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LOCAL DERIVATIONS OF REFLEXIVE ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a reflexive algebra in Banach space X such that both $O_+ \neq O$ and $X_- \neq X$ in Lat $\mathcal{A}$, the invariant subspace lattice of $\mathcal{A}$, then every derivation of $\mathcal{A}$ into itself is spatial. Furthermore, if X is additionally reflexive, then the set of all inner derivations of $\mathcal{A}$ into itself is topologically algebraically reflexive.

1. INTRODUCTION

If $\mathcal{B}$ is a Banach algebra, we say that a linear transformation $\Phi : \mathcal{B} \to \mathcal{B}$ is a local derivation if for every $B \in \mathcal{B}$ there is a derivation $\delta_B : \mathcal{B} \to \mathcal{B}$ depending on B, such that $\Phi(B) = \delta_B(B)$. There are three recent publications [2, 3, 5] where some conditions under which every local derivation (particularly, norm-continuous local derivation) is a derivation are given.

In [3], Kadison considered local derivations on von Neumann algebras. He proved that each norm-continuous local derivation of a von Neumann algebra $\mathcal{B}$ into a dual $\mathcal{B}$-bimodule $\mathcal{M}$ is a derivation.

Larson and Sourour [5] proved that every local derivation of $B(X)$ into itself is a derivation, where $B(X)$ denotes the algebra of all bounded linear operators on a complex Banach space X.

In [2] Han Deguang and Wei Shuyun proved that every norm-continuous local derivation of the nest algebra Alg $\mathcal{N}$ is a derivation, where $\mathcal{N}$ is a nest on a reflexive Banach space X such that for any $0 \subset M \in \mathcal{N}$ and $X \supset N \in \mathcal{N}$, there exist $E_1 \in (0, M]$ and $E_2 \in (N, X]$, such that $(E_1)_- \subset E_1$ and $(E_2)_- \subset E_2$, where $(E, F] = \{K \in \mathcal{N} : E \subset K \subseteq F\}$, where "$\subset$" denotes proper inclusion of subspaces.

The main purpose of this paper is to show that if $\mathcal{A}$ is a reflexive algebra in reflexive Banach space X such that both $O_+ \neq O$ and $X_- \neq X$ in Lat $\mathcal{A}$, the invariant subspaces lattice of $\mathcal{A}$, then every norm-continuous local inner derivation of $\mathcal{A}$ into itself is an inner derivation.

The results stated above can be related to the concept of topological algebraic reflexivity of linear spaces of operators. This notion has appeared in different contexts (see [1], [4] and [6]). We will adopt the terminology and notation of [4]. Recall them as follows: Let X be a Banach space over the complex field and $B(X)$ denote the algebra of all bounded linear operators on X. If $\mathcal{T}$ is a subset of $B(X)$, we

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write $\mathrm{ref}_{\mathrm{at}}(\mathcal{T}) = \{T \in B(X) : Tx \in \mathcal{T}v,\ x \in X\}$, where $\mathcal{T}x = \{Sx : S \in \mathcal{T}\}$. The set $\mathcal{T}$ is said to be topologically algebraically reflexive if $\mathcal{T} = \mathrm{ref}_{\mathrm{at}}(\mathcal{T})$. Therefore our result stated above can be restated that the set of all inner derivations of $\mathcal{A}$ is topologically algebraically reflexive.

2. Preliminaries and notations

In what follows we denote by X a fixed complex Banach space. The usual notation Lat $\mathcal{B}$ will denote the lattice of invariant subspaces for a subset $\mathcal{B} \subseteq (X)$, and Alg $\mathcal{L}$ will denote the algebra of bounded linear operators leaving invariant every member of a family $\mathcal{L}$ of subspaces. $\mathcal{B}$ is reflexive if $\mathcal{B} = \mathrm{ref}\ \mathcal{B}$, where $\mathrm{ref}\ \mathcal{B} = \{T \in B(X) : Tx \in [\mathcal{B}x], x \in X\}$, $[\cdot]$ denotes the norm closure.

For a lattice $\mathcal{L}$ of subspaces of X, if $N \in \mathcal{L}$, we denote $\bigwedge\{M \in \mathcal{L} : N \not\subseteq M\}$ by N_- and $\bigwedge\{M \in \mathcal{L} : M \not\subseteq N\}$ by N_+.

For a subset $\mathcal{T} \subseteq X$, $\mathcal{T}^{\perp} = \{f \in X^* : f(\mathcal{T}) = \{0\}\}$, where X^* is the dual space of X. If $x \in X$ and $f \in X^*$, the rank one operator $u \mapsto f(u)x$ is denoted by $x \otimes f$. If M is a subspace of X and $T \in B(X)$, the restriction of T on M is denoted by $T|_M$.

Let $\mathcal{B}$ be an algebra and $\mathcal{D}$ be a bimodule of $\mathcal{B}$. A derivation δ from $\mathcal{B}$ into itself is a linear mapping such that $\delta(AB) = A\delta(B) + \delta(A)B$ for all A and B in $\mathcal{B}$. δ is said to be inner (resp. spatial) if there exists an element $T \in \mathcal{B}$ (resp. $T \in \mathcal{D}$) such that $\delta(A) = TA - AT$ holds for all A in $\mathcal{B}$.

The following lemma will get repeated use.

Lemma 2.1 ([7])**.** *If $\mathcal{L}$ is a subspace lattice, then $x \otimes f \in$ Alg $\mathcal{L}$ if and only if there exists an element $L \in \mathcal{L}$ such that $x \in L$ and $f \in (L_-)^{\perp}$.*

3. Local derivations

Throughout this section, $\mathcal{A}$ will be a reflexive algebra in Banach space X such that both $O_+ \neq O$ and $X_- \neq X$ in Lat $\mathcal{A}$.

Theorem 3.1. *Every derivation of $\mathcal{A}$ into itself is spatial.*

Proof. Let $\delta : \mathcal{A} \to \mathcal{A}$ be a derivation. Let $x_0 \in X$ and $f_0 \in (X_-)^{\perp}$ be chosen so that $f_0(x_0) = 1$. Then $x_0 \otimes f_0 \in \mathcal{A}$ and $x \otimes f_0 \in \mathcal{A}$ for all x in X.

Define a map $T : X \to X$ by

$$Tx = \delta(x \otimes f_0)x_0.$$

Then T is linear. Since $A(x \otimes f_0) = Ax \otimes f_0$, we have $\delta(Ax \otimes f_0) = \delta(A) \cdot x \otimes f_0 + A \cdot \delta(x \otimes f_0)$. Applying both operators in this equation to x_0, we get $\delta(A)x = TAx - ATx$. This is true for every x, so $\delta(T) = TA - AT$.

We now prove that T is bounded. For every operator A in $\mathcal{A}$, the transformation $TA - AT$ is bounded. On the other hand for every rank one operator $R = u \otimes f$ in $\mathcal{A}$, the transformation TR is bounded, since $TR = Tu \otimes f$. Therefore we must have that $RT = TR - \delta(R)$ is bounded for every rank one operator R in $\mathcal{A}$. It is easy to see that T has a closed graph and hence is bounded. □

Remark. Now we only know that every derivation of $\mathcal{A}$ into itself is spatial, but the question that if every derivation of $\mathcal{A}$ into itself is inner is uncertain.

In what follows, let ϕ be a linear transformation from $\mathcal{A}$ into $\mathcal{A}$ with the property that for each $T \in \mathcal{A}$, there exist operators A_T and B_T in $\mathcal{A}$ such that $\phi(T) = A_T T + T B_T$.

Theorem 3.2. *For each $N \in \operatorname{Lat} \mathcal{A}$, there exist linear mappings $A_N : N \to N$ and $C_N : (N_-)^\perp \to (N_-)^\perp$ such that $\phi(x \otimes f) = x \otimes C_N f + A_N x \otimes f$ for every $x \in N$ and every $f \in (N_-)^\perp$.*

The proof of this theorem is similar to that of the related results of [5]. So we omit it.

For X and O_+, there exist linear mappings A_X, C_X, A_{O_+} and C_{O_+} satisfying the conclusion of Theorem 3.2. We have

Lemma 3.3. *Both C_X and A_{O_+} are bounded. Moreover, if ϕ is additionally norm-continuous, then A_X and C_{O_+} are also bounded.*

Proof. Consider a rank one projection $P = x \otimes f$, where $x \notin X_-$ and $f \in (X_-)^\perp$, so $f(x) = 1$ and $P \in \mathcal{A}$.

By Theorem 3.2, we have

$$\phi(P) = A_X x \otimes f + x \otimes C_X f. \tag{3.1}$$

By the hypothesis on ϕ, there exist operators S and T in $\mathcal{A}$ such that

$$\phi(I - P) = S(I - P) + (I - P)T.$$

Therefore $P\phi(I - P)P = 0$, and so

$$P\phi(P)P = P\phi(I)P. \tag{3.2}$$

Using (3.1), we have that $P\phi(P)P = f(A_X x)P + C_X f(x)P$. Furthermore $P\phi(I)P = f(\phi(I)x)P$. So equation (3.2) yields

$$f(A_X x) + C_X f(x) = f(\phi(I)x) \tag{3.3}$$

for all $x \notin X_-$ and $f \in (X_-)^\perp$ such that $f(x) = 1$. For arbitrary $x \notin X_-$ and $f \in (X_-)^\perp$ we can write f as a linear combination of two linear functionals f_1 and $f_2 \in (X_-)^\perp$ satisfying $f_1(x) = f_2(x) = 1$. Therefore equation (3.3) holds for all $x \notin X_-$ and $f \in (X_-)^\perp$.

Let $\{f_n\}$ be a sequence in $(X_-)^\perp$ such that $f_n \to f$ and $C_X f_n \to g$ for some f and g in $(X_-)^\perp$. Then we have

$$f_n(A_X x) + C_X f_n(x) = f_n(\phi(I)x).$$

Hence $f(A_X x) + g(x) = f(\phi(I)x)$ and so $g(x) = C_X f(x)$ for any $x \notin X_-$.

As g and $C_X f \in (X_-)^\perp$, it is obvious that $g(x) = C_X f(x)$ for any $x \in X_-$. Then we have $g(x) = C_X f(x)$ for any $x \in X$, and so $g = C_X f$. This shows that C_X has a closed graph and hence is bounded.

The proof of the boundedness of A_{O_+} goes similarly.

Moreover, if ϕ is additionally norm-continuous, it is easy to see that both C_{O_+} and A_X are also closed and hence bounded. □

Lemma 3.4. *For any $N \in \operatorname{Lat} \mathcal{A}$ such that $O_+ \subseteq N \subset X$, there exists $\lambda_N \in C$, the complex field, such that $A_X|_N = A_N + \lambda_N I|_N$ and $C_N|_{(X_-)^\perp} = C_X + \lambda_N I|_{(X_-)^\perp}$.*

Proof. We can easily have that for any $x \in N$ there is $\lambda_x \in C$ such that

$$(A_X - A_N)x = \lambda_x x.$$

Since $A_X - A_N$ is linear on N, now the lemma goes easily. □

Lemma 3.5. *If ϕ is additionally norm-continuous, then there exist linear operators A in $B(X)$ and C in $B(X^*)$ such that*

$$\phi(x \otimes f) = Ax \otimes f + x \otimes Cf \tag{3.4}$$

for every $x \in X$ and $f \in (X_-)^\perp$ or for every $x \in O_+$ and $f \in X^*$. Particularly, A is in $\mathcal{A}$.

Proof. Define $A = A_X$ and $C = C_{O_+}$. Using that $A_{O_+} = (A_X - \lambda_{O_+} I)|_{O_+}$ and $C_X = (C_{O_+} - \lambda_{O_+} I)|_{(X_-)^\perp}$ we have that both A and C are linear and satisfy equation (3.4). Using the closed graph theorem, we get that A and C are bounded.

Particularly, by the conclusion of Lemma 3.4, for every $N \in \operatorname{Lat} \mathcal{A}$, we can obtain that $A(N) \subseteq N$, i.e. $A \in \mathcal{A}$. □

Lemma 3.6. *If X is reflexive and ϕ is norm-continuous additionally, then there exist operators $A \in \mathcal{A}$ and $B \in B(X)$ such that $\phi(x \otimes f) = Ax \otimes f + x \otimes fB$ for any $x \in X$ and $f \in (X_-)^\perp$ or for any $x \in O_+$ and $f \in X^*$.*

Proof. By Lemma 3.5, there exist operators $A \in \mathcal{A}$ and $C \in B(X^*)$ such that $\phi(x \otimes f) = Ax \otimes f + x \otimes Cf$ for any $x \in X$ and $f \in (X_-)^\perp$ or for any $x \in O_+$ and $f \in X^*$.

By the reflexivity of X, there exists an operator $B \in B(X)$ such that $B^* = C$. The lemma now follows easily. □

Theorem 3.7. *The hypotheses on X and ϕ are the same as those of Lemma 3.6. Then there exist A and B in $\mathcal{A}$ such that*

$$\phi(T) = AT + TB \quad \textit{for every } T \textit{ in } \mathcal{A}.$$

Proof. By Lemma 3.6, we have that $\phi(F) = AF + FB$ where F is a finite linear combination of rank one operators of the form $x \otimes f$ with $x \in X$ and $f \in (X_-)^\perp$ or $x \in O_+$ and $f \in X^*$. And $A \in \mathcal{A}$ and $B \in B(X)$ are as in Lemma 3.6.

Let $\psi : \mathcal{A} \to B(X)$ be defined by $\psi(T) = AT + TB$ for every $T \in \mathcal{A}$, and let $\phi_0 = \psi - \phi$. We must show that $\phi_0 = 0$.

It is obvious that $\phi_0(F) = 0$ for every above-mentioned F.

Fix $T \in \mathcal{A}$ such that $\phi_0(T) \neq 0$, then there is a nonzero vector $x \notin X_-$ with $\phi_0(T)x \neq 0$. And so there exists a nonzero functional $f \in (X_-)^\perp$ such that $f(x) = 1$. So $P = x \otimes f$ is a projection in $\mathcal{A}$.

(1) Let $g \notin (O_+)^\perp$. Then there is $y \in O_+$ such that $g(y) = 1$. We have that $Q = y \otimes g$ is a projection in $\mathcal{A}$. For each $T \in \mathcal{A}$, there exist A_T and B_T in $B(X)$ such that

$$\phi_0((I-Q)T(I-P)) = A_T(I-Q)T(I-P) + (I-Q)T(I-P)B_T.$$

Then

$$Q\phi_0((I-Q)T(I-P))P = 0.$$

But $T - (I-Q)T(I-P)$ has the form of F, so $\phi_0(T - (I-Q)T(I-P)) = 0$, i.e. $\phi_0(T) = \phi_0((I-Q)T(I-P))$. Hence $Q\phi_0(T)P = 0$ and then

$$g(\phi_0(T)x) = 0 \quad \text{for each } g \notin (O_+)^\perp.$$

(2) Now let $g \in (O_+)^\perp$. If $\phi_0(T)x \in O_+$, then $g(\phi_0(T)x) = 0$. If $\phi_0(T)x \notin O_+$ and $g(\phi_0(T)x) \neq 0$, there is $h \notin (O_+)^\perp$ such that $h(\phi_0(T)x) = 0$. Then $g + h \notin (O_+)^\perp$ and $g + h(\phi_0(T)x) \neq 0$. This is in contradiction with (1).

It follows that $g(\phi_0(T)x) = 0$ for every $g \in X^*$. And so $\phi_0(T)x = 0$, a contradiction. □

Corollary 3.8. *If X is a reflexive Banach space, then the set of all inner derivations of $\mathcal{A}$ into itself is topologically algebraically reflexive.*

Proof. If δ is a local inner derivation, then for each $A \in \mathcal{A}$ there exists A_T in $\mathcal{A}$ such that $\delta(T) = A_T T - T A_T$ and so $\delta(I) = 0$. Now this corollary follows easily. □

Corollary 3.9 ([2])**.** *If X is a reflexive Banach space and $\mathcal{N}$ is a nest on X such that both $O_+ \neq O$ and $X_- \neq X$ in $\mathcal{N}$, then $\mathcal{D}(\mathcal{N})$, the set of all derivations of* Alg $\mathcal{N}$*, is topologically algebraically reflexive.*

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