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ON LIE ALGEBRAS WITH NONINTEGRAL q-DIMENSIONS

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Abstract. In a recent paper an author has suggested a series of dimensions which include as first terms dimension of a vector space, Gelfand-Kirillov dimenision and superdimension. In terms of these dimensions the growth of free polynilpotent finitely generated Lie algebras has been specified. All these dimensions are integers. In this paper we study for all levels $q = 2, 3 \dots$ what numbers $\alpha > 0$ can be a q-dimension of some Lie (associative) algebra.

Introduction: Series of dimensions

Let A be a Lie (associative) algebra over a field K, generated by a finite set X; in this case we write $A = \mathrm{alg}(X)$ (moreover, all algebras in this paper are finitely generated). Denote by $A(X, n)$ the subspace spanned by all monomials in X of length not exceeding n (including identity if A is associative). Denote

$$\gamma_A(n) = \gamma_A(X, n) = \dim_K A(X, n),$$
$$\lambda_A(n) = \gamma_A(n) - \gamma_A(n-1),$$

where $\dim_K$ stands for dimension of a vector space over K. Growth of a function $\gamma_A(n)$ is an important characteristic of A.

On functions $f : \mathbb{N} \to \mathbb{R}^+$, where $\mathbb{R}^+ = \{\alpha \in \mathbb{R} \mid \alpha > 0\}$, we consider the following partial order: $f(n) \preceq g(n)$ iff there exist $N > 0,\ C > 0$, such that $f(n) \leq g(Cn),\ n \geq N$. An equivalence $f(n) \sim g(n)$ means that $f(n) \succeq g(n),\ f(n) \preceq g(n)$. It is easy to verify that for another generating set X' we have an equivalent function $\gamma_A(X, n) \sim \gamma_A(X', n)$. Sometimes it is better to use another partial order: $f(n) \ll g(n)$ iff there exists $N > 0$, such that $f(n) \leq g(n),\ n \geq N$.

The growth less then exponential (in sense $\preceq$) is called subexponential. If it is also greater than any polynomial growth, then it is called intermediate. To study the intermediate growth of Lie and associative algebras the following series of dimensions has been suggested [1], [2]. Denote by iteration

$$\ln^{(1)} n = \ln n; \quad \ln^{(q+1)} n = \ln(\ln^{(q)} n),\ q = 1, 2, \dots.$$

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Consider a series of functions $\Phi^q_\alpha(n),\ q = 1, 2, 3, \dots,$ of a natural argument with the parameter $\alpha \in \mathbb{R}^+$:

$$\begin{aligned}
&\Phi^1_\alpha(n) = \alpha,\\
&\Phi^2_\alpha(n) = n^\alpha,\\
&\Phi^3_\alpha(n) = \exp(n^{\alpha/(\alpha+1)}),\\
&\Phi^q_\alpha(n) = \exp\Big(\frac{n}{(\ln^{(q-3)} n)^{1/\alpha}}\Big); \quad q = 4, 5, \dots.
\end{aligned}$$

Supppose that $f(n)$ is any positive valued function of a natural argument. We define the (*upper*) *dimension of level* $q,\ q = 1, 2, 3, \dots,$ and *lower dimension of level* q by

$$\begin{aligned}
\mathrm{Dim}^q f(n) &= \inf\{\alpha \in \mathbb{R}^+ \mid f(n) \preceq \Phi^q_\alpha(n)\} = \inf\{\alpha \in \mathbb{R}^+ \mid f(n) \ll \Phi^q_\alpha(n)\},\\
\underline{\mathrm{Dim}}^q f(n) &= \sup\{\alpha \in \mathbb{R}^+ \mid f(n) \succeq \Phi^q_\alpha(n)\} = \sup\{\alpha \in \mathbb{R}^+ \mid f(n) \gg \Phi^q_\alpha(n)\}
\end{aligned}$$

(infimums as well as supremums do coincide [2]). We also call these dimensions q-dimensions.

Suppose that A is a finitely generated algebra with $\gamma_A(n)$ as above. We define the *q-dimension* (*lower q-dimension*), $q = 1, 2, 3, \dots,$ of A by

$$\mathrm{Dim}^q A = \mathrm{Dim}^q \gamma_A(n), \quad \underline{\mathrm{Dim}}^q A = \underline{\mathrm{Dim}}^q \gamma_A(n).$$

By the remark above these dimensions do not depend on generating set X. If $0 < \underline{\mathrm{Dim}}^q A \le \mathrm{Dim}^q A < \infty$ then we say that A belongs to *level* q.

Recall that 1-dimension coincides with ordinary dimension of a vector space over the field K: $\mathrm{Dim}^1 A = \underline{\mathrm{Dim}}^1 A = \dim_K A$. Dimensions of level 2 are exactly upper and lower Gelfand-Kirillov dimensions [3]. Dimensions of level 3 correspond to superdimensions of [4] up to normalization (see [2]). Other properties of these two dimensions can be found in a monograph [5] and review [6].

Dimensions of levels $q = 4, 5, \dots$ correspond to growths which are subexponential but are greater than any function $\exp(n^\beta),\ \beta < 1$. Such growths were not known and had not been studied until [1].

The first main result of [1], [2] specifies the growth of a universal enveloping algebra $U(L)$ provided that the growth of a Lie algebra L is known.

Theorem 1 ([1], [2])**.** *Let L be a finitely generated Lie algebra with* $\mathrm{Dim}^q L = \alpha > 0,\ q = 1, 2, \dots.$ *Also for $q \ge 3$ suppose that* $\underline{\mathrm{Dim}}^q L = \alpha$ *and for $q = 2$ suppose that* $\underline{\mathrm{Dim}}^2 \lambda_L(n) = \alpha - 1,\ \alpha \ge 1$. *Then*

$$\underline{\mathrm{Dim}}^{q+1} U(L) = \mathrm{Dim}^{q+1} U(L) = \alpha.$$

The proof was outlined in [1] as well as the proof of a theorem on growth of solvable Lie algebras. In [2] we give detailed proofs and also we prove a more general result on the growth of a free polynilpotent finitely generated Lie algebra. If $\mathcal{M}$ is a variety of Lie algebras then by $F(\mathcal{M}, k)$ we denote its free algebra of rank k. (For theory of varieties see the monograph [7].) Then the second main result of [2] is read as follows.

Theorem 2 ([2])**.** *Let $L = F(\mathcal{N}_{s_q} \dots \mathcal{N}_{s_2} \mathcal{N}_{s_1}, k),\ q \ge 2,$ be a free polynilpotent Lie algebra of rank $k,\ k \ge 2$. Then*

$$\underline{\mathrm{Dim}}^q L = \mathrm{Dim}^q L = s_2 \dim_K F(\mathcal{N}_{s_1}, k).$$

1. Main results

We see that all q-dimensions for free polynilpotent Lie algebras are integers. For the Gelfand-Kirillov dimension of associative algebras (recall that it coincides with dimension of level 2) the following facts are known:

Theorem 3 ([4])**.** *For any $\sigma \in [2, +\infty)$ there exists a two-generated associative algebra A with Gelfand-Kirillov dimension as follows:* $\underline{\mathrm{Dim}}^2 A = \mathrm{Dim}^2 A = \sigma$.

Also Gelfand-Kirillov dimension may equal 1 and 0 (latter are finite-dimensional algebras over the field). There is a gap: $\mathrm{Dim}^2 A \notin (0,1)$ for associative (Lie) algebra A because either $\gamma_A(n+1) \geq \gamma_A(n) + 1$, $\forall n$, hence $\gamma_A(n) \geq n$, or $\exists n\ \gamma_A(n+1) = \gamma_A(n)$ which easily implies that $A(X, N) = A(X, n)\ \forall N \geq n$. It is more interesting that there is another gap:

Theorem 4 (Bergman [5])**.** *The Gelfand-Kirillov dimension of an associative algebra does not belong to the interval* $(1, 2)$.

As for dimension of level 3 (which coincides with superdimension up to normalization) only the following fact is known:

Theorem 5 ([4])**.** *For any $\sigma \in (0, 1]$ there exists a two-generated associative algebra A with* $\mathrm{Dim}^3 A = \sigma$.

In this paper we study nonintegral dimensions for different levels $q = 2, 3, \ldots$ for Lie and associative algebras. For $\alpha \in \mathbb{R}$ let $\{\alpha\}$ be the least integer greater or equal than α. As the first result we prove

Proposition 1. *For any $\sigma \in (b-1, b)$, $b \geq 2$, there exists a two-generated Lie algebra $H \in \mathcal{N}_b\mathcal{A}$ with* $\underline{\mathrm{Dim}}^2 H = \mathrm{Dim}^2 H = \sigma$.

Remark. By Theorem 2 for $L = F(\mathcal{N}_b\mathcal{A}, 2)$ one has $\underline{\mathrm{Dim}}^2 L = \mathrm{Dim}^2 L = 2b$, but Proposition fills only segment $[1, b]$. Thus we suggest the question for which numbers $\sigma \in [b, 2b]$ does there exist a two-generated Lie algebra $H \in \mathcal{N}_b\mathcal{A}$ with $\underline{\mathrm{Dim}}^2 H = \mathrm{Dim}^2 H = \sigma$? □

In view of Proposition 1 the following fact is interesting.

Lemma 3. *Let $L \in \mathcal{A}^2$ be a finitely generated Lie algebra. Then* $\mathrm{Dim}^2 L$ *is an integer.*

The main result of this paper is as follows.

Theorem 6. *For levels $q = 2, 3 \ldots$ and any $\sigma \in [1, +\infty)$ there exists a two-generated Lie algebra $H \in \mathcal{A}^{q-2}\mathcal{N}_s\mathcal{A}$, $s = \{\sigma\}$ with* $\mathrm{Dim}^q L = \underline{\mathrm{Dim}}^q L = \sigma$.

Remarks. For level $q = 2$ the gap $\sigma \in (0, 1)$ is natural (see above). In case $q = 2$ it is interesting that unlike associative algebras Lie algebras have no gap $(1, 2)$ (compare Theorem 4). As for higher levels $q = 3, 4, \ldots$ we have an open problem whether for all $\sigma \in (0, 1)$ there exist Lie algebras with $\mathrm{Dim}^q L = \underline{\mathrm{Dim}}^q L = \sigma$. □

By applying Theorem 1, and adding results of Theorems 3,4,5 one has

Corollary. *For levels $q = 1, 2, 3, \dots$ there exist finitely generated associative algebras A with the following nonzero finite q-dimensions:*

$$\underline{\mathrm{Dim}}^q A = \mathrm{Dim}^q A \in \begin{cases} \mathbb{N}, & q = 1, \\ 1,\ [2, +\infty) & q = 2, \\ (0, +\infty) & q = 3, \\ (1, +\infty) & q = 4, 5, \dots. \end{cases}$$

Remark. For high levels $q = 4, 5, \dots$ we also have intervals $\sigma \in (0, 1)$ for further study. □

2. Nonintegral growth of level 2 (Gelfand-Kirillov dimension)

Let $L = L(x, y)$ be a free Lie algebra with free generators x, y where $y > x$. Now we need some facts [8]. The basis of L is formed by "regular monomials" which uniquely correspond to "regular words" in free associative algebra $A = A(x, y)$. Suppose that words in A are ordered lexicographically from the left. Then regular words $a \in A$ are exactly those that are greater than all cyclic conjugates (i.e. $a = a_1 a_2$ implies that $a > a_2 a_1$). For a regular word $a \in A$ there is a unique arrangement of brackets $[a] \in L$ that makes it into a regular monomial. Linear order is naturally extended to monomials of L.

Now we fix an integer $b \geq 2$ and set $I = \mathrm{Id}_L(y)$, $\widetilde{L} = L/I^{b+1}$, $\rho : L \to \widetilde{L}$. Denote

$$a = a(\alpha_1, \alpha_2 \dots, \alpha_s) = y x^{\alpha_1} y x^{\alpha_2} \dots y x^{\alpha_s}, \quad s = 1, 2, 3, \dots,$$
$$\deg a = s + \alpha_1 \cdots + \alpha_s, \quad \deg_y a = s, \quad \alpha_i(a) = \alpha_i,\ i = 1, \dots, s.$$

In case $s = 0$ we have only one regular word $a = x$. Then one has a basis

$$\widetilde{L} = \bigoplus_{n=1}^{\infty} \widetilde{L}_n; \quad \widetilde{L}_n = \langle\, \rho([a]) \,|\, a = a(\alpha_1, \alpha_2, \dots, \alpha_s) \text{ is a regular word with } \deg a = n,\ 0 \leq s \leq b \rangle$$

(first letter of a being y by regularity).

Suppose that $\Psi_s(n)$ denotes the number of regular words $a \in A$ with $\deg a = n$, $\deg_y a = s$. Then by the Witt formula [8, 2.2.8]:

$$\Psi_s(n) = \frac{1}{n} \sum_{m|n,\, m|s} \mu(m) \frac{(n/m)!}{(s/m)!\,((n-s)/m)!} \approx \frac{n^{s-1}}{s!},$$

$$\gamma_{\widetilde{L}}(n) = \sum_{s=0}^{b} \sum_{m=1}^{n} \Psi_s(m) \approx \frac{n^b}{b\, b!} \sim^d n^b, \tag{1}$$

where $f(n) \approx g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$, $f(n) \sim^d g(n)$ means that $0 < \mathrm{Dim}^q f(n) = \underline{\mathrm{Dim}}^q f(n) = \mathrm{Dim}^q g(n) = \underline{\mathrm{Dim}}^q g(n) < \infty$ for some level q, and $\mu(n)$ is a Möbius function.

Lemma 1. *Let $v = [a(\alpha_1, \dots, \alpha_s)]$ be a regular monomial. Then*

$$[v, x] = [a(\alpha_1, \dots, \alpha_{s-1}, \alpha_s + 1)] + \textit{ linear combination of lower monomials},$$

where $[a(\alpha_1, \dots, \alpha_{s-1}, \alpha_s + 1)]$ is regular.

Proof. We consider $L(x, y)$ to be naturally embedded into $A(x, y)$. For $c \in A$ by $\widehat{c} \in A$ we denote the leading word of c first relative to the degree and then

lexicographically. By [8, 2.2.4]: $v = a(\alpha_1, \dots, \alpha_s) + v^*$, where $v^* \in A$ is the combination of lower words of the same polydegree. Then

$$[v, x] = [yx^{\alpha_1} \dots yx^{\alpha_s} + v^*, x] = yx^{\alpha_1} yx^{\alpha_2} \dots yx^{\alpha_s+1} - xyx^{\alpha_1} \dots yx^{\alpha_s} + v^* x - xv^*,$$

where the first summand is the regular word [8, Lemma 2.1.7]. Evidently $\widehat{[v,x]} = a(\alpha_1, \dots, \alpha_{s-1}, \alpha_s + 1)$ and $v' = [v, x] - [a(\alpha_1, \dots, \alpha_s + 1)]$ is the combination of lower words. Since $v' \in L$, it can be expressed via lower regular monomials.

Fix $0 < \theta < 1$, $N = N(b, \theta)$ such that $n/b - n^\theta$ is increasing for $n \geq N$ and consider

(2)
$$\Xi = \{a = a(\alpha_1, \dots, \alpha_b) \,|\, a \text{ is a regular word}, 0 < \alpha_1 < \frac{n}{b} - n^\theta,\ n = \deg a \geq N\}.$$

□

Lemma 2. *There exists a map $\phi : a \mapsto \phi(a) \in L,\ a \in \Xi$, with*

1) *$\phi(a) = [a] +$ lower monomials of the same polydegree, $a \in \Xi$.*
2) *$\langle \rho\phi(\Xi) \rangle$ is an ideal of $\widetilde{L}$.*
3) *$\{\rho\phi(a) \,|\, a \in \Xi\}$ is linearly independent.*

Proof. 1) We proceed by induction on $\alpha_1(a) + \dots + \alpha_{b-1}(a) = k$. Suppose that $\phi(a)$ for all $a \in \Xi,\ \alpha_1(a) + \dots + \alpha_{b-1}(a) \leq k$ are constructed. Consider all $a \in \Xi$ with $\alpha_1(a) + \dots + \alpha_{b-1}(a) = k + 1$. Also we fix $\alpha_1, \dots, \alpha_{b-1}$; among all these elements we choose the word with minimal $\alpha_b(a)$. Then we set $\phi(a) = [a]$. By [8, 2.1.7] $ax^j,\ j = 1, 2, \dots$, are regular words. Since $n/b - n^\theta$ is the increasing function we have also that $ax^j \in \Xi,\ j = 1, 2, \dots$. For all of them we set

$$\phi(ax^j) = [a](\mathrm{ad}x)^j = [ax^j] + \text{lower monomials},$$

the latter equality being true by Lemma 1. In this way we exhaust all of Ξ.

2) $[\phi(a), y] \in \mathrm{Ker}\rho$ by construction of $\widetilde{L}$ and $[\phi(a), x] = \phi(ax)$ by construction of ϕ.

3) Follows by triangular decomposition of $\phi(\Xi)$ via regular monomials. □

Proposition 1. *For any $\sigma \in (b-1, b),\ b \geq 2$, there exists a two-generated Lie algebra $H \in \mathcal{N}_b\mathcal{A}$ with $\underline{\mathrm{Dim}}^2 H = \mathrm{Dim}^2 H = \sigma$.*

Proof. Set $H = H(x, y) = \widetilde{L}/\langle \rho\phi(\Xi) \rangle$. Since $[v, w],\ v, w \in \widetilde{L}$, necessarily contains y and all monomials with $b + 1$ letters y are trivial one has $H \in \mathcal{N}_b\mathcal{A}$.

Now let us compute the growth in H. By Lemma 2 the basis of H is formed by all $\rho(a)$, where $a = a(\alpha_1, \dots, a_s)$ are regular words either with $s < b$ or $s = b,\ a \notin \Xi$. By (1) the number of words of the first type with degree not exceeding n is approximately $n^{b-1}/((b-1)(b-1)!)$.

Suppose that $f(n)$ is the number of words of the second type with the degree n. For $a = a(\alpha_1, \dots, \alpha_b)$ by regularity we have $\alpha_1 \leq \alpha_2, \dots, \alpha_1 \leq \alpha_b$; hence by (2) also $n/b - n^\theta \leq \alpha_j,\ j = 1, \dots, b$. We evaluate an upper bound for $f(n)$ by the number of cases for which we can choose $b-1$ letters y among the following number of places: $n - 1 - b(n/b - n^\theta) < bn^\theta$; hence

$$f(n) \leq \binom{bn^\theta}{b-1} \approx \frac{b^{b-1}}{(b-1)!} n^{\theta(b-1)}.$$

To get the lower bound for $f(n)$ we consider the number of regular words of the second type with $n/b - n^\theta \le \alpha_1 < n/b - n^\theta/2$. If we take $\alpha_i \ge \alpha_1 + 1,\ i = 2, \dots, b$, then evidently $a(\alpha_1, \dots, \alpha_b)$ is regular. Thus we may set $\alpha_i \ge n/b - n^\theta/2$. For α_1 one has $n^\theta/2$ possibilities; and α_1 being fixed, the number of places we choose $b-2$ letters y is evaluated by $n - 2 - b(n/b - n^\theta/2) = bn^\theta/2 - 2$. Thus

$$f(n) \ge \frac{n^\theta}{2}\binom{bn^\theta/2 - 2}{b-2} \approx \frac{b^{b-2}}{2^{b-1}(b-2)!} n^{\theta(b-1)}.$$

Now we have

$$\sum_{m=1}^{n} f(m) \sim^d n^{\theta(b-1)+1}.$$

To get the desired dimension we set $\theta = \frac{\sigma-1}{b-1} \in (\frac{b-2}{b-1}, 1)$ provided that $\sigma \in (b-1, b)$. □

Corollary. *For any $\sigma \ge 1$ there exists a two-generated Lie algebra $H \in \mathcal{N}_b\mathcal{A}$ with $\underline{\mathrm{Dim}}^2 H = \mathrm{Dim}^2 H = \sigma$ for $b = \{\sigma\}$.*

Proof. If $\sigma = 1$ then we consider an algebra

$$H = \langle a, b_i,\ i = 1, 2, \dots \mid [a, b_i] = b_{i+1};\ [b_i, b_j] = 0\rangle = \mathrm{alg}(a, b_1) \in \mathcal{A}^2.$$

If $\sigma = b \ge 2$ is an integer then we consider $\widetilde{L}$ from above. Otherwise we apply Proposition. □

Lemma 3. *Let $L \in \mathcal{A}^2$ be a finitely generated Lie algebra. Then $\mathrm{Dim}^2 L = \underline{\mathrm{Dim}}^2 L$ is an integer.*

Proof. Suppose that L is generated by $X = \{x_1, \dots, x_k\}$. Then it is spanned by the following left-normed monomials [7]:

$$[x_{i_1}, x_{i_2}, x_{i_3} \dots, x_{i_t}], \quad x_{i_1} > x_{i_2} \le x_{i_3} \le \dots \le x_{i_t}.$$

We view L as a right module over L (and its enveloping algebra $U(L)$):

$$L_{U(L)} = \sum_\alpha V_\alpha, \quad V_\alpha = [x_{i_1}, x_{i_2}] \cdot U(L), \quad \alpha = \{(i_1, i_2) | 1 \le i_2 < i_1 \le k\}.$$

Since we can arbitrarily permute $x_{i_3}, \dots, x_{i_t}$ (see [8], [7]), we view V_α as the module over a commutative algebra $U(L/L^2)$ and also as an exact module over its homomorphic image T_α. Let X_α be the image of X under this homomorphism. Then

$$\max_\alpha \gamma_{T\alpha}(X_\alpha, n-2) \le \gamma_L(X, n) \le \sum_\alpha \gamma_{T\alpha}(X_\alpha, n-2).$$

Recall that the Gelfand-Kirillov dimension of a commutative algebra is always an integer and it coincides with lower Gelfand-Kirillov dimension. Therefore $\mathrm{Dim}^2 L = \underline{\mathrm{Dim}}^2 L$ and this number is equal to the maximum of finitely many Gelfand-Kirillov dimensions of T_α. □

3. Nonintegral growth of high level

Proof of Theorem 6. The proof is obtained by induction on q. The case q=2 is settled in Proposition 1.

Now we suppose that for $q-1$ we have constructed the Lie algebra $L \in \mathcal{A}^{q-1}\mathcal{N}_s\mathcal{A}$ generated by $X = \{x, y\}$ with $\mathrm{Dim}^{q-1}L = \underline{\mathrm{Dim}}^{q-1}L = \sigma$. Now let $H = \langle z\rangle \mathrm{wr} L$ be an abelian wreath product, where $\langle z\rangle$ is an one-dimensional Lie algebra. One has [7]:

$$H = Z \oplus L,\ Z = zU(L);\quad [h_1, h_2] = z_1 \circ l_2 - z_2 \circ l_1 + [l_1, l_2];$$
$$\text{where } h_i = z_i + l_i;\quad z_i \in Z,\ l_i \in L;\ i = 1, 2,$$

where $z \circ l,\ z \in Z,\ l \in L$ denotes an action in the free $U(L)$-module $Z = zU(L)$.

Next we set $\overline{y} = z + y,\ \overline{X} = \{x, \overline{y}\},\ Y = \{x, y, z\}$ and take $\overline{L} = \mathrm{alg}(\overline{X}) \subset \overline{H}$. By the same argument as in [2]

$$\gamma_{\overline{L}}(\overline{X}, n) \le \gamma_H(Y, n) \le \gamma_{U(L)}(X, n-1) + \gamma_L(X, n). \tag{3}$$

To get the lower bound let us suppose that we have an element $0 \ne f \in Z \cap \overline{L}$. Let $f \in \overline{L}(\overline{X}, t)$. Then

$$\overline{L}(\overline{X}, n) \supset f \circ U(L)(X, n-t);\quad \gamma_{\overline{L}}(\overline{X}, n) \ge \gamma_{U(L)}(X, n-t). \tag{4}$$

Now (3),(4) along with Theorem 1 and properties of q-dimensions [2] yield that $\mathrm{Dim}^q\overline{L} = \underline{\mathrm{Dim}}^q\overline{L} = \sigma$.

Next we show how we can find f. By construction $\overline{Z} = (\dots((\overline{L}^2)^{s+1}\underbrace{)^2\dots)^2}_{q-3 \text{ times}} \subset Z$. We only need to show that $\overline{Z} \ne 0$.

First we consider that $q = 3$. For simplicity we take $s = 2$. We denote

$$a(y, \Lambda) = \Big[[yx^\alpha], \big[[yx^\beta], [yx^\gamma]\big]\Big],\qquad \Lambda = (\alpha, \beta, \gamma),\ \alpha < \beta < \gamma.$$

We choose $a(y, \Lambda) \notin \Xi$ (see (2)) and compute

$$\begin{aligned} f =& a(\overline{y}, \Lambda) = g(z, y, \Lambda) + a(y, \Lambda), \quad \text{where} \\ & g(z, y, \Lambda) = zx^\alpha[yx^\beta yx^\gamma] - zx^\beta[yx^\gamma][yx^\alpha] + zy^\gamma[yx^\beta][yx^\alpha] \in Z. \end{aligned}$$

By virtue of identity $a(y, \Lambda) = 0$, and $0 \ne f \in Z \cap \overline{L}$.

In case $q = 4$ we use notations above and set

$$\overline{\overline{L}} = \mathrm{alg}(\overline{\overline{y}}, x) \subset \langle \overline{z}\rangle \mathrm{wr}\overline{L},\qquad \overline{\overline{y}} = \overline{z} + \overline{y},\ \overline{\overline{Z}} = \overline{z}U(\overline{L}).$$

If we set $\Lambda' = (\alpha', \beta', \gamma'), \alpha' < \beta' < \gamma',\ a(y, \Lambda') \notin \Xi$ then we have

$$\begin{aligned} \overline{f} =& [a(\overline{\overline{y}}, \Lambda), a(\overline{\overline{y}}, \Lambda')] \\ =& [g(\overline{z}, \overline{y}, \Lambda) + a(\overline{y}, \Lambda), g(\overline{z}, \overline{y}, \Lambda') + a(\overline{y}, \Lambda')] \\ =& g(\overline{z}, \overline{y}, \Lambda)a(\overline{y}, \Lambda') - g(\overline{z}, \overline{y}, \Lambda')a(\overline{y}, \Lambda) + [a(\overline{y}, \Lambda), a(\overline{y}, \Lambda')]. \end{aligned}$$

The second part is trivial and $\overline{f} \ne 0$ is the desired element provided that Λ, Λ' consist of different elements. If $q > 4$ then we proceed further in the same way. $\square$

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