( h^0(W, \mathcal{O}_W(d)) \) is a polynomial in \( d \) of degree \( n \) for large \( d \) and \( h^0(T, \mathcal{O}_T(d)) \) is a polynomial in \( d \) of degree \( n - (r - 1) \geq 2 \) for large \( d \).

Now we will verify iii). Let \( s = \dim \text{Pic}(T) + 2 \). If \( d_2 \) is sufficiently large, \( \mathcal{O}_T(d_2) \) has the property that if \( p_1, \ldots, p_s \) are any distinct closed points in \( T \), then
\[
h^0(T, \mathcal{O}_T(d_2) \otimes \mathcal{O}_T(-p_1 - \ldots - p_s)) = h^0(T, \mathcal{O}_T(d_2)) - s.
\]
If \( D, C \) are as in iii), and \( p_1, \ldots, p_s \) are distinct closed points of \( D \), then
\[
h^0(T, \mathcal{O}_T(C)) = h^0(T, \mathcal{O}_T(d_2)) \otimes \mathcal{O}_T(-D))
\leq h^0(T, \mathcal{O}_T(d_2) \otimes \mathcal{O}_T(-p_1 - \ldots - p_s))
< h^0(T, \mathcal{O}_T(d_2)) - \dim \text{Pic}(T) - 1
\]
and iii) holds.

By i) we have a natural exact sequence
\[
(1) \quad 0 \to H^0(W, \mathcal{O}_W(d_2) \otimes I_T) \to H^0(W, \mathcal{O}_W(d_2)) \to H^0(T, \mathcal{O}_T(d_2)) \to 0.
\]

Let \( H \to \mathbf{P}(H^0(T, \mathcal{O}_T(d_2))^*) \) be the universal family parametrizing the subschemes of \( T \) given by vanishing of sections of \( H^0(T, \mathcal{O}_T(d_2)) \). Let
\[
V' = \mathcal{V}(H^0(W, \mathcal{O}_W(d_2))^*) - \mathcal{V}(H^0(W, \mathcal{O}_W(d_2) \otimes I_T)^*).
\]
That is, \( V' \) is the complement of \( \mathcal{V}(H^0(W, \mathcal{O}_W(d_2)^*)) \) in
\[
V = \mathcal{V}(H^0(W, \mathcal{O}_W(d_2))^*).\]
(1) gives a natural surjection
\[ \lambda : V' \to \mathbf{P}(\mathcal{H}^0(T, \mathcal{O}_T(d_2))^*) \]
such that \( X_{V'} = \lambda^*(H) \).

We have \( V \cap F = \mathbf{V}(\mathcal{H}^0(W, \mathcal{O}_W(d_2) \otimes I_I^*) \), so that \( \text{codim}_V(V \cap F) \geq 2 \) by ii) and assertion 4) follows.

If \( \zeta \in V \) is the generic point of a codimension 1 subvariety of \( V \), then \( \zeta \in V' \) by ii), and the closure of \( \lambda(\zeta) \) has codimension \( \leq 1 \) in \( \mathbf{P}(\mathcal{H}^0(T, \mathcal{O}_T(d_2))^*) \). Let \( \alpha = \lambda(\zeta) \). By Lemma 4 \( X_{\zeta} \) is geometrically reduced over \( k(\zeta) \) if and only if \( H_\alpha \) is geometrically reduced over \( k(\alpha) \).

Let \( B' \) be the closure of \( \alpha \) in \( \mathbf{P}(\mathcal{H}^0(T, \mathcal{O}_T(d_2))^*) \). There is a finite radicial morphism \( \tau : B \to B' \) such that if \( \beta \) is the generic point of \( B \), \( (H_\beta)_{\text{red}} \) is geometrically reduced over \( k(\beta) \) (cf. IV.4.6.6 EGA). If \( H_\beta \) is not reduced, then there exists a dense open \( U \subset B \) and a flat map \( (H_\beta)_{\text{red}} \times_B U \to U \) such that the fibers over closed points of \( U \) are pairwise distinct subschemes of \( T \), each given by the vanishing of a section \( H^0(T, \mathcal{O}_T(C)) \) for some effective divisor \( C \) on \( T \) with \( h^0(T, \mathcal{O}_T(d_2) \otimes \mathcal{O}_T(-C)) > 0 \), and where each fiber has a common Hilbert polynomial \( P \). Let \( \text{Hilb}^P \) be the component of the Hilbert scheme of \( T \) of subschemes with the Hilbert polynomial \( P \).

There exists an immersion \( U \to \text{Hilb}^P \) such that \( (H_\beta)_{\text{red}} \times_B U \) is the pullback of the universal family over \( \text{Hilb}^P \).

There is a morphism \( \gamma : \text{Hilb}^P \to \text{Pic}(T) \) where the fiber containing the point corresponding to the subscheme \( C \) is \( \mathbf{P}(\mathcal{H}^0(T, \mathcal{O}_T(C))^*) \). By iii),
\[ \dim \text{Hilb}^P \leq \dim \mathbf{P}(\mathcal{H}^0(T, \mathcal{O}_T(d_2))^*) - 2. \]
Hence \( \dim(B') = \dim(U) \leq \dim \mathbf{P}(\mathcal{H}^0(T, \mathcal{O}_T(d_2))^*) - 2 \). This shows that \( \lambda(\beta) = \alpha \) has codimension \( \geq 1 \) in \( \mathbf{P}(\mathcal{H}^0(T, \mathcal{O}_T(d_2))^*) \), a contradiction, so that \( X_{\zeta} \) is geometrically reduced over \( k(\zeta) \). \( \text{codim}_V(V \cap E) \geq 2 \) so that \( \text{codim}_V(E) \geq 2 \).

**Proof of Theorem 1.** First suppose that \( k \) is algebraically closed. Consider the map \( f : X \to Y \) defined before Proposition 8. By Proposition 8 the assumptions of Corollary 7 are satisfied. Hence there is a surjection \( \pi_1(X_1) \to \pi_1(X_\eta) = \pi_1(Z) \). \( X_1 \subset W \otimes_k k(a_1^t, a_t^t) \) is a smooth irreducible complete intersection of dimension \( \geq 1 \) under the assumptions of 1) and of dimension \( \geq 2 \) under the assumptions of 2). The composite map
\[ \pi_1(X_1) \to \pi_1(Z) \to \pi_1(W) \]
is a surjection under the assumptions of 1) and is an isomorphism under the assumptions of 2) by SGA2 XII 3.5 and Proposition 7.3.2 [M]. If \( k \) is not algebraically closed the conclusions of 1) and 2) now hold by SGA1 IX 6.1 or Theorem 8.1.1 [M].

**References**


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