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NOT ALL JULIA SETS ARE QUASI-SELF-SIMILAR

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ABSTRACT. We show that there exist rational functions, whose Julia set fails to be quasi-self-similar.

1.

One of the conspicuous features of the Julia sets of rational functions is that small parts of them look very much like some large parts. Sullivan has introduced a proper concept to describe the situation: the quasi-self-similarity. He also established the quasi-self-similarity of the Julia sets of all hyperbolic rational functions [1, Theorem 8.6], [3, Theorem 7], [8, p. 742]. One of the open problems listed at the end of [3] asks, whether the same is true of all rational functions of degree ≥ 2. The purpose of the present note is to show that this is not the case.

2.

Let $c \in (0,1]$. A set E in the euclidean n-space $\mathbb{R}^n$ is c-porous if each closed ball $\overline{B}^n(x,r) \subset \mathbb{R}^n$ contains a point z such that the open ball $B^n(z,cr)$ does not meet E; E is porous if it is c-porous for some c (see e.g. [10]). For instance, Cantor sets with constant ratio in $\mathbb{R}^n$ are porous in $\mathbb{R}^n$. Given $k > 0$, we let ϕ_k stand for the similarity map $x \mapsto kx$, $x \in \mathbb{R}^n$. A nonempty set $E \subset \mathbb{R}^n$ is called K-quasi-self-similar if there is an $r_0 > 0$ such that, given any closed ball $\overline{B}^n(x,r)$ with $t = r_0/r > 1$, there exists a K-quasi-isometry $f : \phi_t(\overline{B}^n(x,r) \cap E) \to E$, i.e., f satisfies

$$(1) \qquad \frac{1}{K}|y-z| \leq |f(y)-f(z)| \leq K|y-z| \quad \text{for all } y,z \in \phi_t(\overline{B}^n(x,r) \cap E).$$

Quasi-self-similarity means K-quasi-self-similarity for some $K \geq 1$. See [1, p. 121], [3, p. 65], [4, p. 183]. The constant r_0 is called a standard size of E [4, p. 183]. We are going to show that quasi-self-similarity implies porosity under some mild restrictions.

Lemma. *Let $E \subset \mathbb{R}^n$ be a compact, nowhere dense, quasi-self-similar set. Then E is porous in $\mathbb{R}^n$.*

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Proof. Let K be a constant of quasi-self-similarity of E, and let r_0 be a standard size of E. Assume that E fails to be porous. Then we find a sequence (x_j) of points in $\mathbb{R}^n$ and a sequence (r_j) of radii such that if $B^n(z_j, r'_j)$ is the maximal open ball with $z_j \in \overline{B}^n(x_j, r_j)$ and $B^n(z_j, r'_j) \cap E = \varnothing$, then r'_j/r_j converges to 0. Another way to express this state of affairs is to say that $A_j = \phi_{t_j}(\overline{B}^n(x_j, r_j) \cap E) - \phi_{t_j}(x_j)$ converges to $\overline{B}^n(0, r_0) = \overline{B}^n(r_0)$ in the Hausdorff metric, defined in the space of nonempty compact subsets of $\mathbb{R}^n$. We have used the standard notation $A - y = \{x - y | x \in A\}$ and the abbreviation $t_j = r_0/r_j$. We will see that this convergence property permits us to construct a K-quasi-isometry from $\overline{B}^n(r_0)$ into E.

Obviously r_j tends to 0 as $j \to \infty$. Hence we may assume that $r_j < r_0$ for all $j \geq 1$. By assumption, there exists a K-quasi-isometry $f_j : A_j \to E$ for each $j \geq 1$. Let $C = \{y_i | i \in N\}$ be a countable dense subset of $\overline{B}^n(r_0)$. Since $A_j \to \overline{B}^n(r_0)$ in the Hausdorff metric, we find for each $i \in N$ a sequence (y_{ij}) such that $y_{ij} \in A_j$ for all i, j and $\lim_{j\to\infty} y_{ij} = y_i$. Since E is compact, there is a subsequence $(y_{1j_{1k}})$ of (y_{1j}) such that $\lim_{k\to\infty} f_{j_{1k}}(y_{1j_{1k}})$ exists. Similarly, we can find a subsequence (j_{2k}) of (j_{1k}) such that $\lim_{k\to\infty} f_{j_{2k}}(y_{2j_{2k}})$ exists. We proceed in this manner. Finally, thanks to the well-known diagonal process, we find a strictly increasing sequence of indices, say (j_k), such that $\lim_{k\to\infty} f_{j_k}(y_{ij_k})$ exists for each $i \in N$. Hence we are in a position to define

$$f : C \to \mathbb{R}^n, \qquad f(y_i) = \lim_{k\to\infty} f_{j_k}(y_{ij_k}).$$

Obviously $f(C) \subset E$.

We next verify that f is a K-quasi-isometry. Let $x, y \in C$, say $x = y_h$, $y = y_i$, and let $\varepsilon > 0$. Pick $k \in N$ such that $|y_h - y_{hj_k}| < \varepsilon$, $|y_i - y_{ij_k}| < \varepsilon$, $|f(y_h) - f_{j_k}(y_{hj_k})| < \varepsilon$ and $|f(y_i) - f_{j_k}(y_{ij_k})| < \varepsilon$. Since $f_{j_k} : A_{j_k} \to E$ is a K-quasi-isometry, we have

$$|f_{j_k}(y_{hj_k}) - f_{j_k}(y_{ij_k})| \leq K|y_{hj_k} - y_{ij_k}|.$$

Hence

$$\begin{aligned} |f(x) - f(y)| = |f(y_h) - f(y_i)| &< 2\varepsilon + K|y_{hj_k} - y_{ij_k}| \\ &< 2\varepsilon + K(2\varepsilon + |y_h - y_i|) = 2(1+K)\varepsilon + K|x - y|. \end{aligned}$$

Letting $\varepsilon \to 0$ gives the right-hand side of (1). The proof of the left-hand inequality is similar.

Now f is uniformly continuous in a dense subset of $\overline{B}^n(r_0)$. This implies that f admits a continuous extension $f^* : \overline{B}^n(r_0) \to E$. It is again a simple matter to verify that f^* is a K-quasi-isometry. But this contradicts the assumption that E is nowhere dense. The proof is complete. □

3.

It is not too difficult to show, making use of net measures, that the Hausdorff dimension of every set, which is porous in $\mathbb{R}^n$, is less than n. See e.g. [9, p. 127]. Hence we deduce

Corollary 1. *Let $E \subset \mathbb{R}^n$ be a compact, nowhere dense, quasi-self-similar set. Then $\dim_H(E)$, the Hausdorff dimension of E, is less than n.*

Consider now the family of complex quadratic polynomials

$$f_c(z) = z^2 + c, \qquad c \in \mathbb{C},$$

and let $J(f_c)$ denote the Julia set of f_c. Note that $J(f_c)$ is a compact nowhere dense subset of $\mathbb{C}$, because ∞ belongs to the Fatou set of f_c. Shishikura [5], [6] has recently shown that there are values of c for which $\dim_H(J(f_c)) = 2$. More precisely, there is a residual subset F of the boundary of the Mandelbrot set such that if $c \in F$, then $\dim_H(J(f_c)) = 2$. Hence we have

Corollary 2. *There are values $c \in \mathbb{C}$ such that $J(f_c)$ fails to be quasi-self-similar.*

Remark 1. As mentioned above, the Julia set of any hyperbolic rational function is quasi-self-similar (see [2, pp. 89–93] for basic properties of hyperbolic rational maps). It follows from Corollary 1 that $\dim_H(J(f)) < 2$ for such functions. Sullivan [8, Theorem 4] has deduced this result relying on properties of conformal measures defined on Julia sets.

Remark 2. Of course, the same question can be proposed in the context of finitely generated Kleinian groups; that is, is the limit set of any finitely generated Kleinian group of $\overline{\mathbb{R}}^n$ quasi-self-similar? The answer is again in the negative. This follows, in view of Corollary 1, from a result of Sullivan [7], according to which there are finitely generated Kleinian groups of $\overline{\mathbb{R}}^2$ whose limit set has Hausdorff dimension two. Note, however, that the Hausdorff dimension of a geometrically finite Kleinian group of $\overline{\mathbb{R}}^n$ is always less than n [9, Theorem D].

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